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Ultra-Linear Amplifiers

IN recent years, and particularly in the U.S.A., the so-called ultra-linear amplifier has become popular among quality enthusiasts. A great deal has been written about it and a good many performance figures have been published which do seem to show some reduction of non-linearity distortion as compared with similar triode, tetrode or pentode amplifiers[1]. No serious attempt at explaining why this result should occur seems to have been made, however.

The arrangement is shown in Fig. 1 applied to a single output valve (a pair in push-pull is usually adopted) and it can be seen that it differs from a normal output stage only in that the screen grid is connected to a tapping on the output transformer primary. If this tapping were at the h.t. end of the winding, the screen-grid would be joined to h.t. and the stage would be an ordinary tetrode or pentode one. If it were at the anode end of the winding, the screen-grid would be joined to the anode, and the stage would be a normal triode one.

What happens is that a portion of the alternating voltage developed on the anode is fed back to the screen grid and the amount of feedback depends on the position of the tapping on the winding. By varying the amount of feedback, a gradual transition between the limiting triode and tetrode conditions is obtainable. Now a triode is generally thought to be more linear than a pentode and it is sometimes said that this can be accounted for by the linearizing action of feedback. A triode can be regarded as merely a tetrode with 100% feedback to the screen-grid.

If this were all, the ultra-linear stage would represent an intermediate condition between the triode and the tetrode and one would expect it to be less linear than the triode and more linear than the tetrode. On this basis, there is nothing to account for its being more linear than either.

The flaw in this argument is the supposition that the ordinary ideas about negative feedback hold good when they are applied to this special case of feedback to an electrode other than the input electrode. The normal ideas have been developed on the basis of the feedback and the input being applied to a common electrode so that the feedback voltage and the input voltage are additive. It is well known that feedback then always tends to improve the linearity. When separate electrodes are used, however, it can be shown that feedback will introduce curvature into an otherwise straight characteristic.

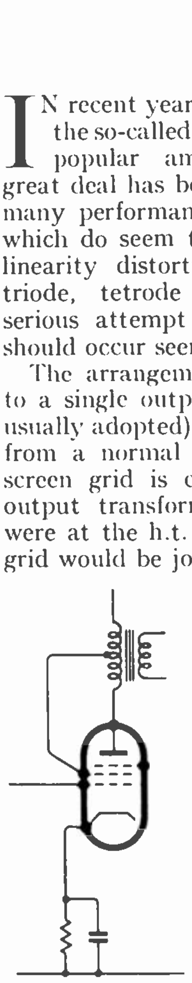

Fig. 1

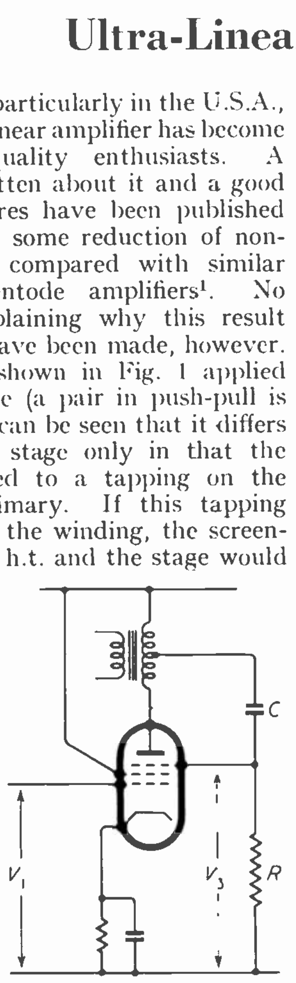

Fig. 2

[1] "Amplifiers and Superlatives" by D. T. N. Williamson and P. J. Walker, *Wireless World*, September 1952, p. 357.

A modified form of the ultra-linear amplifier is shown in Fig. 2; in this, a fraction β of the anode voltage is applied through the CR coupling to the suppressor grid. It is well known that in a pentode the cathode current is

$$i_k = a_1 + b_1 V_1$$

within the limits of a linear approximation. It is also well known that it is independent of V_3 which serves only to control the division of current between screen-grid and anode, so that

$$i_a = i_k\,(a_3 + b_3 V_3)$$

Therefore, we can write

$$i_a = (a_1 + b_1 V_1)(a_3 + b_3 V_3) \quad \ldots \quad (1)$$

In Fig. 2, the anode voltage is $-i_a R_a$, where R_a is the anode load, and so $V_3 = -\beta i_a R_a$, and (1) becomes

$$i_a = \frac{a_3(a_1 + b_1 V_1)}{1 + \beta b_3 R_a\,(a_1 + b_1 V_1)}$$

$$= A[a_1 + b_1\,(1 - a_1 B)\,V_1 - b_1^2 B\,(1 - a_1 B)\,V_1^2 + b_1^3 B^2\,(1 - a_1 B)\,V_1^3 \ldots\ldots] \quad \ldots \quad (2)$$

where $A = a_3/(1 + a_1 b_3\,\beta R_a)$

and $B = b_3\,\beta R_a/(1 + a_1 b_3\,\beta R_a)$

Feedback to the suppressor grid thus introduces curvature into an otherwise linear characteristic. If the input V_1 is a sine wave, the anode current will contain all harmonics of it.

Feedback to the screen-grid of a valve is, of course, not quite the same thing as feedback to the suppressor grid. However, as is well known, the mutual conductance does depend upon the screen voltage and so the equation for this condition must be rather like (1); there must be a term involving the product of the control- and screen-grid voltages. If this product term is present, then the equation for screen-grid feedback will be of the same general form as (2).

It would thus appear that feedback to an electrode other than the input electrode is a bad thing. However, we have only so far shown that it makes an otherwise linear stage non-linear. There is the possibility that the kind of non-linearity which it introduces is inverse to that which occurs naturally in valves. If this possibility is a reality, then a critical degree of feedback to screen or suppressor grid would improve the linearity of a stage.

Let us suppose that the input grid has a non-linear action, but the feedback grid is still linear, so that the equation for anode current is

$$i_a = (a_1 + b_1 V_1 + c_1 V_1^2 + d_1 V_1^3)(a_3 + b_3 V_3)$$

and, as before, $V_3 = -\beta i_a R_a$. Then

$$i_a = \frac{a_3(a_1 + b_1 V_1 + c_1 V_1^2 + d_1 V_1^3)}{1 + \beta b_3 R_a(a_1 + b_1 V_1 + c_1 V_1^2 + d_1 V_1^3)}$$

$$= A\,\frac{a_1 + b_1 V_1 + c_1 V_1^2 + d_1 V_1^3}{1 + B(b_1 V_1 + c_1 V_1^2 + d_1 V_1^3)}$$

where A and B have the same values as before. This equation can be expanded to the form

$$i_a = A\,[a_1 + (b_1 - a_1 B)V_1 - (1 - a_1 B)(b_1^2 B - c_1)V_1^2 + \{B^2 b_1^3(1 - a_1 B) - B b_1 c_1(1 - 2a_1 B) - B(a_1 d_1 + b_1 c_1) + d_1\}\,V_1^3 \ldots]$$

retaining terms only up to the cube.

It is at once evident that it may be possible to choose β so that the coefficient of V_1^3 is zero. It is not worth while to attempt to work out the condition for this. The equation would be complex and of little practical utility because we have assumed a linear control of anode current by the feedback grid and this is unlikely to be present in practice.

With feedback to a grid other than the signal grid, however, the possibility exists of being able so to adjust the amount of feedback that a particular harmonic can be eliminated from the output. In the ultra-linear amplifier, two valves are used in push-pull so that all even harmonics are, ideally, eliminated.

The third harmonic, which is normally the most important remaining one, can, again ideally, be eliminated by a critical adjustment of feedback to the screen-grids. The distortion present should thus be confined to odd-order harmonics higher than the third. Superficially, therefore, the ultra-linear amplifier would appear to be an improvement on ordinary types. We say superficially because we have not examined the amplitudes of the residual harmonics and the ability to eliminate a third harmonic would confer no benefit if it meant that the higher harmonics became of comparable amplitude.

There is no reason to suppose that this is the case, however, for the evidence of measurement does support the claims made for the ultra-linear amplifier. Our aim here has been merely to show the mechanism by which this improvement is achieved, for we have felt that it has remained too long unexplained.

W. T. C.

TUNABLE TEMPERATURE-COMPENSATED REFERENCE CAVITY

With Separate Tuning and Compensating Mechanisms

By M. S. Wheeler

SUMMARY.—A geometry is proposed for a tunable temperature-compensated microwave cavity having mechanically-separate actions in tuning and in compensation. The compensation is considered first at one frequency where it is shown that, in using practical materials, the compensated frequency change with temperature is non-linear. By the proper choice of such materials, however, the characteristics may be made linear, allowing for greater perfection in frequency compensation. Based on this linear characteristic, the requirements for compensation over a frequency range are derived. An equivalent circuit, which is justified by experimental results, is described giving information about this type of mechanism.

Introduction

FIXED-TUNED frequency reference cavities, having good frequency stability, have been made in production numbers since World War II. The principal example of this type of device is the 1Q* series of 3-cm valves, having a vacuum seal to make atmospheric effects upon frequency negligible and a bimetal temperature compensator to minimize thermal effects. As more and more fixed frequencies are required, it is reasonable to ask if similar precision could be attained in a tunable cavity.

Of the many ways in which it would be possible to achieve this end, engineering compromise may lead one to separate the functions of tuning and temperature compensation, producing the tuning action in a low-sensitivity region of the cavity where relatively large motion is required per Mc/s of temperature compensation. This condition gives the maximum precision of control in tuning but allows for a physically-small compensating mechanism from which only small motions are possible. One might also choose a nosed-in resonator giving small physical size, and freedom from unwanted modes as compared with straight cylindrical geometry. A vacuum envelope is, of course, required because of the importance of atmospheric changes upon frequency.

The major design problem is temperature compensation across the tuning range. But in order to describe the theory of operation, let us first consider a fixed-tuned device, and later find the effects of changing the design frequency.

1. Fixed-Tuned Cavity

It has been shown[1] that if a cavity of resonant frequency f is made of arbitrary geometry of a material having a linear coefficient of expansion, a change in temperature Δt produces a frequency change given by:

$$\Delta f = -\alpha f \Delta t \qquad \ldots \quad (1)$$

In addition, the cavity may be temperature-compensated by having a frequency-sensitive surface that can be distorted, producing an opposing linear frequency change of R Mc/s per mil† of motion. This action may be produced by a bimetal of temperature coefficients α_1 and α_2 and of effective length l, resulting in a total frequency change given by:

$$\Delta f = -\alpha f \Delta t + Rl(\alpha_2 - \alpha_1)\Delta t \qquad \ldots \quad (2)$$

Equation (2) predicts a linear relation between temperature change and the resulting frequency change. By design, then, it would be possible to choose the bimetal length l, for metals of given expansion coefficients, α_1 and α_2, such that Δf would vanish in equation (2). For a fixed frequency device, the compensation would be perfectly accomplished and no frequency change would be observed. It is known, however, that metal expansion is not precisely linear. For the precision required in this case, it becomes necessary to describe more accurately the bimetal behaviour. This is understandable when it is noted that the separate terms in equation (2) amount to about 12 Mc/s for copper devices used at 9,000 Mc/s over 75°C temperature range, and yet we are to ask that the difference remain within a few tenths of a Mc/s of zero.

With this is mind, the temperature coefficients may be expanded in a Taylor series about room temperature:

$$\left.\begin{aligned} \alpha &= \alpha_0 + \beta\,\Delta t + \ldots\ldots \\ \alpha_1 &= \alpha_{01} + \beta_1\,\Delta t + \ldots\ldots \\ \alpha_2 &= \alpha_{02} + \beta_2\,\Delta t + \ldots\ldots \end{aligned}\right\} \qquad \ldots \quad (3)$$

where the α_0 terms are the normal linear temperature coefficients at room temperature, and the β are second-order‡ coefficients, which are

* The 1Q23 for example is a 9280-Mc/s transmission cavity with a loaded Q of 2000 and 5 db of insertion loss. Frequency change is limited to ± 0·3 Mc/s for a 75°C temperature change.

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† Mil is used in this case as one thousandth of an inch.

‡ Note that β as defined in Equation (3) is twice as great as the 2nd-order coefficient used in some handbooks, such as "Handbook of Chemistry and Physics", 35th edition, Chemical Rubber Publication Company, p. 2835.

usually much smaller than the α_0. If equation (2) is written in differential form with the substitution of two terms of the expansion of equation (3) and the result integrated, one finds upon rewriting in the form of equation (2):

$$\Delta f = -f\left(\alpha_0 + \beta\frac{\Delta t}{2}\right)\Delta t + lR\left(\alpha_{02} + \beta_2\frac{\Delta t}{2} - \alpha_{01} - \beta_1\frac{\Delta t}{2}\right)\Delta t. \quad (4)$$

In this equation, Δf and Δt are again the frequency and temperature change respectively, with respect to room temperature. It may be seen that the frequency shift is no longer linear with the temperature change but it is a quadratic through the origin as illustrated by curve 1 in Fig. 1.

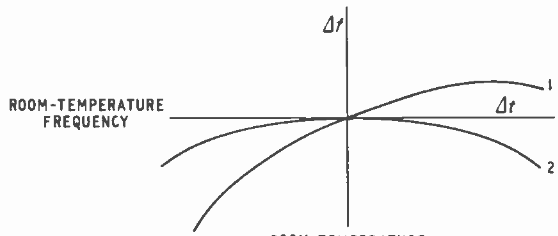

Fig. 1. *Frequency change with temperature: 1, General cavity having both 1st- and 2nd-order error in compensation; 2, Cavity having correct 1st-order compensation.*

Following the same procedure as before, the bimetal length may be chosen such that the linear terms in equation (4) vanish leaving a quadratic residual:

$$\Delta f = -\beta f\frac{\Delta t^2}{2} + lR\left(\beta_2 - \beta_1\right)\frac{\Delta t^2}{2}\ \ldots \quad (5)$$

where

$$l = \frac{f\alpha_0}{R\left(\alpha_{02} - \alpha_{01}\right)} \quad (6)$$

This is symmetrical about room temperature, as shown by curve 2 in Fig. 1. Equation (5) may be written in a more useful form:

$$\Delta f = \frac{\alpha_0\left(\beta_2 - \beta_1\right) - \beta\left(\alpha_{02} - \alpha_{01}\right)}{\alpha_{02} - \alpha_{01}} f\frac{\Delta t^2}{2}. \quad (7)$$

It can be seen that by the proper selection of temperature coefficients, the quadratic residual could also be made to vanish. Thus,

$$\frac{\alpha_0}{\beta} = \frac{\alpha_{02} - \alpha_{01}}{\beta_2 - \beta_1} \quad (8)$$

A useful terminology might be introduced to speak of the linearly-perfect compensation, defined by equation (6), as first-order temperature compensation and the quadratic correction, defined by equation (8), as the second-order temperature compensation. From the above, it can be seen that first-order compensation depends upon cavity dimensions and materials, but that second-order compensation depends only upon the materials used. In practice, one will first choose a combination of metals satisfying equation (8) because there will be no bimetal length which will eliminate the second-order effect. The bimetal length will then be set by equation (6). In a group of production cavities having the normal accumulation of mechanical tolerances, the first-order frequency shift will not be precisely zero, sometimes being slightly positive and sometimes slightly negative. The frequency-temperature relation will then be, as a result, a family of approximately straight lines through the origin.

One interesting result of equation (7), in which correct first-order compensation is assumed, is that the residual frequency change of a copper cavity compensated with, say copper and invar, is identical with the residual frequency change of an invar cavity, compensated with copper and invar. This shows that, except for the fact that first-order compensation is easier to control with a low-expansion cavity, there is no improvement in optimum performance over the higher-expansion cavity compensated with the same materials.

Let us assume, now, that materials for a reference cavity have been properly chosen by equation (8) resulting in a negligible second-order frequency shift over the temperature range that is required of the design. We may then return to the linear frequency-shift equation and inquire what errors are introduced when frequency is added as a variable.

2. Tunable Cavity Requirements

At any one frequency, the device might be compensated perfectly by equation (6) as in any fixed-tuned cavity. Considering this a design centre, then, one might inquire how compensation will change as the valve is tuned away from this point. As described in the introduction, the tuning and compensating mechanisms are being considered separately, so there will be no change in the effective length of the bimetal during tuning. The sensitivity of the cavity to the compensation action, R, however, is a complicated frequency function depending upon cavity dimensions. Using the subscript, 0, to indicate design centre, substitutions in equation (2) give

$$\Delta f = -f\alpha\,\Delta t + \frac{R}{R_0} f_0\,\alpha\,\Delta t \quad (9)$$

where

$$R_0 = \frac{f_0\,\alpha}{l\left(\alpha_2 - \alpha_1\right)} \quad (10)$$

and R is a frequency function.

The ideal variation of R, then, to give good compensation at all frequencies would be

$$R = \frac{f}{f_0} R_0 \quad \ldots \quad (11)$$

Considerable experimental work on a modified double-nosed cavity showed that it is possible to choose a cavity geometry that will approximate equation (11) over a 5% or 10% band. Such a design is shown in Figs. 2 and 3. The upper part of the reference cavity containing the compensating mechanism is similar to the 1Q series of fixed-frequency valves[2]; that is, the entire structure, except for the invar centre rod in the upper section, is made of copper and metals having expansion coefficients equal to copper. The invar rod is mounted rigidly at the top of the drawing and flexes the thin copper diaphragm forming the upper radio-frequency boundary of the resonator proper. Frequency compensation is thus achieved by relative motion of the cavity nose and side wall. The frequency sensitivity R of the cavity to this motion is relatively high, so that motion here is limited to the order of one thousandth of an inch for a 75°C temperature change.

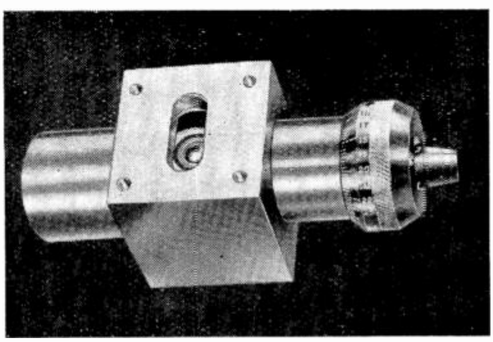

Fig. 2. *Westinghouse experimental tunable reference cavity WX-3339.*

To give mechanical tuning in a region of low-frequency sensitivity, the lower section is moved axially by a screw mechanism acting through a metal bellows. The frequency sensitivity to this motion is relatively low, so that motion here is of the order of a half an inch for a 10% tuning range. The rather specialized geometry of this tuner has been chosen, mostly by trial and error, in a manner to best approximate equation (11) over the design bandwidth. Considerable help may be had, however, in the choice of the tuner geometry by an equivalent circuit consideration which will be given briefly.

3. Tunable Cavity Equivalent Circuit

The frequency variable R is a microwave characteristic of the radio-frequency cavity. It is the rate of change of resonant frequency with respect to small cavity distortion produced by the bimetal in compensation action. Because of the complex cavity geometry, it is impractical to find a field solution for R. An equivalent circuit can be deduced from the appearance of the

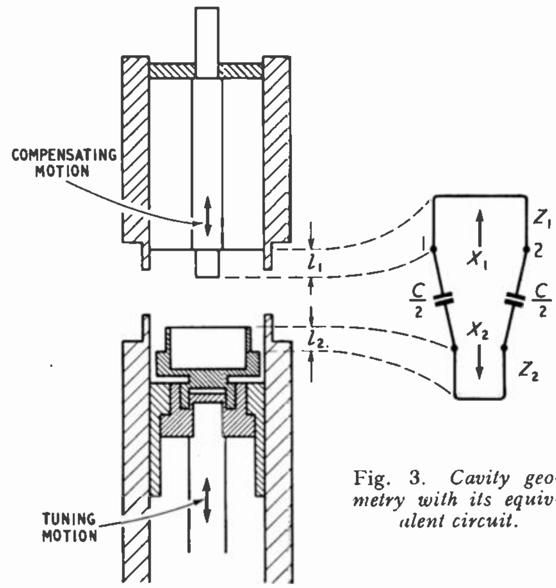

Fig. 3. *Cavity geometry with its equivalent circuit.*

geometry which gives numerical results close to those which are actually measured. As in any equivalent circuit, this is then the justification of its use. Fig. 3 shows that equivalence and defines the circuit constants including two shorted transmission lines of lengths l_1 and l_2 and characteristic impedances Z_1 and Z_2.

An initial feeling for the equivalence of the above circuit and geometry may be had by noting that the upper cavity nose is a short piece of shorted coaxial line. With the lower tuner completely withdrawn, this has a resonant frequency of its own ($\lambda_1 = 4l_1$). This is understandable if it is explained that the circular waveguide section below the nose is cut off to all modes in the tuning range. The lower nose (call it a nose, if you please, from its electrical similarity with the upper nose) also has resonance of its own ($\lambda_2 = 4l_2$) which is independent of the tuner position. This is similarly a shorted length of coaxial line having a lower characteristic impedance than the upper section and generally a slightly lower resonant wavelength. The large hole in the centre of the lower nose is below cut off for all modes and contributes only to C, the nose-to-nose capacitance. In compensation, the action of the bimetal alters the position of the upper nose by flexing a diaphram in the upper wall. Predominantly, this is equivalent to a change in the line length. In addition, but to a lesser order, the motion of the upper nose changes the nose-to-nose capacitance. The frequency sensitivity, then, is given by the sum of these two effects.

$$R = \frac{df}{dm} = \frac{\delta f}{\delta l_1}\frac{dl_1}{dm} + \frac{\delta f}{\delta C}\frac{dC}{dm} \quad \ldots \quad \ldots \quad (12)$$

where m is linear motion of the upper nose.

To derive R, one first writes an expression for resonance of the entire system by equating the susceptance at the terminals (1, 2) Fig. 3 to zero.

$$B_{12} = \frac{1}{x_1} + \frac{1}{x_2 - 1/\omega C} = 0 \quad \ldots \quad \ldots \quad (13)$$

Solving for the resonant angular frequency;

$$\omega = 1 \bigg/ \left[C \left\{ Z_1 \tan \frac{\omega l_1}{v} + Z_2 \tan \frac{\omega l_2}{v} \right\} \right] \quad (14)$$

Here ω is the angular frequency and v is the wave velocity in the shorted transmission lines.

The two required partial derivatives of ω are

$$\frac{\delta\omega}{\delta l_1} = \frac{\omega Z_1}{v} \frac{\sec^2 \frac{l_1\omega}{v}}{C\left(Z_1 \tan \frac{l_1\omega}{v} + Z_2 \tan \frac{l_2\omega}{v}\right)^2 + \frac{l_1 Z_1}{v}\sec^2\frac{l_1\omega}{v} + \frac{l_2 Z_2}{v}\sec^2\frac{l_2\omega}{v}} \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

and

$$\frac{\delta\omega}{\delta C} = \frac{-\left(Z_1 \tan \frac{l_1\omega}{v} + Z_2 \tan \frac{l_2\omega}{v}\right)}{C^2\left(Z_1 \tan \frac{l_1\omega}{v} + Z_2 \tan \frac{l_2\omega}{v}\right)^2 + C\left(\frac{l_1 Z_1}{v}\sec^2\frac{l_1\omega}{v} + \frac{l_2 Z_2}{v}\sec^2\frac{l_2\omega}{v}\right)} \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

The equivalent circuit capacitance, C, is considered to be variable to give the mechanical tuning of the device just as tuner motion varies the nose-to-nose capacitance in the actual cavity without affecting the line lengths. It is convenient, however, to use wavelength, λ_0, as the independent variable in these equations and eliminate capacitance with the aid of equation (14). This gives

$$\frac{\delta\omega}{\delta l_1} = \frac{4\pi^2 v}{\lambda_1^2} \frac{\left(\frac{\lambda_1}{\lambda_0}\right)^2 \sec^2\left(\frac{\lambda_1}{\lambda_0}\frac{\pi}{2}\right)}{\left\{\tan\left(\frac{\lambda_1}{\lambda_0}\frac{\pi}{2}\right) + \frac{\lambda_1}{\lambda_0}\frac{\pi}{2}\sec^2\left(\frac{\lambda_1}{\lambda_0}\frac{\pi}{2}\right)\right\} + \frac{Z_2}{Z_1}\left\{\tan\left(\frac{\lambda_2}{\lambda_0}\frac{\pi}{2}\right) + \frac{\lambda_2}{\lambda_0}\frac{\pi}{2}\sec^2\left(\frac{\lambda_2}{\lambda_0}\frac{\pi}{2}\right)\right\}} \quad \ldots \quad \ldots \quad (17)$$

and

$$\frac{\delta\omega}{\delta C} = \frac{4\pi^2 v^2 Z_1}{\lambda_1^2} \frac{-\left(\frac{\lambda_1}{\lambda_0}\right)^2 \left\{\tan\left(\frac{\lambda_1}{\lambda_0}\frac{\pi}{2}\right) + \frac{Z_2}{Z_1}\tan\left(\frac{\lambda_2}{\lambda_0}\frac{\pi}{2}\right)\right\}}{\left\{\tan\left(\frac{\lambda_1}{\lambda_0}\frac{\pi}{2}\right) + \frac{\lambda_1}{\lambda_0}\frac{\pi}{2}\sec^2\left(\frac{\lambda_1}{\lambda_0}\frac{\pi}{2}\right)\right\} + \frac{Z_2}{Z_1}\left\{\tan\left(\frac{\lambda_2}{\lambda_0}\frac{\pi}{2}\right) + \frac{\lambda_2}{\lambda_0}\frac{\pi}{2}\sec^2\left(\frac{\lambda_2}{\lambda_0}\frac{\pi}{2}\right)\right\}} \quad \ldots \quad (18)$$

The two parts of equation (12) should be discussed separately. It was indicated that the first member $\frac{\delta f}{\delta l_1}\frac{dl_1}{dm}$ was the first-order term and practice finds this so. In fact, the second-order term vanishes at one extreme of the tuning range. In the first term, $\frac{dl_1}{dm}$ is constant and would appear to be about 1/2 from inspection of the compensating mechanism. That is, motion of the bimetal mechanism effectively increases the outside diameter of the coaxial line but not that of the centre conductor. Measurements on a Westinghouse experimental valve, WX-3339, show the constant to be close to 1/3. The variable factor in the first-order term, $\delta f/\delta l_1$, with two arbitrary parameters, λ_1/λ_2 and Z_2/Z_1, may be plotted, giving families of curves as shown in Figs. 4 and 5.

That the first-order term is the type of relation required to minimize equation (9) is demonstrated from equation (17). Consider the two lines to be of equal length and the tuning range limited to the asymptotic region where the tangent terms become negligible and the secant terms approach unity. This is typical of the characteristics shown in Fig. 5. Equation (17) reduces to

$$\frac{\delta\omega}{\delta l_1} = \frac{4\pi^2 v}{\lambda_1^2} \frac{\lambda_1/\lambda_0}{\pi/2\,(1 + Z_2/Z_1)} \quad \ldots \quad \ldots \quad (19)$$

This shows that compensation rate, R, is proportional to frequency, which is all that is required by equation (11).

The second-order term, then, should be looked upon as a source of error and something to minimize. This is the reason for the large cup in the centre of the WX-3339 tuner nose. It can be seen from physical reasoning that this would reduce, to a large extent, the rate of change of capacitance with respect to compensating motion. To get a quantitative measure of this error term is not so simple as in the case of the first-order term. The difficulty is that, unlike the first-order term, the total derivative is not constant. In fact, calculation shows it to be the principal variable in the second-order term and like the total derivative in the first-order term, this factor is not obtainable from the equivalent circuit. If it becomes necessary to know the rate of change of capacitance with respect to compensating motion, quantitatively, it can be determined by measurement of resonant frequency with respect to tuner position. It is noticed that, with the geometrical arrangement shown in Fig. 3, an increment of compensating motion has an identical effect upon the capacitance as the same increment of tuner motion. The rate of change of capacitance with respect to tuner motion could then be calculated from the resonant frequency versus tuner position characteristic with the help of equation (14).

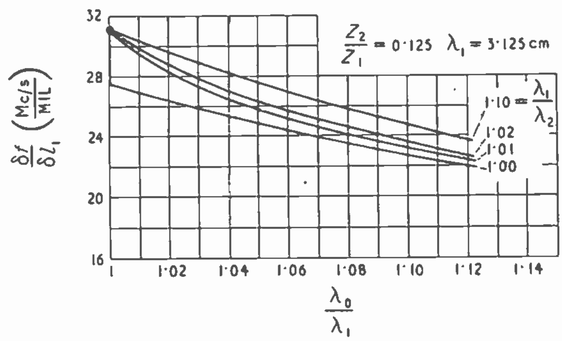

Fig. 4. *Frequency sensitivity for constant line impedance.*

4. Experimental Results

It is found experimentally that the inclusion of the second-order term in the compensation formula equation (7) is well justified. The second-order residual may be predicted to within 10% or 20% from hand-book values of expansion coefficients. With the proper choice of materials to minimize to second-order error, the compensation sensitivity (R Mc/s per mil) has been measured mechanically over a 10% tuning range in the 3-cm band. This predicted a cavity with no frequency shift at the design centre and a maximum of 0·5 Mc/s for 75°C temperature rise at two frequency extremes. Actual temperature measurements justified this calculation to within 15%. Of course, over a smaller tuning range a corresponding improvement can be made in the temperature compensation. It is likely that this does not represent optimum performance, although this result comes from a year's search for the preferred tuner nose geometry.

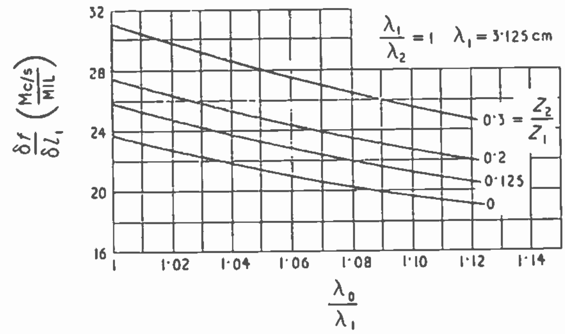

Fig. 5. *Frequency sensitivity for equal nose lengths.*

Finally, it is found that, if dl_1/dm and dC/dm are obtained by measurement (it has been explained that they are not derivable from the equivalent circuit) and used to calculate the frequency sensitivity by equation (12), this result agrees with measurements to within 5% over the higher-frequency half of a 10% tuning range. Best agreement over this half of the tuning range should be expected because in this region the two noses are physically well separated, avoiding as much as possible higher-order waves on the short transmission lines.

Acknowledgment

Curve plotting for this paper was done by Miss S. Hamalian and some of the experimental results were obtained by E. R. Mittelman of the Westinghouse Electron Tube Division.

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STABILITY OF OSCILLATION IN VALVE GENERATORS

By A. S. Gladwin, Ph.D., D.Sc.

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SUMMARY.—A comprehensive theory of stability is developed which is applicable to a large class of harmonic oscillators. The paper is concerned mainly with the 4-terminal regenerative type of oscillator with grid-leak bias but the analysis applies also to circuits with fixed grid-bias and to 2-terminal oscillators. All known forms of instability appear as special cases of the general theory and some new forms are predicted.

Stability is determined by the nature of the roots of a characteristic equation and stability criteria are obtained in the form of inequalities between the parameters of the amplifier and feedback network. A modified form of Nyquist locus-diagram can also be used. When the feedback network is symmetrical with respect to the oscillation frequency the characteristic equation can be factorized to give independent criteria for frequency and amplitude stability.

Hysteresis effects and periodic instability are analysed in detail In addition to the general treatment, specific forms of the various parameters which appear in the stability criteria are worked out for a three-halves-law amplifier with a semi-linear or exponential grid-current characteristic. Quantitative experimental confirmation is provided.

1. Introduction

THIS investigation concerns the dynamical stability of free oscillations in generators producing a nearly-sinusoidal voltage waveform (harmonic oscillators) and is limited to those types known as 'separable' oscillators, which consist of a non-linear amplifier or resistance whose behaviour is independent of time, and a passive linear impedance network. Under certain conditions the frequency or amplitude of oscillation may become unstable and change to a new value or vary over a range of values. The object of this inquiry is to elucidate these conditions in their most general form.

Over the past 35 years the subject has attracted much attention. Early work on discontinuous amplitude changes was carried out by Appleton and van der Pol[1] and by Greaves[2]. Van der Pol also studied a form of discontinuous frequency instability[3]. Accounts of later work are given in recent publications[4-8].

All the early work, and most of the later, dealt with simple types of circuits, and it was usually assumed that the behaviour of the amplifier or non-linear resistance could be represented by a few terms of a power series. It was also supposed that the working point of the amplifier was independent of oscillation amplitude; i.e., the non-oscillatory part of the voltage wave was of fixed amplitude.

In modern oscillators the amplifier characteristic can seldom be represented by a low-degree polynomial. Moreover, grid-leak bias or some other form of amplitude control is invariably used. This causes the amplifier working point to vary with oscillation amplitude and may also give rise to a periodic instability sometimes known a 'squegging'.

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Van Slooten[9, 10] examined this type of instability by setting up a second-order differential equation for the oscillation amplitude, stability being determined by the roots of the auxiliary equation. A simple form of circuit and valve characteristic was assumed.

A different approach was proposed by Edson[11] who suggested that the Nyquist criterion could be applied to determine the stability of oscillation amplitude by considering the changes in magnitude and phase experienced by a small sinusoidal modulation of the amplitude when transmitted through the amplifier and feedback network.

These two methods are limited to oscillators having feedback networks with a response characteristic symmetrical about the oscillation frequency; e.g., a high-Q antiresonant circuit. In asymmetrical networks, amplitude changes are accompanied by frequency changes which the methods are unable to take into account. Other suggestions, some of which are considered later, have been put forward. An account of these can be found in Edson's book[12].

In their different ways all the methods so far proposed are restricted in scope, and the diversity of treatment obscures the fundamental unity of all forms of instability. The piecemeal treatment also suggests that there may exist types of instability hitherto overlooked. A new and more general theory seems desirable which would include all possible types of instability as special cases and be applicable to any kind of network or amplifier. It should also be possible to obtain numerical results for practical oscillators without having to make violent assumptions and drastic approximations. This would allow a theoretical and experimental check to be made on the validity of any proposed stability criterion.

2. List of Symbols

$A(p)$ = Amplitude stability function. See (8.2)
b_a, b_g = Constants of the valve. See (4.19) and (4.21)
$D(p)$ = Stability function for the oscillator. See (6.4)
$F(p)$ = Frequency stability function. See (8.2)
G_n = Fourier coefficients of the grid–anode transconductance. See (5.10)
H = $1 - K + K/Y$
i_a, i_g = Anode and grid currents
i_{a1}, i_{g1} = Anode and grid currents of fundamental frequency
i_{ad}, i_{gd} = Perturbations of anode and grid currents
k = Fraction of total current reaching the anode
K = $-V_g/V_{g1}\,|\,1 + Z_I/\mu Z_T\,|$
N = Parameter of the steady state. See (4.13) and (4.14)
p = $a + j\omega_d$, complex frequency
q = $\exp j\theta$
R_a, R_g = Anode and grid circuit d.c. resistances
r_g = Grid input resistance at fundamental frequency
S_n = Fourier coefficients of the grid input conductance. See (5.1)
u_n = Complex amplitudes of transient grid voltage components. See (5.2)
v_a, v_{a1} = Anode voltage, anode voltage of frequency ω_0.
V_e = Amplitude of equivalent grid voltage of frequency ω_0. See (4.5)
v_g, V_g = Grid voltage, mean grid voltage
v_{g1}, V_{g1} = Grid voltage and amplitude of grid voltage at frequency ω_0
v_{ad}, v_{gd} = Perturbations of anode and grid voltages
V_{ca}, V_{cg} = Grid voltages required to cut off anode and grid currents
Y = V_g/V_{ca}
Z_i, Z_0, Z_t = $R_i + jX_i$, etc. Open-circuit input, output and transfer impedances of the feedback network at frequency ω_0
Z_I, Z_θ, Z_T = Modified values of Z_i, etc. See (4.11)
Z_i^+, Z_i^0, Z_i^- = $Z_i(p + j\omega_0)$, $Z_i(p)$, $Z_i(p - j\omega_0)$, etc.
Z_E = $R_E + jX_E = (Z_T + Z_I/\mu)/(1 - kZ_T/r_g)$
θ = Phase angle between cathode current and grid voltage
μ = Amplification factor of the valve
ω_0 = Oscillation frequency. (Fundamental)
$*$ = Complex conjugate value

3. Method of Solution

The method consists in examining the stability of the possible steady states of the oscillator. By a 'possible steady state' is meant a condition in which the system is in equilibrium with a periodic wave of constant amplitude and frequency. Stability is determined by the nature of the disturbance produced when a possible steady state is momentarily perturbed by a small external force, equilibrium being stable if the amplitude of the disturbance decreases with time. If equilibrium is stable, the possible steady state may be an actual steady state but, if equilibrium is unstable, the possible steady state cannot be realized physically.

When the disturbance is small the oscillator behaves towards it like a linear network with time-varying parameters. Now the transients in any linear system can be represented as the sum of a number of independent normal modes and to find the form of these modes in an oscillator it is sufficient to consider the simplest example.

Suppose that the oscillator consists of a 2-terminal linear passive network with an admittance $Y(D)$, $(D = d/dt)$, connected across a suitable non-linear negative-resistance element having a current–voltage relation given by $i = f(v)$. Let ω_0 be the frequency of oscillation and let the steady-state oscillation voltage across the network and non-linear resistance be v_s. If a small disturbance v_d is added to v_s the current in the non-linear resistance becomes $f(v_s + v_d) = f(v_s) + v_d f'(v_s)$. The current due to v_d is therefore $i_d = v_d f'(v_s)$. Since v_s is periodic $f'(v_s)$ is also periodic, and i_d can therefore be written as $i_d = gv_d$, where

$$g = \sum_{0}^{\infty} g_r \cos(r\omega_0 t + \phi_r) \quad \ldots \quad \ldots \quad (3.1)$$

Thus the non-linear resistance behaves towards v_d like a linear resistance with a conductance g varying periodically at the oscillation frequency.

By Kirchhoff's law the current flowing into the linear network is equal and opposite to the current entering the non-linear resistance, since the total currents at the branch points must be zero. But the network current due to v_d is $Y(D)v_d$. Hence

$$Y(D)v_d = -gv_d \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

In a linear network with constant parameters the general transient is the sum of a number of elementary waves each of the form $\exp.(at)\cos(\omega_d t + \theta)$. Inspection of (3.1) and (3.2) shows that the elementary transient in an oscillator must be different from this, for although the current in the linear network would be of the same form, modified only in amplitude and phase, the current gv_d in the non-linear resistance would be the sum of an infinite number of waves of different frequencies. The inspection also suggests that a possible solution might be

$$v_d = \exp.(at)\sum_{-\infty}^{\infty} V_n \cos\left\{(n\omega_0 + \omega_d)t + \theta_n\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

The currents in both linear and non-linear branches would then have components of all the frequencies $n\omega_0 + \omega_d$, and by choosing suitable values for the coefficients in (3.3) these currents could be made equal and opposite to satisfy (3.2).

From the analytical point of view, $Y(D)$ is a rational function of D and so (3.2) is a linear differential equation with periodic coefficients. Floquet's theory of such equations[16] shows that the solutions are of the form (3.3). Solutions in which $n\omega_0$ is replaced by $nk\omega_0/m$, k and m being integers, are also theoretically possible but it can be shown that these require exact relations between the coefficients g_r and must therefore be excluded on physical grounds.

Equation (3.3) represents the terminal voltage of one transient normal mode but equation (3.2) can usually be satisfied for more than one value of a and ω_d. The general disturbance can therefore be represented as the sum of a set of normal-mode voltages each similar to (3.3). In the more convenient complex notation (3.3) becomes

$$v_d = V_d \sum_{-\infty}^{\infty} u_n \exp.\ (p + jn\omega_0)\ t \qquad (3.4)$$

where $p = a + j\omega_d$ and the coefficients u_n are complex numbers. The actual voltage is the real part of this expression. With each normal mode is associated a characteristic complex frequency p, and the criterion for stability is therefore that all possible values of p should have negative real parts since this ensures that v_d ultimately vanishes.

By writing $n + m$ for n in (3.4) it is seen that if p is a solution then so is $p + jm\omega_0$, m being any integer. The imaginary part of p may therefore be restricted to the range $-\frac{1}{2}\omega_0 < \omega_d < \frac{1}{2}\omega_0$. Also, by writing $-n$ for n it follows that if p is a solution then so is p^* (complex conjugate).

To calculate p, the sums of the currents of like frequencies in the network and non-linear resistance are equated to zero. At any particular (complex) frequency $p + jq\omega_0$, the current in the linear network has the (complex) amplitude $V_d u_q Y\ (p + jq\omega_0)$, but the current in the non-linear resistance depends on all the values of $V_d u_n$, and, in fact, may easily be shown to have the (complex) amplitude

$$\tfrac{1}{2}\sum_{r=0}^{\infty} g_r \left\{ u_{q+r} \exp.\ (-j\phi_r) + u_{q-r} \exp.\ (j\phi_r) \right\}.$$

So, corresponding to every value of q is an equation containing all the values of u_n. Thus an infinite set of linear simultaneous equations for the unknowns u_n is obtained, the coefficients of u_n being functions of p and the quantities g_r and ϕ_r. When these equations are written in canonical form the condition for compatibility[18] is that the infinite determinant formed by the coefficients should vanish. This gives a determinantal equation from which the values of p could in principle be calculated. The equation is a more general form of Hill's determinantal equation[17]. However, to make the calculation practicable the equation must be simplified.

In many oscillators the steady-state voltage is nearly sinusoidal because the network impedance at the harmonic frequencies is very small. In calculating the transient behaviour it is therefore permissible to neglect all terms associated with harmonic frequencies. This means that the expression for v_d can be restricted to three terms

$$v_d = V_d \sum_{-1}^{1} u_n \exp.\ (p + jn\omega_0)\ t\ \ldots \qquad (3.5)$$

Only the first three terms in the Fourier series (3.1) for g need then be taken into account and the infinite determinant is reduced to its three central rows and columns. (A similar approximation was used by Hill in the solution of an astronomical problem.)

When (3.5) is adopted, the previously-noted restriction on the range of ω_d becomes obligatory in order to confine the transient frequencies to the appropriate range.

For regenerative oscillators of the type shown in Fig. 1 (which are the main subject of the paper) the procedure is similar but the algebra is naturally more complicated. The amplifier has input, output, and transfer conductances each of the form (3.1) and three functions are needed to specify the performance of the feedback network, namely, the open-circuit input impedance Z_i between terminals 1 and 3, the open-circuit output impedance Z_0 between 2 and 3, and the open-circuit transfer impedance Z_t between 1–3 and 2–3.

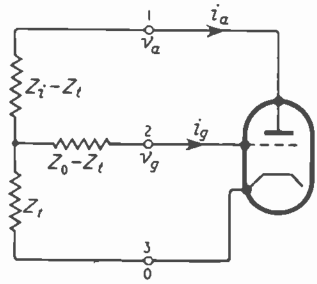

Fig. 1. *Oscillator circuit.*

Both anode and grid transient voltages are of the form (3.5), and since the input and output currents and voltages of the amplifier are related through the network impedances, all currents and voltages can be expressed in terms of a single variable—the amplifier input (grid) voltage.

For simplicity, the power supplies and the grid-bias arrangement are not shown in Fig. 1, and a triode is depicted although the analysis applies equally to tetrodes and pentodes. The values of Z_i and Z_0 for d.c. are taken as R_a and R_g, and these will usually be equal to the h.t. decoupling resistance and the grid-leak resistance. Z_t is zero for d.c.; i.e., there is no path for d.c. between anode and grid except via the cathode terminal.

Before stability can be investigated the possible steady states of the oscillator must be calculated.

4. Steady-State Amplitude and Frequency

The method used here is an extension of a previous analysis [19] and the general procedure—sometimes known as the method of equivalent linearization—is to consider only the constant and fundamental-frequency components of the currents and voltages. When, as is usual, the amplifier input voltage is nearly sinusoidal the method yields a good approximation even when the currents are markedly non-sinusoidal.

The grid–cathode voltage is first considered. Let this be

$$v_g = V_{g1} \cos \omega_0 t + V_g \quad .. \quad .. \quad (4.1)$$

V_g is the grid-bias voltage which may be derived either from a fixed source of e.m.f. or from the flow of grid current through the grid-leak resistance R_g. The second arrangement is examined first. If the anode voltage is not too small or does not vary too greatly, the grid current is well approximated by a single-valued function of v_g only

$$i_g = e\,(v_g).$$

When v_g is periodic i_g is also periodic. Thus

$$i_g = I_g + I_{g1} \cos \omega_0 t + \text{etc.}$$

The constant component I_g flows through R_g to produce the bias voltage. If all the bias is produced in this way Fourier analysis gives

$$V_g = -I_g R_g = -(R_g/\pi) \int_0^\pi e\,(V_{g1} \cos x + V_g)\, dx \quad (4.2)$$

For any given value of V_{g1} this is an equation for V_g. Since i_g has harmonic components, the assumption that v_g is sinusoidal implies that R_g is shunted by a capacitor of suitable size.

Due to the flow of grid current the amplifier has an a.c. input resistance r_g given by the ratio of the fundamental components of grid voltage and current. Thus

$$V_{g1} = r_g I_{g1} = (2r_g/\pi) \int_0^\pi e\,(V_{g1} \cos x + V_g) \cos x dx \quad (4.3)$$

from which r_g may be determined. If the function $e\,(v_g)$ is such that V_g is proportional to V_{g1} then r_g is constant and conversely. When the bias voltage is obtained from a fixed source and is sufficient to prevent the flow of grid current, $r_g = \infty$. When the fixed bias voltage is insufficient to stop grid current r_g is still given by (4.3), but (4.2) is not then valid.

To calculate the anode current i_a it is assumed that the cathode current i_k is a function of $v_g + v_a/\mu$, v_a being the anode–cathode voltage and μ a constant. For a triode $i_a = i_k - i_g$ and for multigrid valves i_a is usually a fairly constant fraction of this. Thus in all cases

$$i_a = f\,(v_g + v_a/\mu) - k i_g$$

where $f\,(v)$ is a single-valued function of v. For triodes $k = 1$ and for many pentodes $k \approx 0{\cdot}8$. It is more convenient to write

$$i_a = f\,(v_g + v_{aa}/\mu - V_{ca}) - k i_g \quad .. \quad (4.4)$$

where v_{aa} is the alternating component of v_a, and V_{ca} depends on the mean anode voltage. If the anode circuit contains a decoupling resistance, V_{ca} will vary with the oscillation amplitude. This is discussed in Appendix 3. Meanwhile V_{ca} is assumed to be constant.

The alternating anode voltage is assumed to be sinusoidal.

Let

$$v_g + v_{aa}/\mu = V_g + V_e \cos(\omega_0 t + \theta) \quad (4.5)$$

Then

$$i_a = f\{V_e \cos(\omega_0 t + \theta) + V_g - V_{ca}\} - k i_g \quad (4.6)$$

Let $K = -V_g/V_e$ and $Y = V_g/V_{ca}$. By Fourier analysis the component of i_a of fundamental frequency is

$$i_{a1} = gh\,(K,Y)\, V_e \cos(\omega_0 t + \theta) - k V_{g1} \cos \omega_0 t / r_g \quad (4.7)$$

where

$$V_e gh\,(K,Y) = (2/\pi) \int_0^\pi f\,(V_e \cos x + V_g - V_{ca}) \cos x dx \quad (4.8)$$

g is an arbitrary positive constant having the measure of conductance which can be chosen to give a suitable scale for the function $h\,(K,Y)$.

The actual currents and voltages are now replaced by their complex amplitudes, the reference quantity being the grid alternating voltage. Using the same symbols as for the real voltages (4.7) becomes

$$i_{a1} = (v_{g1} + v_{a1}/\mu)\, gh\,(K,Y) - k v_{g1}/r_g \quad (4.9)$$

Now in the impedance network of Fig. 1 the currents and voltages are connected by the following relations:

$$\left.\begin{aligned} i_{g1} &= v_{g1}/r_g \\ i_{a1} &= -v_{g1}/Z_t - i_{g1} Z_0/Z_t = -v_{g1}(1 + Z_0/r_g)/Z_t \\ v_{a1} &= v_{g1} Z_i/Z_t + Z_n i_{g1} = v_{g1}(Z_i/Z_t + Z_n/r_g) \\ \text{where } Z_n &= Z_i Z_0/Z_t - Z_t \end{aligned}\right\} (4.10)$$

Suppose that the network is modified by connecting a resistance r_g between 2 and 3. The new network could be replaced by a three-element network similar to the original but with modified values for the elements. Denoting the new elements by $Z_I\ Z_\theta\ Z_T$

$$\left.\begin{aligned} Z_I &= Z_i - Z_t^2/(r_g + Z_0) = (Z_i + Z_n Z_t/r_g)/(1 + Z_0/r_g) \\ Z_\theta &= Z_0/(1 + Z_0/r_g) \\ Z_T &= Z_t/(1 + Z_0/r_g) \\ Z_N &= Z_I Z_\theta/Z_T - Z_T = Z_n \end{aligned}\right\} (4.11)$$

It is noted that $Z_\theta/Z_T = Z_0/Z_t$, and if $Z_n = 0$ then also $Z_I/Z_T = Z_i/Z_t$.

From (4.10) and (4.11), $i_{a1} = -v_{g1}/Z_T$ and $v_{a1} = v_{g1} Z_I/Z_T$.

Substituting these into (4.9) gives

$$1 = -gh\,(K,Y)\,(Z_T + Z_I/\mu) + k Z_T/r_g$$

and finally putting

$$Z_E = (Z_T + Z_I/\mu)/(1 - kZ_T/r_g) \qquad (4.12)$$

gives

$$1 = -\, gh\,(K,Y)\, Z_E$$

This is the equation defining the steady state. Writing $Z_E = R_E + jX_E$, and similarly for Z_I and Z_T, and equating separately the real and imaginary parts of the equation gives

$$1 = Nh\,(K,Y) \quad .. \quad (4.13)$$

where

$$N = -\, gR_E = -\, g\,(R_T + R_I/\mu)/(1 - kR_T/r_g) \qquad (4.14)$$

and $\qquad X_E = 0$

$$\text{or } X_T + X_I/\mu + k\,(R_I X_T - R_T X_I)/\mu r_g = 0 \qquad (4.15)$$

If $f\,(v)$ is an increasing function of v then from (4.8) $gh\,(K,Y)$ is always positive. Hence R_E and so also R_T must be negative.

Equation (4.14) gives the amplitude, and (4.15) the frequency of oscillation but the equations are not completely independent. r_g will usually vary more or less with the oscillation amplitude and μ and k may also vary slightly. The frequency of oscillation will generally be affected by all these changes. For the frequency to be independent of amplitude and of changes in the amplifier it is necessary that Z_i, Z_0 and Z_t should be entirely resistive at the frequency ω_0. If this condition is satisfied (4.11) shows that $X_T = X_I = 0$ whatever the value of r_g and the frequency equation (4.15) is then independent of μ, k and r_g.

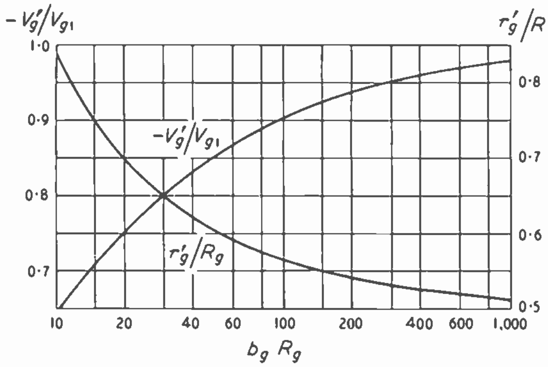

Fig. 2. *Grid bias and grid resistance of diode rectifier.*

In simple oscillators only one frequency is possible but, with more complicated networks, two or more values of ω_0 may be found to satisfy (4.15). These values will be frequencies of possible steady states only if corresponding values of Y can be found to satisfy the amplitude equation (4.14).

The parameters K and θ are found as follows. When expressed in complex form the voltage $V_e \cos(\omega_0 t + \theta)$ has an amplitude $V_e \exp. j\theta$. On comparing (4.7) and (4.9) it is seen that $V_e \exp. j\theta = v_{g1} + v_{a1}/\mu = v_{g1}\,(1 + Z_I/\mu Z_T)$. Hence, by taking the modulus of both sides,

$$K = -\, V_g/V_e = -\, V_g/V_{g1}\,|\,1 + Z_I/\mu Z_T\,| \quad (4.16)$$

Also

$$\left.\begin{array}{l} \exp. j\theta = (1 + Z_I/\mu Z_T)/\,|\,1 + Z_I/\mu Z_T\,| \\ \text{and} \\ \exp. -j\theta = (1 + Z_I^*/\mu Z_T^*)/|\,1 + Z_I/\mu Z_T\,| \end{array}\right\} (4.17)$$

(complex conjugate value)

When Z_T is resistive Z_I must also be resistive [from (4.15)].

$$\text{Then} \quad K = -\, V_g/V_{g1}\,(1 + R_I/\mu R_T) \text{ and } \theta = 0 \quad .. \quad .. \quad .. \quad (4.18)$$

To obtain working formulae, particular forms of the functions $e\,(v_g)$ and $f\,(v)$ must be studied. The process of solution will therefore be illustrated with reference to those forms which occur most often in practice. Details of the calculations are given in Appendix 1. The results are as follows.

It is first assumed that the grid-bias voltage is obtained by a grid-leak and capacitor arrangement. When the oscillation amplitude is fairly large grid current flows mainly when v_g is positive, and if the minimum anode voltage is well above the maximum grid voltage a fair approximation to i_g is

$$\left.\begin{array}{ll} i_g = e\,(v_g) = b_g\,(v_g - V_{cg}) & \text{when } v_g > V_{cg} \\ \quad = 0 & \text{when } v_g \leqslant V_{cg} \end{array}\right\} (4.19)$$

b_g and V_{cg} are measurable constants. Then

$$\left.\begin{array}{l} V_g = V_g'\,(1 - V_{cg}/V_{g1}) \\ r_g = r_g'\,(1 + V_{cg}/V_{g1}) \end{array}\right\} (4.20)$$

These formulae are valid for small values of V_{cg}/V_{g1}. Fig. 2 shows $-\,V_g'/V_{g1}$ and r_g'/R_g as functions of $b_g R_g$. These are the familiar curves for the diode rectifier.

When the amplifier operates in the space-charge-limited condition the cathode current follows a three-halves power law

$$\left.\begin{array}{ll} f\,(v) = b_a v^{3/2} & \text{when } v > 0 \\ \quad = 0 & \text{when } v \leqslant 0 \end{array}\right\} (4.21)$$

with $\quad v = V_e \cos(\omega_0 t + \theta) + V_g - V_{ca}$

b_a is a constant and V_{ca} is the grid voltage required to reduce the anode current to zero when the anode voltage is constant. The quantity g in (4.7) is taken as

$$g = 1\tfrac{1}{2} b_a\,(-\,V_{ca})^{\frac{1}{2}}$$

$$\text{Then} \quad N = -\, 1\tfrac{1}{2} b_a\,(-\,V_{ca})^{\frac{1}{2}} R_E \quad .. \quad (4.22)$$

Fig. 3 shows N as a function of K and Y calculated from (4.8).

To find the amplitude and frequency of oscillation approximate values of V_g/V_{g1} and r_g are found from Fig. 2 by neglecting V_{cg}. Z_I and Z_T are then calculated and ω_0 is found by solving

(4.15). K and N can next be evaluated from (4.16) and (4.22) and finally Y is read off from Fig. 3. A value for V_{g1} is thus obtained from which V_{cg}/V_{g1} can be found, and by repeating the procedure, taking V_{cg} into consideration, better approximations for V_{g1} and ω_0 can be obtained.

The results will be correct only if the expressions for i_a and i_g are valid. Slight departures from the three-halves law at low currents produce little error, but large deviations may occur if the minimum anode voltage approaches the 'knee' in a pentode or is comparable with the maximum grid voltage in a triode.

Large amplitudes require a small value of K, but good regulation is obtained when K is large; i.e., the change of amplitude with load is small. Large values of R_g and small values of μ give large values of K. Since $f(v) = 0$ when $v < 0$ it follows from (4.7) and (4.8) that $V_e + V_g - V_{ca} > 0$ for otherwise the anode current would always be zero. Dividing through by V_e the inequality becomes $1 - K + K/Y > 0$. Hence if $K > 1$ the upper limit of Y is $K/(K - 1)$ when the grid current is negligibly small. However the three-halves law would almost certainly not be followed for these extreme values. Other features of the graphs are discussed in later Sections.

When the grid-bias voltage is obtained from a fixed source and is large enough to stop the flow of grid current Fig. 3 can again be used to find V_{g1}. Y is now a constant, N can be evaluated as before and the corresponding value of K read off from the graphs. V_{g1} follows from K.

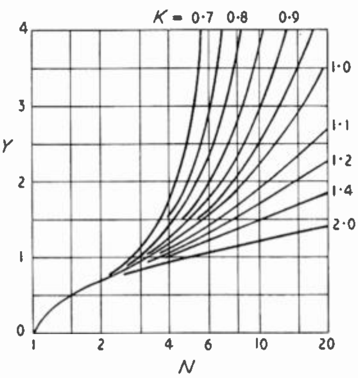

Fig. 3. *Graphs for oscillation amplitude.*

If the grid-bias voltage is fixed but insufficient to stop grid current a direct solution (by successive approximations) is laborious, but indirect solutions are easily obtained by observing that in an oscillator with grid-leak bias the steady state is unchanged if the grid leak is replaced by a source of e.m.f. equal to $-I_gR_g$.

In studying behaviour at the threshold of oscillation the values of V_g and r_g for vanishingly small amplitudes are required. The grid current characteristic has the exponential form

$$i_g = I_0 \exp. (v_g/V_0) \quad \ldots \quad \ldots \quad (4.23)$$

where I_0 and V_0 are positive constants. Then

$$r_g = -R_gV_0/V_g \quad \ldots \quad \ldots \quad (4.24)$$

Fig. 4 shows $-V_g/V_0$ as a function of R_gI_0/V_0.

It is seen that the value of r_g may be many times less than for the semi-linear form of characteristic.

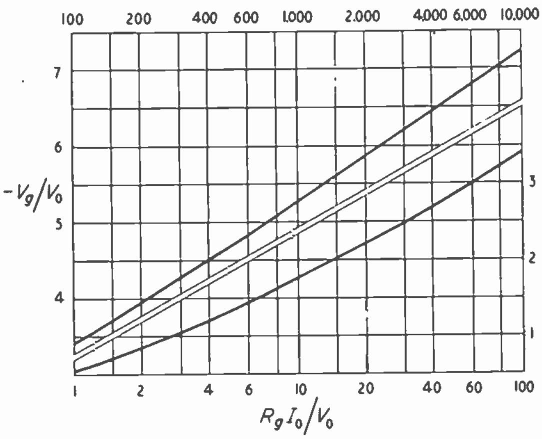

Fig. 4. *Grid bias (exponential characteristic).*

5. Transient Grid and Anode Currents

Let a small disturbance v_{gd} be added to the steady-state grid voltage v_g. The grid current is

$$e(v_g + v_{gd}) = e(v_g) + v_{gd}e'(v_g)$$

The change in grid current produced by v_{gd} is $i_{gd} = v_{gd}e'(v_g)$. Since $v_g = V_{g1} \cos \omega_0 t + V_g$, $e'(v_g)$ can be expressed as a cosine Fourier series in $\omega_0 t$. In complex notation this is

$$e'(v_g) = \sum_{-\infty}^{\infty} S_n \exp. jn\omega_0 t$$

where $S_n = (1/\pi) \int_0^\pi e'(V_{g1} \cos x + V_g) \cos nx dx$

$$\ldots \quad \ldots \quad \ldots \quad (5.1)$$

Following Section 3 the transient voltage is written as

$$v_{gd} = V_d \sum_{-1}^{1} u_n \exp. (p + jn\omega_0) t \qquad (5.2)$$

The transient current has an infinite number of terms but only those having the same frequencies as v_{gd} need be considered. Thus

$$i_{gd} = V_d \sum_{-1}^{1} w_n \exp. (p + jn\omega_0) t \qquad (5.3)$$

w_n being complex numbers. Equating (5.3) to the terms of like frequencies in the product $v_{gd}e'(v_g)$ gives three equations connecting the values of w_n and u_n. These are conveniently expressed in matrix notation.

$$\begin{bmatrix} w_1 \\ w_0 \\ w_{-1} \end{bmatrix} = \begin{bmatrix} S_0 & S_1 & S_2 \\ S_1 & S_0 & S_1 \\ S_2 & S_1 & S_0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_0 \\ u_{-1} \end{bmatrix} \quad .. \quad (5.4)$$

From (4.2) (4.3) and (5.1) the following relations are obtained.

$$\left.\begin{aligned} (S_0 - S_2)\, r_g &= 1 \\ S_1 R_g &= -(1 + S_0 R_g)\, dV_g/dV_{g1} \\ (1 + S_0 R_g)(S_0 + S_2) - 2S_1^2 R_g &= (1 + S_0 R_g)\,(1 - (V_{g1}/r_g)\, dr_g/dV_{g1})/r_g \end{aligned}\right\} \quad .. \quad .. \quad (5.5)$$

The first expression is found by integrating (4.3) by parts, and the second and third by differentiating (4.2) and (4.3) with respect to V_{g1}. These equations are independent of the form of $e\,(v_g)$, but it may be assumed that $e'\,(v_g)$ is always positive, for a negative slope would indicate an unstable form of characteristic. With this restriction it can be shown that

$$S_0 > 0, \quad S_0 + S_2 > 0, \quad S_0 - S_2 > 0, \quad S_0\,(S_0 + S_2) - 2S_1^2 > 0 \quad .. \quad (5.6)$$

The first three inequalities follow at once from the definition of S_n and the fourth can be proved by applying Schwarz's inequality[21]. (5.5) and (5.6) are required for use in later Sections.

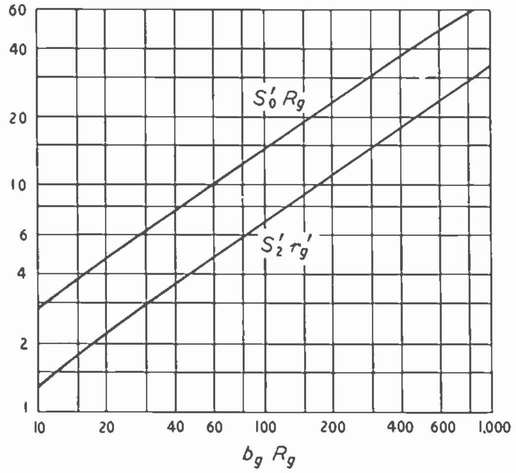

Fig 5. *Grid conductance functions.*

When the grid-bias voltage is supplied from a fixed source and is large enough to stop grid current, $S_0 = S_1 = S_2 = 0$.

If the grid-current characteristic has the semi-linear form of (4.19) it is shown in Appendix 2 that

$$\left.\begin{aligned} S_2/(S_0 - S_2) = S_2 r_g &= (1 + 0{\cdot}72 V_{cg}/V_{g1})\, S_2' r_g' \\ S_2/S_1 &= -(1 - V_{cg}/V_{g1})\, V_g'/V_{g1} + V_{cg}/V_{g1} \\ S_0 R_g &= S_0' R_g\,(1 - 0{\cdot}36 V_{cg}/V_{g1}) \end{aligned}\right\} \quad (5.7)$$

Fig. 5 shows $S_0' R_g$ and $S_2' r_g'$ plotted against $b_g R_g$. S_2/S_1 is found by using Fig. 2.

At the threshold of oscillation when V_{g1} is vanishingly small, the differential coefficients of V_g and r_g appearing in (5.5) are of major importance. It is shown in Appendix 2 that when the grid current has the exponential form (4.23)

$$\left.\begin{aligned} dV_g/dV_{g1} &= nV_{g1}/V_g, \quad dr_g/dV_{g1} = m r_g V_{g1}/V_g^2 \\ n &= (V_g/V_0)^2/2(1 - V_g/V_0) \\ m &= -(V_g/V_0)^2\,(1 + V_g/V_0)/4\,(1 - V_g/V_0) \end{aligned}\right\} (5.8)$$

Next the transient anode current is considered. Let small disturbances v_{gd} and v_{ad} be added to the steady-state grid and anode voltages, and let the change in anode current so produced be i_{ad}. Then from (4.4) and (4.5)

$$i_{ad} = (v_{gd} + v_{ad}/\mu)\, f'\,[V_e \cos(\omega_0 t + \theta) + V_g - V_{ca}] - k i_{gd}$$

f' can obviously be expressed as a cosine Fourier series in $\omega_0 t + \theta$. Hence

$$i_{ad} + k i_{gd} = (v_{gd} + v_{ad}/\mu) \sum_{-\infty}^{\infty} G_n \exp.\, jn\,(\omega_0 t + \theta) \quad (5.9)$$

where

$$G_n = (1/\pi) \int_0^\pi f'\,(V_c \cos x + V_g - V_{ca}) \cos nx\,dx \quad (5.10)$$

As before, only terms of the three principal frequencies in i_{ad} and v_{ad} need be considered. Let

$$\left.\begin{aligned} i_{ad} &= V_d \sum_{-1}^{1} x_n \exp.\,(p + jn\omega_0)\, t \\ v_{ad} &= V_d \sum_{-1}^{1} y_n \exp.\,(p + jn\omega_0)\, t \end{aligned}\right\} (5.11)$$

Using (5.2), (5.3), and (5.11), $i_{ad} + k i_{gd}$ is equated to the terms of like frequency on the r.h.s. of (5.9) to give

$$\begin{bmatrix} x_1 + kw_1 \\ x_0 + kw_0 \\ x_{-1} + kw_{-1} \end{bmatrix} = \begin{bmatrix} G_0 & G_1 q & G_2 q^2 \\ G_1 q^{-1} & G_0 & G_1 q \\ G_2 q^{-2} & G_1 q^{-1} & G_0 \end{bmatrix} \begin{bmatrix} u_1 + y_1/\mu \\ u_0 + y_0/\mu \\ u_{-1} + y_{-1}/\mu \end{bmatrix} (5.12)$$

in which $q = \exp.\, j\theta \quad .. \quad .. \quad .. \quad (5.13)$

Expressions for q and q^{-1} are given by (4.17).

Many of the stability criteria can be simply expressed in terms of the coefficients G_n. The following relations are required in later Sections.

$$G_0 - G_2 = g/N = -1/R_E$$
$$= -(1 - kR_T/r_g)/(R_T + R_I/\mu) \quad (5.14)$$

This is obtained by integrating (4.8) by parts and using (4.14) and (5.10). It is independent of the form of $f(v)$. If the restriction that $f'(v)$ should not be negative is imposed, the following inequalities can be proved in the same way as for S_n.

$$G_0 > 0, \quad G_0 + G_2 > 0, \quad G_0 - G_2 > 0,$$
$$G_0 (G_0 + G_2) - 2G_1^2 > 0 \quad .. \quad .. \quad (5.15)$$

In Appendix 2 the values of G_0, G_1 and G_2 for a three-halves-law amplifier are calculated. Fig. 6 shows $G_2/(G_0 - G_2) = -G_2R_E$ and $-G_1R_E$ plotted as functions of the parameter $H = 1 - K + K/Y$. From these graphs the values of G_2/G_1 and G_0R_E are easily obtained. When V_{g1} is very small

$$\left.\begin{aligned} G_0 &= g(1 - Y)^{\frac{1}{2}} = -1/R_E, \\ G_1 &= -Y/4KR_E(1 - Y) \\ G_2 &= Y^2/32K^2R_E(1 - Y)^2 \end{aligned}\right\} \quad .. \quad (5.16)$$

When V_{g1} is large

$$G_2/G_1 = \frac{(1 - 15H/16 + 35H^2/256 + \ldots)}{(1 - 3H/16 - 5H^2/256 + \ldots)} \quad (5.17)$$

It is also shown in the Appendix that

$$G_1 > 0 \text{ and } G_1^2 - G_0G_2 > 0 \quad .. \quad (5.18)$$

In the foregoing analysis it has been tacitly assumed that when the amplifier is a tetrode or pentode the screen-grid voltage remains constant. If this condition is not satisfied the theory must be modified in the way discussed in Appendix 4.

6. Determinantal Equation for *p*

In the preceding Section the transient grid and anode currents were expressed in terms of the corresponding voltages and the amplifier parameters, but these currents and voltages are also related through the impedances of the feedback network. Written in operational form, these relations are

$$i_{ad} = -\{1/Z_t(D)\} v_{gd} - \{Z_0(D)/Z_t(D)\} i_{gd}$$
$$v_{ad} = \{Z_i(D)/Z_t(D)\} v_{gd} + Z_n(D)\, i_{gd}$$
$$.. \quad .. \quad .. \quad (6.1)$$

Now $Z(D) \exp.(p + jn\omega_0)t$
$$= \exp.(p + jn\omega_0)t\, Z(p + jn\omega_0)$$

Substituting in these equations for v_{gd}, i_{gd}, v_{ad}, and i_{ad} according to (5.2), (5.3), and (5.11), gives

$$x_n = -u_n/Z_t(p + jn\omega_0) - w_nZ_0(p + jn\omega_0)/Z_t(p + jn\omega_0)$$
$$y_n = u_nZ_i(p + jn\omega_0)/Z_t(p + jn\omega_0) + w_nZ_n(p + jn\omega_0)$$

In order to conserve space the following notation is adopted

$$Z_i(p + j\omega_0) = Z_i^+,$$
$$Z_i(p) = Z_i^0, \; Z_i(p - j\omega_0) = Z_i^-$$

with similar expressions for Z_0, Z_t, etc.

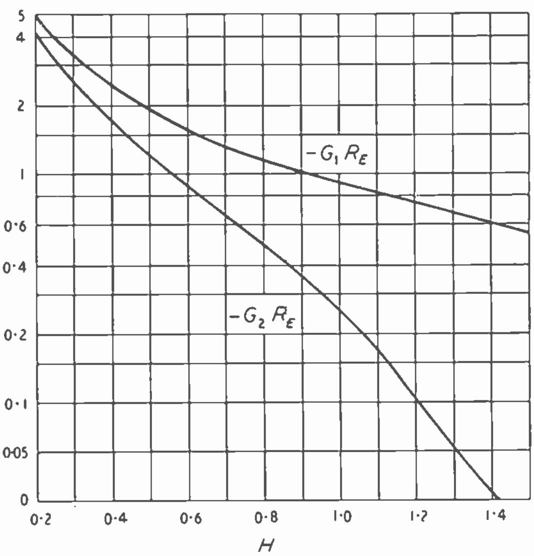

Fig. 6. *Anode conductance functions.*

It is convenient to refer to Z_i^+, Z_i^-, etc., as the high-frequency impedances, and to Z_i^0, etc., as the low-frequency impedances. Impedances without superscripts are meant, as in previous Sections, to refer to the oscillation frequency; e.g., $Z_T = Z_T(j\omega_0)$. When $p = 0$, $Z_i^0 = R_a$ and $Z_0^0 = R_g$ (the anode-decoupling and grid-leak resistances). When $p = \pm\frac{1}{2}j\omega_0$ the 'low' frequencies merge with the 'high' frequencies, but in most practical oscillators all the impedances are then negligible.

The above expressions for x_n and y_n are now substituted in (5.12) and both sides of the equation premultiplied by the diagonal matrix

$$\begin{bmatrix} Z_t^+ & 0 & 0 \\ 0 & Z_0^0 & 0 \\ 0 & 0 & Z_t^- \end{bmatrix}$$

The result is

$$\begin{bmatrix} u_1 + w_1(Z_0^+ - kZ_t^+) \\ u_0 + w_0(Z_0^0 - kZ_t^0) \\ u_{-1} + w_{-1}(Z_0^- - kZ_t^-) \end{bmatrix} + \begin{bmatrix} G_0Z_t^+ & G_1Z_t^+q & G_2Z_t^+q^2 \\ G_1Z_t^0q^{-1} & G_0Z_t^0 & G_1Z_t^0q \\ G_2Z_t^-q^{-2} & G_1Z_t^-q^{-1} & G_0Z_t^- \end{bmatrix} \begin{bmatrix} u_1(1 + Z_i^+/\mu Z_t^+) + w_1Z_n^+/\mu \\ u_0(1 + Z_i^0/\mu Z_t^0) + w_0Z_n^0/\mu \\ u_{-1}(1 + Z_i^-/\mu Z_t^-) + w_{-1}Z_n^-/\mu \end{bmatrix} = 0$$

Finally the w terms are substituted according to (5.4) to give a set of equations for u_n.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \\ a_{-1} & b_{-1} & c_{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_0 \\ u_{-1} \end{bmatrix} = 0 \quad \ldots \quad (6.2)$$

The coefficients are as follows:

$$\begin{aligned}
a_1 &= 1 + G_0 (Z_t^+ + Z_i^+/\mu) + S_0 (Z_0^+ - kZ_t^+) + G_0S_0Z_n^+Z_t^+/\mu + G_1S_1qZ_n^0Z_t^+/\mu + G_2S_2q^2Z_n^-Z_t^+/\mu \\
b_1 &= G_1qZ_t^+ (1 + Z_i^0/\mu Z_t^0) + S_1 (Z_0^+ - kZ_t^+) + G_0S_1Z_n^+Z_t^+/\mu + G_1S_0qZ_n^0Z_t^+/\mu + G_2S_1q^2Z_n^-Z_t^+/\mu \\
c_1 &= G_2q^2Z_t^+ (1 + Z_i^-/\mu Z_t^-) + S_2 (Z_0^+ - kZ_t^+) + G_0S_2Z_n^+Z_t^+/\mu + G_1S_1qZ_n^0Z_t^+/\mu + G_2S_0q^2Z_n^-Z_t^+/\mu \\
a_0 &= G_1q^{-1}Z_t^0 (1 + Z_i^+/\mu Z_t^+) + S_1 (Z_0^0 - kZ_t^0) + G_0S_1Z_n^0Z_t^0/\mu + G_1S_0q^{-1}Z_n^+Z_t^0/\mu + G_1S_2qZ_n^-Z_t^0/\mu \\
b_0 &= 1 + G_0 (Z_t^0 + Z_i^0/\mu) + S_0 (Z_0^0 - kZ_t^0) + G_0S_0Z_n^0Z_t^0/\mu + G_1S_1Z_t^0 (q^{-1}Z_n^+ + qZ_n^-)/\mu \\
c_0 &= G_1qZ_t^0 (1 + Z_i^-/\mu Z_t^-) + S_1 (Z_0^0 - kZ_t^0) + G_0S_1Z_n^0Z_t^0/\mu + G_1S_0qZ_n^-Z_t^0/\mu + G_1S_2q^{-1}Z_n^+Z_t^0/\mu \\
a_{-1} &= G_2q^{-2}Z_t^- (1 + Z_i^+/\mu Z_t^+) + S_2 (Z_0^- - kZ_t^-) + G_0S_2Z_n^-Z_t^-/\mu + G_1S_1q^{-1}Z_n^0Z_t^-/\mu + G_2S_0q^{-2}Z_n^+Z_t^-/\mu \\
b_{-1} &= G_1q^{-1}Z_t^- (1 + Z_i^0/\mu Z_t^0) + S_1 (Z_0^- - kZ_t^-) + G_0S_1Z_n^-Z_t^-/\mu + G_1S_0q^{-1}Z_n^0Z_t^-/\mu + G_2S_1q^{-2}Z_n^+Z_t^-/\mu \\
c_{-1} &= 1 + G_0 (Z_t^- + Z_i^-/\mu) + S_0 (Z_0^- - kZ_t^-) + G_0S_0Z_n^-Z_t^-/\mu + G_1S_1q^{-1}Z_n^0Z_t^-/\mu + G_2S_2q^{-2}Z_n^+Z_t^-/\mu
\end{aligned} \quad (6.3)$$

Neglecting the trivial case of $u_1 = u_0 = u_{-1} = 0$, the condition for the consistency of equations (6.2) is that the determinant of the coefficient matrix should vanish[18]. The auxiliary condition that at least one of the minors should not vanish is always satisfied. Since all the coefficients are functions of p the determinant is also a function of p. Let

$$D(p) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \\ a_{-1} & b_{-1} & c_{-1} \end{vmatrix} \quad \ldots \quad \ldots \quad (6.4)$$

The condition for consistency is then

$$D(p) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6.5)$$

This is the characteristic equation of the oscillator the roots of which are the characteristic frequencies of the transient normal modes. When the determinant is multiplied out, the general expression for $D(p)$ is of extreme length even allowing for some cancellation of terms. In most practical examples, however, the expression can be greatly simplified, in one way or another, by making suitable approximations, but these depend on the particular form of oscillator and the type of instability being studied.

(To be continued)

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ACTIVE LADDER NETWORK ANALYSIS

By Haim D. Polishuk, M.Sc.

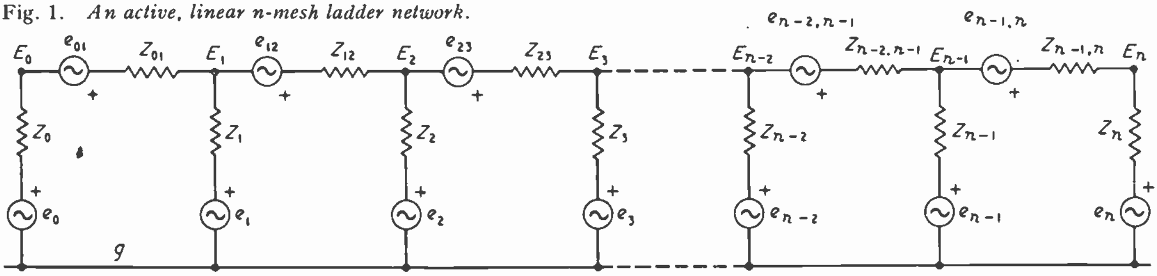
SUMMARY.—An expression is developed, based on Thévenin's Theorem, for the voltage appearing across the final 'rung' of an active linear n-mesh ladder network, which yields a set of recurring admittance coefficients. It is shown how the compact representation of these coefficients sets the pattern for a straightforward and economical computational procedure of analysis. Examples are given, which illustrate the general applicability of the method offered to the analysis of all networks of the conventional ladder geometry.

Introduction

A VARIETY of analytical methods as well as fundamental theorems have been proposed for the analysis of electrical networks. The classical among these methods appears to be by means of determinants, based on Kirchhoff's laws and derived in terms of either the network junctions or meshes. Determinants provide a very generalized and concise way of expressing the required operations and results in the solution of networks. However, the algebraic manipulations involved and expansions of high-order determinants of multi-mesh networks do not constitute any rapid or straightforward process, and a considerable amount of care must be exercised to avoid errors in the repeated operations of reduction or in proper sign.

In dealing with active, asymmetrical, linear and time-invariant networks the primary problem that commonly presents itself is the determination of a definite circuit parameter or physical quantity, such as the current flowing through a branch, the voltage or impedance across a junction-pair, etc., by means of which the performance of the system may be studied or quantitatively evaluated.

Fig. 1. *An active, linear n-mesh ladder network.*



The following treatment concerns itself with an analysis of a particular type of network, derivatives of which, nevertheless, characterize a considerable array of equivalent practical circuits encountered in the fields of radio-communication and electronics, by a method aimed at effecting computational economy. The voltage appearing across the final, n-th, 'rung' of an active, linear, unbalanced ladder network is expressed as a function of all the individual, randomly-distributed sources of e.m.f., in a process of network reduction by means of the successive application of Thévenin's Theorem to each of the n meshes in turn. This results in a set of admittance coefficients, the evaluation of which completes the quantitative determination of the required voltage. These coefficients turn out to be simple recurrent algebraic expressions of impedance and admittance products that provide an immediately applicable, compact and easily remembered process of obtaining the desired solution directly from the network or its equivalent form. Knowledge of this end-potential enables one to arrive rapidly at each of the other $n - 1$ junction potentials directly from Kirchhoff's current equations without actually solving them.

This fundamental mode of attack which purposely avoids the use of advanced matrix techniques should, it is believed, be more significant to the engineer who is interested in a practical method of solution. It should also be noted the proposed method is free from the inevitable redundancy inherent in the classical analysis by determinants, is readily adapted to practical computations, and becomes particularly useful when dealing with multi-mesh networks of the given general configuration.

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General Theory

An active ladder structure is shown in Fig. 1, composed of generators (all operating at any one given frequency) and linear bilateral impedances, appearing in both the series and shunt branches of an n-mesh network.

Let Z_k ($k = 0, 1, 2, \ldots, n$) denote shunt impedances, and e_k the shunt generators, all terminating in the common reference-junction

g (which is of known potential). The series generators* and impedances, bridging every k-th and $(k+1)$-th terminals, will be denoted respectively by $e_{k,k+1}$ and $Z_{k,k+1}$. Furthermore, let V_{gk} and Z_{gk} signify, respectively, the equivalent Thévenin generator voltage and impedance; namely, the open-circuited voltage between g and any k-th terminal when the network to the right of these terminals is disconnected, in series with the impedance which would be measured looking back to the left of terminals $g - k$ with all generators inactive and replaced by their internal impedances.

One formal approach to the numerical solution of the given network is to apply Kirchhoff's voltage law to each of the n meshes in turn, to write down the resulting n linear simultaneous equations and to solve for each of the n loop currents by means of Cramer's rule, or by the method of systematic elimination of the variables. By the nature of the given geometry, the resulting determinant formed by the contour and mutual impedances of the n meshes will be a symmetric continuant which will, therefore, considerably simplify the otherwise formidable manipulations required of a completely expanded solution.

In terms of the unknown junction potentials E_k (measurable between terminals g and any k), and considering the k-th junction $(1 \leqslant k \leqslant n-1)$, the steady-state equilibrium equation can be expressed as

$$y_{k-1,k}E_{k-1} + (y_{k-1,k} + y_k + y_{k,k+1})E_k - y_{k,k+1}E_{k+1} = e_{k-1,k}y_{k-1,k} + e_k y_k - e_{k,k+1}y_{k,k+1} \quad .. \quad (1)$$

where any $y_k = Z_k^{-1}$ and $y_{k,k+1} = Z_{k,k+1}^{-1}$.

It is seen from (1) that if the end-potential E_n were known, then the entire set of junction potentials could easily be extracted through successive substitutions of known E_{k+1} and E_k in the respective junction equations, to yield E_{k-1}, by going from terminal n progressively backwards to terminal 0.

Thévenin's Theorem suggests that the structure given in Fig. 1 may be condensed into the one illustrated in Fig. 2 where a Thévenin equivalent generator is replaced across terminals g and $n-1$; and furthermore, that V_{gn} identically equals E_n. Equating short-circuit currents, it follows that

$$\frac{V_{gn}}{Z_{gn}} = \frac{e_{n-1,n} + V_{g,n-1}}{Z_{n-1,n} + Z_{g,n-1}} + \frac{e_n}{Z_n} = \frac{\frac{e_n}{Z_n}\left(1 + \frac{Z_{n-1,n}}{Z_{g,n-1}}\right) + \frac{e_{n-1,n}}{Z_{g,n-1}} + \frac{V_{g,n-1}}{Z_{g,n-1}}}{1 + \frac{Z_{n-1,n}}{Z_{g,n-1}}}, \quad (2)$$

but,

$$\frac{1}{Z_{gn}} = \frac{1}{Z_n} + \frac{1}{Z_{n-1,n} + Z_{g,n-1}} = \frac{\frac{1}{Z_n}\left(1 + \frac{Z_{n-1,n}}{Z_{g,n-1}}\right) + \frac{1}{Z_{g,n-1}}}{1 + \frac{Z_{n-1,n}}{Z_{g,n-1}}}, \quad .. \quad (3)$$

hence, substituting (3) into (2) gives,

$$V_{gn} = \frac{\frac{e_n}{Z_n}\left(1 + \frac{Z_{n-1,n}}{Z_{g,n-1}}\right) + \frac{e_{n-1,n}}{Z_{g,n-1}} + \frac{V_{g,n-1}}{Z_{g,n-1}}}{\frac{1}{Z_n}\left(1 + \frac{Z_{n-1,n}}{Z_{g,n-1}}\right) + \frac{1}{Z_{g,n-1}}}. \quad (4)$$

This last result is also obtainable by applying the dual of Thévenin's Theorem, or what is known as Norton's Theorem, to the situation depicted in Fig. 2.

If just a 2-mesh network $(n = 2)$ is considered,

$$V_{g2} = \frac{\frac{e_2}{Z_2}\left(1 + \frac{Z_{12}}{Z_{g1}}\right) + \frac{e_{12}}{Z_{g1}} + \frac{V_{g1}}{Z_{g1}}}{\frac{1}{Z_2}\left(1 + \frac{Z_{12}}{Z_{g1}}\right) + \frac{1}{Z_{g1}}}, \quad (5)$$

but, from (2),

$$\frac{V_{g1}}{Z_{g1}} = \frac{\frac{e_1}{Z_1}\left(1 + \frac{Z_{01}}{Z_{g0}}\right) + \frac{e_{01}}{Z_{g0}} + \frac{V_{g0}}{Z_{g0}}}{1 + \frac{Z_{01}}{Z_{g0}}} \quad .. \quad (6)$$

Substituting (6) into (5) and rearranging, yields,

$$V_{g2} = \frac{\left[\frac{e_2}{Z_2}\left(1 + \frac{Z_{12}}{Z_{g1}}\right) + \frac{e_{12}}{Z_{g1}}\right]\left(1 + \frac{Z_{01}}{Z_{g0}}\right) + \frac{e_1}{Z_1}\left(1 + \frac{Z_{01}}{Z_{g0}}\right) + \frac{e_{01}}{Z_{g0}} + \frac{V_{g0}}{Z_{g0}}}{\left[\frac{1}{Z_2}\left(1 + \frac{Z_{12}}{Z_{g1}}\right) + \frac{1}{Z_{g1}}\right]\left(1 + \frac{Z_{01}}{Z_{g0}}\right)},$$

$$= \frac{\frac{e_0 + e_{01} + e_{12}}{Z_0} + \frac{e_1 + e_{12}}{Z_1}\left(1 + \frac{Z_{01}}{Z_{g0}}\right) + \frac{e_2}{Z_2}\left(1 + \frac{Z_{01}}{Z_{g0}}\right)\left(1 + \frac{Z_{12}}{Z_{g1}}\right)}{\frac{1}{Z_0} + \frac{1}{Z_1}\left(1 + \frac{Z_{01}}{Z_{g0}}\right) + \frac{1}{Z_2}\left(1 + \frac{Z_{01}}{Z_{g0}}\right)\left(1 + \frac{Z_{12}}{Z_{g1}}\right)}, \quad .. \quad .. \quad (7)$$

* The reference polarities appended to generators assume an arbitrary potential rise, in the direction of positive current flow (clockwise for loop currents); affixed double-subscripts, when employed, refer to the datum junction g.

noticing that $V_{g0} = e_0$ and $Z_{g0} = Z_0$, while from (3), for $n = 2$,

$$\frac{1}{Z_{g1}}\left(1 + \frac{Z_{01}}{Z_{g0}}\right) = \frac{1}{Z_1}\left(1 + \frac{Z_{01}}{Z_{g0}}\right) + \frac{1}{Z_0} .$$

If a similar process is pursued for a 3-mesh network ($n = 3$), the following result is arrived at:—

$$V_{g3} = \frac{\dfrac{e_0 + e_{01} + e_{12} + e_{23}}{Z_0} + \dfrac{e_1 + e_{12} + e_{23}}{Z_1}\left(1 + \dfrac{Z_{01}}{Z_{g0}}\right) + \dfrac{e_2 + e_{23}}{Z_2}\left(1 + \dfrac{Z_{01}}{Z_{g0}}\right)\left(1 + \dfrac{Z_{12}}{Z_{g1}}\right) + \dfrac{e_3}{Z_3}\left(1 + \dfrac{Z_{01}}{Z_{g0}}\right)\left(1 + \dfrac{Z_{12}}{Z_{g1}}\right)\left(1 + \dfrac{Z_{23}}{Z_{g2}}\right)}{\dfrac{1}{Z_0} + \dfrac{1}{Z_1}\left(1 + \dfrac{Z_{01}}{Z_{g0}}\right) + \dfrac{1}{Z_2}\left(1 + \dfrac{Z_{01}}{Z_{g0}}\right)\left(1 + \dfrac{Z_{12}}{Z_{g1}}\right) + \dfrac{1}{Z_3}\left(1 + \dfrac{Z_{01}}{Z_{g0}}\right)\left(1 + \dfrac{Z_{12}}{Z_{g1}}\right)\left(1 + \dfrac{Z_{23}}{Z_{g2}}\right)} \quad \ldots \quad (8)$$

By the method of mathematical induction one may readily deduce the general expression for the voltage V_{gn} appearing across the end terminals $g - n$ of the n-mesh network of Fig. 1. Thus,

$$\begin{aligned} E_n \equiv V_{gn} &= [(e_0 + e_{01} + e_{12} + \ldots + e_{n-1,n})\,\lambda_0 + (e_1 + e_{12} + \ldots + e_{n-1,n})\lambda_1 \\ &\quad + (e_2 + e_{23} + \ldots + e_{n-1,n})\,\lambda_2 + \ldots + (e_{n-1} + e_{n-1,n})\,\lambda_{n-1} \\ &\quad + e_n\lambda_n]/(\lambda_0 + \lambda_1 + \lambda_2 + \ldots + \lambda_n) \\ &= \sum_{k=0}^{k=n}\left(e_k + \sum_{j=k}^{j=n-1} e_{j,j+1}\right)\lambda_k \Big/ \sum_{k=0}^{k=n}\lambda_k , \end{aligned} \quad \ldots \quad (9)$$

where the coefficients λ_k ($k = 0, 1, 2, \ldots, n$), which are dimensionally admittances, are defined as follows:—

$$\left.\begin{aligned} \lambda_0 &= \frac{1}{Z_0} \\ \lambda_1 &= \frac{1}{Z_1}\left(1 + \frac{Z_{01}}{Z_{g0}}\right) \\ \lambda_2 &= \frac{1}{Z_2}\left(1 + \frac{Z_{01}}{Z_{g0}}\right)\left(1 + \frac{Z_{12}}{Z_{g1}}\right) \\ \lambda_3 &= \frac{1}{Z_3}\left(1 + \frac{Z_{01}}{Z_{g0}}\right)\left(1 + \frac{Z_{12}}{Z_{g1}}\right)\left(1 + \frac{Z_{23}}{Z_{g2}}\right) \\ &\;\;\vdots \\ \lambda_n &= \frac{1}{Z_n}\left(1 + \frac{Z_{01}}{Z_{g0}}\right)\left(1 + \frac{Z_{12}}{Z_{g1}}\right)\ldots\left(1 + \frac{Z_{n-2,n-1}}{Z_{g,n-2}}\right)\left(1 + \frac{Z_{n-1,n}}{Z_{g,n-1}}\right) \\ &= \frac{1}{Z_n}\prod_{k=0}^{k=n-1}\left(1 + \frac{Z_{k,k+1}}{Z_{gk}}\right). \end{aligned}\right\} \quad \ldots \quad (10)$$

Expression (9) which exhibits the principle of superposition as applied to linear network loops, states that the potential across the end terminals $g - n$ constitutes the sum of the component contributions of all active individual loops that pass through these end-terminals, considered separately, in a manner independent of the effects of all other voltage sources not series-connected along each such loop. The various factors involved in the individual coefficients λ_k may now be expanded and regrouped while doing away with the general impedances Z_{gk}. It should be noted in passing, that each of the admittances $Y_{gk} = Z_{gk}^{-1}$ is expressible in a continued fraction. With the shunt branches expressed as admittances and the series branches as impedances, this takes the form,

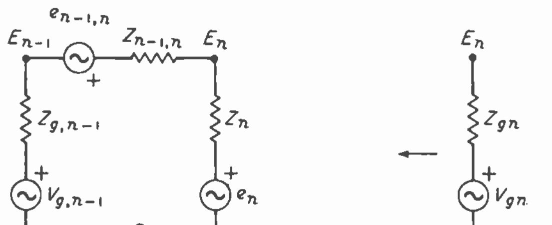

Fig. 2. *Thévenin's equivalent of the circuit of Fig. 1.*

$$Y_{gk} = y_k + \cfrac{1}{Z_{k-1,k} + \cfrac{1}{y_{k-1} + \cfrac{1}{Z_{k-2,k-1} + \cfrac{1}{y_{k-2} + \cfrac{\vdots}{+ \cfrac{1}{Z_{01} + \cfrac{1}{y_0}}}}}}} . \quad \ldots \ldots \quad (11)$$

To transform the expression of λ_k into a series of recurrence functions, use is made of relation (3):—

$$\left.\begin{aligned}
\lambda_0 &= \frac{1}{Z_0} , \\
\lambda_1 &= \frac{1}{Z_1}\left(1 + \frac{Z_{01}}{Z_{g0}}\right) = \frac{1}{Z_1}\left(1 + \lambda_0 Z_{01}\right), \\
\lambda_2 &= \frac{1}{Z_2}\left(1 + \frac{Z_{01}}{Z_{g0}}\right)\left(1 + \frac{Z_{12}}{Z_{g1}}\right) \\
&= \frac{1}{Z_2}\left[\left(1 + \frac{Z_{01}}{Z_{g0}}\right) + \left(1 + \frac{Z_{01}}{Z_{g0}}\right)\left(\frac{1}{Z_{01} + Z_{g0}} + \frac{1}{Z_1}\right) Z_{12}\right] \\
&= \frac{1}{Z_2}\left[1 + \lambda_0 (Z_{01} + Z_{12}) + \lambda_1 Z_{12}\right], \text{ and similarly,} \\
\lambda_3 &= \frac{1}{Z_3}\left[1 + \lambda_0 (Z_{01} + Z_{12} + Z_{23}) + \lambda_1 (Z_{12} + Z_{23}) + \lambda_2 Z_{23}\right], \\
&- - - - - - - - - - - - - \\
\lambda_n &= \frac{1}{Z_n}\Big[1 + \lambda_0 (Z_{01} + Z_{12} + \ldots + Z_{n-1,n}) + \lambda_1 (Z_{12} + Z_{23} + \ldots + Z_{n-1,n}) + \ldots \\
&\qquad + \lambda_{n-2} (Z_{n-2,n-1} + Z_{n-1,n}) + \lambda_{n-1} Z_{n-1,n}\Big] \\
&= \frac{1}{Z_n}\left[1 + \sum_{k=0}^{k=n-1} \lambda_k \left(\sum_{j=k}^{j=n-1} Z_{j,j+1}\right)\right].
\end{aligned}\right\} \quad (12)$$

Examination of (12) reveals the important relationship:—

$$Z_n \lambda_n = Z_{n-1} \lambda_{n-1} + \left(\sum_{k=0}^{k=n-1} \lambda_k\right) Z_{n-1,n} , \quad (13)$$

and while generally,

$$Z_n \lambda_n = Z_{n-1} \lambda_{n-1} \left(1 + \frac{Z_{n-1,n}}{Z_{g,n-1}}\right) , \quad \ldots \quad (14)$$

it can easily be shown that

$$Z_{gn} = Z_n \lambda_n \Big/ \sum_{k=0}^{k=n} \lambda_k . \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Thus, evaluation of the admittance coefficients, preferably by means of (13), completes the solution for E_n.

One may now readily notice that, should all series branches be shorted ($e_{k,k+1} = 0$ and $Z_{k,k+1} = 0$), then, while every $\lambda_k = y_k$, (9) reduces to what is known as Millman's Theorem, namely,

$$E_n = \frac{\sum_{k=0}^{k=n} e_k y_k}{\sum_{k=0}^{k=n} y_k} . \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Recurrence relation (13) may also be normalized and expressed in terms of the dimensionless ratio $\lambda_k/\lambda_0 = \phi_k$. Thus,

$$Z_k \phi_k = Z_{k-1} \phi_{k-1} + \left[\sum_{j=0}^{j=k-1} \phi_j\right] Z_{k-1,k} \; (1 \leqslant k \leqslant n), \quad \ldots \quad \ldots \quad (17)$$

with $\phi_0 = 1$. The introduction of ϕ_k will prove useful should $Z_0 = 0$, although, so far as the evaluation of E_n is concerned the series impedances Z_0 and Z_{01} may always be conveniently interchanged.

Illustrations

The following examples will serve to illustrate the application of the suggested method to two specific cases.

1. A detailed numerical solution of the 7-mesh network shown in Fig. 3 will start by the determination of E_7, the end-potential, for which λ_0 to λ_7, must first be evaluated. Relation (13) states that:—

$$Z_k\lambda_k = Z_{k-1}\,\lambda_{k-1} + (\lambda_0 + \lambda_1 + \ldots + \lambda_{k-1}) \times Z_{k-1,k}\ (1 \leqslant k \leqslant n)$$

with $Z_0\lambda_0 = 1$, and therefore provides a simple and systematic procedure for arriving at any λ.

the total impedance presented at the end-terminals.

2. Given an n-mesh ladder network made up of equal series resistors and equal shunt pure capacitors, excited by a single voltage source $E_0 = e_p(t)$, as shown in Fig. 4, let it be required to determine the distribution of voltage throughout the various branches of this network in the sinusoidal steady-state. Without recourse to mesh or node equations and the use of elimination or substitution procedures, comparing this network with the general one in Fig. 1, it is readily

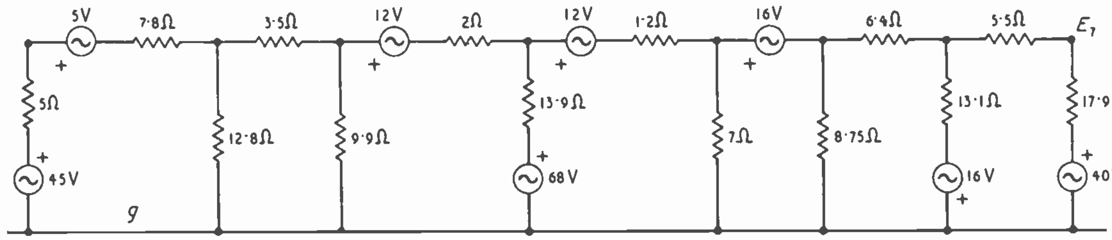

Fig. 3. *A specific example of an active ladder network.*

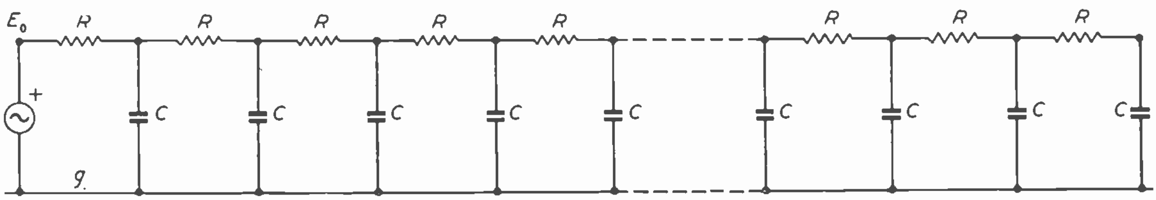

Fig. 4. *The n-section RC filter network.*

By inspection, and listing all arithmetical operations in tabulated order, the exact solution will take the following straightforward form:—

$Z_k\,\lambda_k = Z_{k-1}\,\lambda_{k-1} + [\lambda_0 + \ldots \lambda_{k-1}] \times Z_{k-1,k}$		$\therefore \lambda_k =$
$5{\cdot}0\ \lambda_0$	$= 1$	$\lambda_0 = 0{\cdot}2$
$12{\cdot}8\ \lambda_1 =$	$1 + 0{\cdot}2 \times 7{\cdot}8 = 2{\cdot}56$	$\lambda_1 = 0{\cdot}2$
$9{\cdot}9\ \lambda_2 =$	$2{\cdot}56 + 0{\cdot}4 \times 3{\cdot}5 = 3{\cdot}96$	$\lambda_2 = 0{\cdot}4$
$13{\cdot}9\ \lambda_3 =$	$3{\cdot}96 + 0{\cdot}8 \times 2{\cdot}0 = 5{\cdot}56$	$\lambda_3 = 0{\cdot}4$
$7{\cdot}0\ \lambda_4 =$	$5{\cdot}56 + 1{\cdot}2 \times 1{\cdot}2 = 7{\cdot}0$	$\lambda_4 = 1{\cdot}0$
$8{\cdot}75\ \lambda_5 =$	$7{\cdot}0 + 2{\cdot}2 \times 0 = 7{\cdot}0$	$\lambda_5 = 0{\cdot}8$
$13{\cdot}1\ \lambda_6 =$	$7{\cdot}0 + 3{\cdot}0 \times 6{\cdot}4 = 26{\cdot}2$	$\lambda_6 = 2{\cdot}0$
$17{\cdot}9\ \lambda_7 =$	$26{\cdot}2 + 5{\cdot}0 \times 5{\cdot}5 = 53{\cdot}7$	$\lambda_7 = 3{\cdot}0$

$$\therefore \sum_{k=0}^{k=7} \lambda_k = 8{\cdot}0$$

Applying now expression (9) directly,

$$E_7 = \frac{1}{8}\Big[(45 - 5 - 12 - 12 - 16) \times 0{\cdot}2 + (-12 - 12 - 16) \times 0{\cdot}2 + (-12 - 12 - 16) \times 0{\cdot}4$$

$$+ (68 - 12 - 16) \times 0{\cdot}4 + (-16) \times 1{\cdot}0 + (0) \times 0{\cdot}8 + (-16) \times 2{\cdot}0 + (40) \times 3{\cdot}0\Big]$$

$$= +\,8{\cdot}0 \text{ volts.}$$

Also,

$$Z_{g7} = \frac{Z_7\,\lambda_7}{\sum_{k=0}^{k=7} \lambda_k} = \frac{53{\cdot}7}{8} = 6{\cdot}71\ \Omega,$$

seen from (9) that since only the first voltage source is involved,

$$E_n = \frac{E_0\,\lambda_0}{\sum_{k=0}^{k=n} \lambda_k} = \frac{E_0}{\sum_{k=0}^{k=n} \phi_k},$$

hence,

$$\frac{E_0}{E_n} = \sum_{k=0}^{k=n} \phi_k\ .$$

With $p = j\omega$, it follows that every series resistance divided by any shunt reactance equals pT where $T = RC$. Thus, referring to (17), and taking $Z_0 = 0$, the following sequence of polynomials becomes immediately evident:

E

$\phi_0 = 1$; $\phi_3 = 3(pT) + 4(pT)^2 + (pT)^3$

$\phi_1 = pT$; $\phi_4 = 4(pT) + 10(pT)^2 - 6(pT)^3 + (pT)^4$

$\phi_2 = 2(pT) + (pT)^2$; $\phi_5 = 5(pT) + 20(pT)^2 - 21(pT)^3 + 8(pT)^4 + (pT)^5$, etc.

Examination of the numerical coefficients of these polynomials will suggest that they can also be expressed in terms of the factorial fractions; i.e.,

$$\phi_k = \binom{k}{1}\left(pT\right) + \binom{k+1}{3}\left(pT\right)^2 + \binom{k+2}{5}\left(pT\right)^3 + \ldots + \binom{2k-1}{2k-1}\left(pT\right)^k$$

$$= \sum_{r=1}^{r=k}\binom{k+r-1}{2r-1}\left(pT\right)^r.$$

From elementary algebra it is known that:

$$\binom{a}{b} + \binom{a-1}{b} + \binom{a-2}{b} + \ldots + \binom{b}{b} = \binom{a+1}{b+1}$$

hence, applying this relation when summing all corresponding coefficients of $(pT)^k$,

$$\frac{E_0}{E_n} = \sum_{k=0}^{k=n}\phi_k = 1 + \binom{n+1}{2}\left(pT\right) + \binom{n+2}{4}\left(pT\right)^2 + \binom{n+3}{6}\left(pT\right)^3 + \ldots + \binom{n+n}{2n}\left(pT\right)^n$$

$$= \sum_{r=0}^{n}\binom{n+r}{2r}\left(pT\right)^r,$$

the familiar polynomial expressing the voltage transfer ratio in the analysis of the n-mesh RC filter network. Because of the linearity of the given network one may readily write down all n junction potentials in terms of E_n and thus express all possible driving point and transfer functions.

LOW-FREQUENCY CROSSTALK IN PULSE-PHASE MODULATION

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1. Introduction

IN recent years several papers have dealt with the subject of crosstalk in pulse-amplitude (p.a.m.), pulse-width (p.w.m.) and pulse-phase (p.ph.m.) modulation systems. However, no satisfactory treatment of low-frequency crosstalk in pulse-phase modulation systems has yet appeared.

This was first discussed by Moskowitz, Diven, and Feit[1], who give a formula for the crosstalk when the time constant of the distorting network is short compared with the time interval between pulses. This was extended by Flood[2], who considers the case in which both disturbed and disturbing pulses are modulated. He also obtains an expression for crosstalk when the distorting time constant is long, by considering the effect upon the spectrum of the modulated pulses. It will be shown that this method is unsatisfactory in the case of pulse-phase modulation. Finally, Fagot[3] obtains an expression for crosstalk due to low-frequency distortion, but he makes conflicting approximations which invalidate his results.

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LIST OF SYMBOLS

A	= pulse amplitude
C	= capacitance
dv/dt	= pulse slope at slicing level
f_m	= modulation frequency
f_r	= pulse repetition frequency
$J_n(x)$	= Bessel function of first kind of order n
k	= an integer
K	= RC = time constant of coupling
l	= pulse length
l_d	= peak time modulation
n	= an integer
R	= resistance
t	= time
t_1	= time deviation of a pulse
t_2	= separation of disturbed pulse from nearest disturbing pulse
t_3	= $1/f_r$ = repetition interval
ϕ	= phase angle of modulation
ω_m	= $2\pi f_m$ = angular modulation frequency

2. The Spectrum Approach

A circuit introducing low-frequency distortion will modify each component of the spectrum of a modulated pulse train. The effect of this will be the same as the addition to the pulse-train of a signal, each component of which is equal to the modification of the corresponding component in the spectrum of the pulse train. Cross-talk can be evaluated by considering the modification of those frequency components which involve the modulation frequency.

It can be shown that in pulse-amplitude and pulse-length modulation the only component whose modification is important from the point of view of crosstalk is the modulation-frequency component itself. It has been assumed by Flood[2] that this can be extended to pulse-phase modulation as well. This assumption will now be examined.

Consider the simple coupling circuit of Fig. 1. For a component of amplitude V_1 and frequency ω

$$V_2 = \frac{V_1 R}{R - j/\omega C}$$

$$= \frac{V_1(\omega^2 K^2 + j\omega K)}{1 + \omega^2 K^2} \quad \text{where } K = RC$$

$$\therefore V_2 - V_1 = \frac{V_1(-1 + j\omega K)}{1 + \omega^2 K^2}$$

$$= \frac{jV_1}{\omega K} \text{ if } \omega K \gg 1 \quad \ldots \quad \ldots \quad (1)$$

Thus the distortion is inversely proportional to frequency.

Now the spectrum of a train of phase-modulated pulses is given by Fitch[4] as:

$$F(t) = Alf_r \left\{ 1 + 2\pi l_d f_m \cdot \frac{\sin \pi l f_m}{\pi l f_m} \cdot \cos(2\pi f_m t + \phi - \pi l f_m) \right.$$
$$\left. + 2 \sum_{\substack{k=1 \\ n=-\infty}}^{\infty} J_n(2\pi k l_d f_r) \cdot \frac{\sin \pi l (kf_r + nf_m)}{\pi k l f_r} \cdot \cos[2\pi(kf_r + nf_m)t + n\phi - \pi n l f_m] \right\}$$

From equation (1) the effect of the coupling circuit will be equivalent to adding to the pulse train a signal $F_1(t)$ where

$$F_1(t) = Alf_r \left\{ 1 - \frac{l_d}{K} \cdot \frac{\sin \pi l f_m}{\pi l f_m} \cdot \sin(2\pi f_m t + \phi - \pi l f_m) \right.$$
$$\left. - 2 \sum_{\substack{k=1 \\ n=-\infty}}^{\infty} \frac{J_n(2\pi k l_d f_r)}{2\pi K(kf_r + nf_m)} \cdot \frac{\sin \pi l (kf_r + nf_m)}{\pi k l f_r} \cdot \sin[2\pi(kf_r + nf_m)t + n\phi - \pi n l f_m] \right\} \quad (2)$$

$$\text{Now } J_n(x) = \frac{x^n}{2^n n!} \left\{ 1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \ldots\ldots\ldots \right\}$$

If we assume that

$$x = 2\pi k l_d f_r \ll 1$$

$$\text{Then } J_n(x) = \frac{x^n}{2^n n!}$$

Also if $\pi l (kf_r + nf_m) \ll 1$

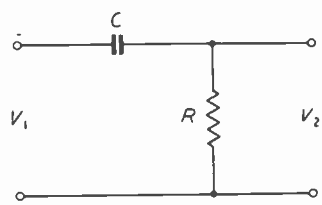
$$\text{Then } \frac{\sin \pi l (kf_r + nf_m)}{kf_r + nf_m} = 1$$

and equation (2) becomes:

$$F_1(t) = Alf_r \left\{ -\frac{l_d}{K} \cdot \sin(2\pi f_m t + \phi - \pi l f_m) \right.$$
$$- \sum_{k=1}^{\infty} \left[\frac{1}{\pi K k f_r} \cdot \sin 2\pi k f_r t \pm \frac{l_d}{K} \cdot \sin[2\pi(kf_r \pm f_m)t \pm \phi \mp \pi l f_m] \right.$$
$$\left. \left. \pm \frac{l_d}{K} \cdot \frac{\pi k l_d f_r}{2} \cdot \sin[2\pi(kf_r \pm 2f_m)t \pm 2\phi \mp 2\pi l f_m] \pm \ldots \right] \right\} \quad (3)$$

Since $2\pi k l_d f_r \ll 1$, the terms where $n > 1$ can be ignored relative to the remainder. The remaining terms which can cause crosstalk are those whose frequencies are f_m and $(kf_r \pm f_m)$. With the approximations which have been used, these all have the same amplitude $Alf_r l_d/K$. Thus it is incorrect to assume that crosstalk can be calculated by ignoring all components other than that of frequency f_m. A similar analysis for p.w.m. and p.a.m. shows that for these types of modulation the assumption is justified so long as $f_m \ll f_r$.

Fig. 1. *The simple RC-coupling circuit considered.*



3. The Waveform Approach

A better method of approach is to consider the distortion of the waveform of a single pulse, and then to sum the effects of all preceding disturbing pulses upon a given disturbed pulse.

In Fig. 2(a) a rectangular pulse is shown. Fig. 2(b) shows the same pulse after transmission through the network of Fig. 1. The amplitude of the 'tail' of the distorted pulse is

$$v_0(t) = -Ae^{-t/K}(e^{-l/K} - 1) \quad \text{where } K = RC$$

$$= -\frac{Al}{K} \cdot e^{-t/K} \text{ if } l \ll K \quad \ldots \quad \ldots \quad (4)$$

If the pulse of Fig. 2 (b) is displaced so that its trailing edge occurs at time t_1, equation (4) becomes

$$v_1(t) = -\frac{Al}{K} \cdot e^{-(t-t_1)/K}$$

and the amplitude change in the 'tail' due to this displacement is

$$v(t) = v_1(t) - v_0(t) = \frac{-Al}{K}[e^{-(t-t_1)/K} - e^{-t/K}]$$

$$= -\frac{Al}{K} \cdot e^{-t/K}(e^{t_1/K} - 1)$$

$$= -\frac{Alt_1}{K^2} \cdot e^{-t/K} \quad \text{if } t_1 \ll K \quad \ldots \quad (5)$$

Now let us consider a train of modulated pulses, and the sum of their effects upon a single unmodulated pulse.

Let $t_1 = l_d \exp. j(-nt_3\omega_m + \phi)$

$t = t_2 + nt_3$

where t_2 = separation of disturbed pulse from nearest disturbing pulse

$t_3 = 1/f_r$ = repetition interval

l_d = peak time modulation

ω_m = angular modulation frequency

ϕ = modulation phase angle

n = an integer

The sum of the disturbing voltages at time t_2 due to all preceding modulated pulses is given by:

$$v_{t_2} = \sum_{n=0}^{\infty} -\frac{All_d}{K^2} \cdot \exp. j(-nt_3\omega_m + \phi) . \exp. -\frac{1}{K}(t_2 + nt_3)$$

$$= -\frac{All_d}{K^2} \cdot e^{-t_2/K} \cdot e^{j\phi} \cdot \sum_{n=0}^{\infty} \exp. -nt_3\left(j\omega_m + \frac{1}{K}\right)$$

This is the sum to infinity of a geometric progression.

$$\therefore v_{t_2} = -\frac{All_d}{K^2} \cdot e^{-t_2/K} \cdot e^{j\phi} \cdot \frac{1}{1 - \exp. -t_3(j\omega_m + 1/K)}$$

$$= -\frac{All_d}{K^2} \cdot e^{-t_2/K} \cdot e^{j\phi} \cdot \frac{1 - e^{-t_3/K}(\cos \omega_m t_3 + j \sin \omega_m t_3)}{1 + e^{-2t_3/K} - 2e^{-t_3/K} . \cos \omega_m t_3}$$

Thus the crosstalk voltage v_{t_2} is sinusoidal, having an angular frequency ω_m and amplitude

$$V_{t_2} = \frac{All_d}{K^2} \cdot e^{-t_2/K} \cdot \frac{[(1 - e^{-t_3/K}\cos \omega_m t_3)^2 + (e^{-t_3/K}\sin \omega_m t_3)^2]^{\frac{1}{2}}}{1 + e^{-2t_3/K} - 2e^{-t_3/K} . \cos \omega_m t_3}$$

$$= \frac{All_d}{K^2} \cdot e^{-t_2/K}(1 + e^{-2t_3/K} - 2e^{-t_3/K}\cos \omega_m t_3)^{-\frac{1}{2}} \quad \ldots \quad \ldots \quad (6)$$

This voltage will be superimposed on the pulse, causing amplitude modulation, and this will be converted into time modulation of the pulse flank by any slicing or squaring process. If the slope of the pulse at slicing level is dv/dt, then:

$$\frac{\text{signal amplitude}}{\text{crosstalk amplitude}} = \frac{l_d}{\delta t} = \frac{l_d}{V_{t_2} . dt/dv}$$

$$= \frac{dv}{dt} \cdot \frac{K^2 e^{t_2/K}}{Al} \cdot (1 + e^{-2t_3/K} - 2e^{-t_3/K} . \cos \omega_m t_3)^{\frac{1}{2}} \quad (7)$$

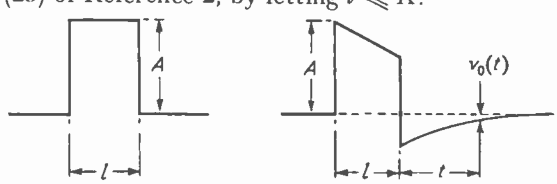
If $\omega_m \ll 1/t_3$

$$\frac{l_d}{\delta t} = \frac{dv}{dt} \cdot \frac{K^2 e^{t_2/K}}{Al} \cdot (1 - e^{-t_3/K}) \quad \ldots \quad \ldots \quad (8)$$

If $K \ll t_3$

$$\frac{l_d}{\delta t} = \frac{dv}{dt} \cdot \frac{K^2 e^{t_2/K}}{Al} \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

This last result can be obtained from equation (23) of Reference 2, by letting $l \ll K$.



(a) (b)

Fig. 2. *Pulse waveform (a) and its shape after an RC-coupling circuit (b).*

If we take the particular case in which the pulse shape is a 'raised cosine',

i.e., $v = 0$ for $-\infty < t < -l$ and $l < t < \infty$

$$v = \frac{A}{2}\left(1 + \cos\frac{\pi t}{l}\right) \quad \text{for } -l < t < l$$

Then the maximum slope is at half height, where $\dfrac{dv}{dt} = \dfrac{\pi A}{2l}$

Substituting in equation (7)

$$\therefore \frac{l_d}{\delta t} = \frac{\pi K^2 e^{t_2/K}}{2l^2} \cdot (1 + e^{-2t_3/K} - 2e^{-t_3/K}\cos \omega_m t_3)^{\frac{1}{2}} \quad (9)$$

These results are only correct for a single distorting network of the nature of a coupling or decoupling circuit. Unfortunately, they cannot easily be extended to several such networks in successive stages of an amplifier, owing to the fact that the pulse 'tail' is no longer exponential, for the addition of the contributions of the various disturbing pulses becomes more difficult. However, they do give a guide to the order of low-frequency response necessary in such cases for the attainment of a given quality of transmission.

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RADIATION FROM AERIALS

By Giorgio Barzilai*

(Polytechnic Institute of Brooklyn, New York.)

SUMMARY.—A formal expression for the complex power radiated by a thin, centre-driven aerial is derived in terms of the vector potential on the surface of the conductor, making no assumption on the distribution of current. A similar expression is then obtained for the complex power radiated by the same aerial, on the assumption of sinusoidal-current distribution. By letting the radius of the conductor approach zero, the asymptotic forms of the two aforesaid expressions are compared.

1. Introduction

It has been known for more than half a century[1,2,3] that, on thin conducting wires, energized at one point by a sinusoidal e.m.f., the current distribution is approximately sinusoidal. The sinusoidal approximation has been used by several authors[4,5,6] to compute the real and reactive power radiated by a thin centre-driven aerial. The method calls for the calculation of the tangential component of the electric field on the surface of the aerial. Schelkunoff has called this approach a radiation paradox, since the tangential electric field on the surface of the conductor, computed on the assumption of sinusoidal current, is inconsistent with the values imposed by the metallic boundaries.

The legitimacy of the said method of calculation has been discussed in the literature[8-13]. For the real power radiated by the aerial the method can be easily justified, at least asymptotically for infinitely thin wires. Less simple, and not usually discussed, is the question relative to the reactive power.

It is the purpose of this paper to derive a formal expression for the complex power radiated by a thin, centre-driven, perfectly conducting aerial, making no assumption on the distribution of current. A similar expression is then obtained for the complex power radiated by the same aerial, on the assumption of sinusoidal current distribution. By letting the radius of the conductor approach zero, the asymptotic forms of the two aforesaid expressions will be compared.

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MS accepted by the Editor, September 1954

2. Statement of the Problem

The aerial in Fig. 1 is assumed to be perfectly conducting, and of circular cross-section. The applied voltage V_0 is assumed to act through the gap d of infinitesimal length. We have

$$E_z = -V_0 \delta(z) \qquad (1)$$

where $E_z = E_z(z)$ is the tangential electric field on the surface of the aerial, and $\delta(z)$ is the Dirac function.

In the following we shall indicate with $I = I(z)$ the current in the aerial, and we shall assume

$$I(\pm h) = 0 \qquad (2)$$

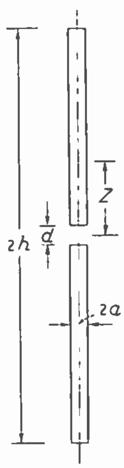

Fig. 1. *Geometry of a centre-driven cylindrical aerial.*

The complex power P radiated can be expressed as follows

$$P = -\int_{-h}^{+h} E_z I^* \, dz = V_0 I^*(0) \qquad (3)$$

where the asterisk indicates the complex conjugate, and use has been made of the r.m.s. values.

If now we assume sinusoidal current distribution, the complex radiated power will be expressed by the integral

$$P_s = -\int_{-h}^{+h} E_{z,s} I_m \sin\theta \, dz \, . \, . \qquad (4)$$

where I_m is the maximum amplitude of the sine wave, $\theta = k(h - |z|)$, $k = 2\pi/\lambda$, λ is the wavelength, and the index s has been added to indicate that the relative quantities refer to sinusoidal current distribution.

Formulae to compute $E_{z,s}$ are given in the literature[14].

Our aim is to find analytical expressions for P and P_s which are suitable for comparison.

3. Formal Solution for the Current

With reference to Fig. 1 we have

$$- V_0\delta(z) = - j\omega\mu A - \frac{dV}{dz} \quad .. \quad (5)$$

$$\frac{dA}{dz} = - j\omega\epsilon V \quad .. \quad .. \quad .. \quad (6)$$

where $A = A(z)$ and $V = V(z)$ are vector and scalar potentials on the surface of the aerial†, ω is the angular frequency, and ϵ and μ are the dielectric constant and the magnetic permeability of the medium surrounding the aerial.

From (5) and (6) we obtain

$$A = B \cos\theta + C \sin\theta \quad .. \quad .. \quad (7)$$

$$V = j\eta\,[B \sin\theta - C \cos\theta]\,\epsilon\,(z) .. \quad (8)$$

where $\eta = \sqrt{\mu/\epsilon} \approx 120\pi$, for free space.

$$\epsilon\,(z) = \begin{matrix} +1 & z > 0 \\ -1 & z < 0 \end{matrix},$$

and B and C are constants of integration.

On the other hand the integral expression of A in terms of I is well known. Equating this expression to the right-hand side of (7) we obtain the following integral equation for the current

$$\frac{1}{4\pi}\int_{-h}^{+h} G(z, z')\, I(z')\, dz' = B \cos\theta + C \sin\theta \quad (9)$$

$$G\,(z, z') = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-jkr}}{r}\, d\alpha \quad .. \quad .. \quad (10)$$

$$r = [(z - z')^2 + 2a^2\,(1 - \cos\alpha)]^{1/2}$$

Equation (9), with a simplified kernel, has been solved by Hallèn[15,16], in terms of the applied voltage V_0. In our case it is more convenient to prescribe the current $I(z_0)$ in an arbitrary point $z = z_0$. V_0 will therefore become an unknown quantity.

It can be shown that the integral on the left of (9) can be written as follows

$$A(z) = \frac{1}{4\pi}\,[\Omega I(z) + f(z)] \quad .. \quad .. \quad (11)$$

$$\Omega = 2 \log_e \frac{2h}{a}$$

In (11), $f(z)$ is a function dependent upon the current distribution, and therefore generally unknown, but it is continuous at $a = 0$.

We shall now express the constants B and C in (7) and (8), in terms of the vector potentials at $z = h$ and $z = z_0$. Taking into account (11) we obtain

$$B = A(h) \qquad C = \frac{1}{4\pi}[I_m\Omega + \phi(z_0)] \quad .. \quad (12)$$

where we put

$$I_m = \frac{I(z_0)}{\sin\theta_0};\ \phi(z_0) = \frac{f(z_0) - f(h)\cos\theta_0}{\sin\theta_0} \quad (13)$$

and $\theta_0 = k(h - |z_0|) \neq (n + 1)\,\pi,\ n = 0, 1, 2, \ldots$

Upon taking into account (11) and (12), (9) can be formally solved as follows

$$I(z) = I_m \sin\theta + \frac{1}{\Omega}[f(h)\cos\theta - f(z) + \phi(z_0)\sin\theta]$$

$$= I_s(z) + I_c(z) \quad .. \quad .. \quad .. \quad (14)$$

The current $I(z)$ can be thought of as the sum of two components: a sinusoidal principal current

$$I_s(z) = I_m \sin\theta, \quad .. \quad .. \quad .. \quad (15)$$

and a complementary current $I_c(z)$, whose definition follows from (14) and (15).

In virtue of the continuity of $f(z)$ at $a = 0$, from (14) it follows that

$$\lim_{a\to 0} I(z) = I_m \sin\theta \quad .. \quad .. \quad (16)$$

It is emphasized that, as Aharoni points out[17], it would be incorrect to conclude from (16) that (15) is the asymptotic solution of (9) for $a \to 0$. The complementary current, in fact, brings a significant contribution to the integral in (9), even in the limit for $a \to 0$. This can be easily proved by inserting (14) into (9) and then taking the limit for $a \to 0$. The details of the proof are evident if one notices that

$$\lim_{a\to 0}\int_{-h}^{+h} G(z, z')\,\frac{f(z')}{\Omega}\, dz' = \lim_{a\to 0}\int_{-h}^{+h} \frac{e^{-jkr_0}}{\Omega r_0}\, f(z')\, dz' = f_s(z)$$

where we put

$$r_0 = [(z - z')^2 + a^2]^{1/2}$$

$$\lim_{a\to 0} f(z) = f_s(z) \quad .. \quad .. \quad .. \quad (17)$$

In virtue of the continuity of $f(z)$ at $a = 0$, an explicit expression[18] for $f_s(z)$ can be found from the definition (11), assuming the current distribution (15). The approximation made on the kernel (10) causes, as Schelkunoff points out[19], an error in the evaluation of the integral of the order of a^2.

4. Analytical Expressions for P and P_s

To compute the power P we recall that

$$V_0 = 2V(0^+) \quad .. \quad .. \quad .. \quad (18)$$

where $V(0^+)$ is the value approached by $V(z)$, when $z \to 0$ from the positive side. (18) follows from (5) if this equation is integrated with

† Because of the one dimensional character of the problem the vector potential can be treated as a scalar quantity.

respect to z from $-b$ to $+b$, and then the limit is taken for $b \to 0$. Using (3), (8), (18), (13) and (14), we obtain

$$P = 2V(0^+)\, I^*(0) = 2jI_m\eta \left\{ A(h) - \left[A(0) + \frac{\Omega}{4\pi} I_c^*(0) \right] \cos kh + \frac{H}{\Omega} \right\} \quad \ldots \quad (19)$$

where $I_c(0)$ is the value of the complementary current at $z = 0$, and H is a function continuous at $a = 0$.

It is easy to cast the power P_s in a form suitable for our purposes. From (4) we obtain

$$P_s = I_m \int_{-h}^{+h} \left[j\omega\mu A_s + \frac{dV_s}{dz} \right] \sin\theta\, dz = 2jI_m\eta \left[A_s(h) - A_s(0)\cos kh \right] \quad \ldots \quad (20)$$

where the index s refers to quantities computed using (15), and the last step in (20) is easily justified by integrating by parts the second integral on the right, twice, first with respect to $(dV_s/dz)dz$, and then with respect to $V_s dz$. In performing this integration we have to recall (6), and to notice that $(d|z|/dz) = \epsilon(z)$, $(d\epsilon(z)/dz) = 2\delta(z)$, and that A_s and V_s are respectively even and odd functions of z. The result (20) can also be obtained by simple inspection of the integral expression (4), if in place of $E_{z,s}$ we substitute the expression for this quantity obtained from the well-known formulae [14] for the field of a sinusoidal filament of current.

By comparing (19) and (20) and keeping in mind (14) and (15), we can write

$$P = P_s - j\Omega\frac{\eta}{\pi} \operatorname{Re} I_c(0) \cos kh + \frac{H'}{\Omega} \quad \ldots \quad (21)$$

where H' is a function of a, continuous at $a = 0$. From (21) we can conclude that asymptotically when $a = 0$

$$P = P_s \qquad \text{if } kh = (2n+1)\frac{\pi}{2} \quad (22)$$

$$P = P_s \qquad \text{if } kh = (n+1)\pi \quad (23)$$

$$\operatorname{Re} P = \operatorname{Re} P_s \qquad \text{if } kh \neq n\frac{\pi}{2} \quad \ldots \quad (24)$$

$$\operatorname{Img} P \underset{\neq}{=} \operatorname{Img} P_s \quad \begin{matrix} z_0 = 0 \\ z_0 \neq 0 \end{matrix}$$

Where Re and Img indicate the real and imaginary parts respectively and $n = 0, 1, 2, \ldots$

The result (22) has been obtained numerically for $n = 0$, and $n = 1$ using Hallèn's expansion[20]. The result (23), which is the typical relationship between input and output power in a uniform transmission line of length $\lambda/4$, can be derived from Schelkunoff's theory[7].

The result (24) states that when $kh \neq n\pi/2$ the difference between the reactive powers approaches zero when $a \to 0$, only if the current on the aerial is prescribed at the input terminals; i.e., $I_c(0) = 0$. In this case, however, both reactive powers approach infinity.

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CORRECTION

In the June Editorial it was stated that a breakdown in any one of the individual transmitters at the B.B.C. f.m. stations would cause a reduction of signal strength of only 3 db. This figure should be 6 db, since one-half of the power is then radiated by only one-half of the aerial and the resulting reduction of aerial gain produces an extra 3-db loss. Similarly, a fault in one-half of the aerial will cause a 6-db drop in the signals for all three programmes, since it not only halves the aerial gain but makes three transmitters ineffective.

POST OFFICE APPOINTMENTS

Sir Gordon Radley has been appointed Director-General of the Post Office in succession to Sir Alexander Little. He is the first engineer to become Director-General.

R. J. P. Harvey and S. D. Sargent have been appointed Deputy Directors-General in succession to Sir Gordon Radley and Sir Dudley Lumley.

NEW BOOKS

Transistor Audio Amplifiers

By RICHARD F. SHEA. Pp. 219 + xiii. Chapman & Hall Ltd., 37 Essex Street, London, W.C.2. Price 52s.

The outstanding feature of this book is that it is so much more practical in its outlook than most books dealing with transistors. It is confined to the junction transistor at audio frequencies. There is a short introductory chapter on theoretical matters and then a lengthy one on transistor parameters. These are derived from the conventional black box and with them some equivalent circuits.

The parameters and equivalent circuit which are chiefly used in the book are not the usual ones. The so-called *h* parameters are used with an equivalent circuit comprising a pair of resistances, a collector circuit constant-current generator and an input-circuit constant-voltage generator. It is claimed that these *h* parameters are not only easier to measure but are less affected by changes of operating conditions than others. The nomenclature, however, is obviously inconvenient, when the same letter with only different subscripts stands for resistance and also for dimensionless quantities, such as current and voltage amplification factors. Some 48 pages in Chapter 2 are devoted to tabulated data on American junction transistors.

Chapter 3 deals with basic amplifier design and Chapter 4 with multi-stage amplifiers. Chapter 5 is entitled "Preamplifiers" and deals with a diversity of subjects, such as noise, impedance considerations, the effect of frequency and gain control. This last well illustrates the practical outlook of the book, for quite a bit of space is devoted to matters which are important in practice, but which are often overlooked in purely theoretical discussions. Chapters 6 and 7 are devoted to power amplification, both class A and class B, and the final chapter is descriptive and, to some extent analytic, of some complete amplifier designs.

Throughout the book the importance of stabilizing the operating point and of temperature are kept well to the fore. It is a book which will undoubtedly be of considerable use to amplifier designers. W. T. C.

Servomechanism Practice

By WILLIAM R. AHRENDT. Pp. 349 + vii. McGraw-Hill Publishing Co. Ltd., 95 Farringdon Street, London, E.C.4. Price 50s.

This book is of an elementary character and is almost entirely devoid of mathematics. The outlook is practical rather than theoretical. After an introductory chapter, descriptive of a simple servomechanism, there are two chapters describing the physical form and characteristics of potentiometers and synchros. Chapters on double-speed synchronization; demodulators and modulators; network and various amplifiers (electronic, magnetic, rotating) follow. Other chapters cover rate generators, servomotors and hydraulic systems. There is a chapter on design and one on manufacture.

The general form of treatment is descriptive and the book forms a useful introduction to the subject.

W. T. C.

Remote Control by Radio (2nd Edition)

By A. H. BRUINSMA. Philips' Technical Library, Popular Series. Pp. 97 + viii. Cleaver-Hume Press Ltd., 31 Wright's Lane, Kensington, London, W.8. Price 8s. 6d.

Precision Electrical Measurements

Proceedings of a Symposium held at the National Physical Laboratory, 17th–20th November 1954. Pp. 329 + xxii. H.M. Stationery Office, York House, Kingsway, London, W.C.2. Price 27s. 6d.

CABMA Register of British Products and Canadian Distributors, 1955-56.

Pp. 760. Published jointly by Kelly's Directories Ltd. and Iliffe & Sons Ltd. for the Canadian Association of British Manufacturers and Agencies. Obtainable in the U.K. from Iliffe & Sons, Dorset House, Stamford Street, London, S.E.1, price 44s.; in Canada, from Managers of British Trade Centres: Royal Bank Building, Toronto; Arrowhead Building, Montreal; and Hall Building, Vancouver.

In its Buyers' Guide, this book provides an alphabetical list of some 4,000 British products available to the Canadian market, together with their suppliers. French equivalents of the headings are listed in a separate Glossary. There is also a directory of 4,500 British firms with details of their Canadian distribution arrangements. A further section enables products to be identified from proprietary names and trade marks.

STANDARD-FREQUENCY TRANSMISSIONS

(Communication from the National Physical Laboratory)

Values for June 1955

Date 1955 June	Frequency deviation from nominal: parts in 10^8		Lead of MSF impulses on GBR 1000 G.M.T. time signal in milliseconds
	MSF 60 kc/s 1429-1530 G.M.T.	Droitwich 200 kc/s 1030 G.M.T.	
1	+0·4	+1	−8·8
2	+0·5	+1	−7·2
3	+0·5	+1	−7·8
4	+0·5	+2	NM
5	+0·5	0	NM
6	+0·4	+2	NM
7	+0·4	+2	NM
8	+0·4	+1	NM
9	+0·5	+1	−4·2
10	+0·5	+1	−2·9
11	+0·5	+1	NM
12	+0·5	+1	NM
13	+0·5	+1	−0·7
14	+0·5	+1	−0·3
15	+0·5	+1	−0·2
16	+0·5	0	NM
17	+0·5	+1	NM
18	NM	+1	NM
19	NM	+1	NM
20	−0·2	+2	+5·7
21	−0·3	+1	+4·1
22	−0·3	+1	+4·4
23	−0·3	+1	+2·6
24	−0·3	+1	+5·7
25	NM	+1	NM
26	NM	+1	NM
27	−0·2	+1	+7·3
28	−0·2	+2	+7·7
29	−0·2	+2	+8·2
30	−0·3	+1	+6·2

The values are based on astronomical data available on 1st July 1955.
NM = Not Measured.

Change in MSF pulse modulation: From the 1st July 1955 the 60th pulse in each minute of pulse modulation is lengthened from 5 msec to 100 msec and the 59th pulse, previously suppressed, is restored.