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ALFRED N. GOLDSMITH, Ph.D.

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On the evening of December 20, 1915, a meeting of the Boston Section was held at the Cruft High Tension Laboratory, Harvard University. Professor J. C. Hubbard presented a paper on "The Effect of Distributed Capacity in Inductance Coils." This paper was discussed by Professors George W. Pierce, W. G. Cady, J. E. Ives, and A. E. Kennelly. At the same meeting there were exhibits of radio apparatus by a number of companies and individuals. The following companies were represented: General Radio Company, W. J. Murdock Company, George S. Saunders and Company, the Clapp-Eastham Company, and the American Radio and Research Corporation. The private exhibitors were the following. Professor A. E. Kennelly showed a "Paul" 800-cycle telephone current generator, a "Drysdale" alternating current potentiometer, and a "Duddell" oscillograph. Messrs. Fulton Cutting and Washington showed a small transmitting set using "Chaffee" gaps operated on 500 volts D. C. Dr. E. L. Chaffee showed a Tesla coil operated by two of his gaps together with a vacuum tube for showing the spark characteristics. Professor George W. Pierce, in addition to a number of instruments, showed his mercury arc oscillator acting as a generator and also as a receiver. In the latter case the signals from Arlington were recorded on tape. Mr. H. E. Rawson showed a wave meter, and a receiving set with which reception from the South San Francisco station was demonstrated. The attendance was one hundred and forty-three.

SEATTLE SECTION

On the evening of September 4, 1915, a meeting of the Seattle Section of the Institute was held, Mr. Robert H. Marriott presiding. The organization of the Seattle Section was discussed by Messrs. Marriott, Cooper, Alexis Paysse (Secretary-Treasurer of the Section), Wolf, and Milligan. The attendance was twenty-six.

On the evening of October 6, 1915, a meeting of the Seattle Section was held in the rooms of the Seattle Chamber of Commerce. Professor Osborne of the University of Washington offered the use of rooms at the University to the Seattle Section. This offer was formally accepted, with thanks. A paper on "Radio Development in the United States from 1899 to 1915" was presented by Mr. Robert H. Marriott. The paper was illustrated by slides. Mr. Roy E. Thompson discussed the paper; after which there was a discussion by a number of the members on allied topics. The attendance was forty-two.

On the evening of November 6, 1915, a meeting of the Seattle Section was held at Denny Hall, University of Washington. Two papers were read by the Chairman, Mr. Marriott. These were "The Effectiveness of the Ground Antenna in Long Distance Reception" by Mr. Robert B. Woolverton and "The Use of Multiphase Radio Transmitters" by Mr. William C. Woodland. A discussion followed. Mr. Robert H. Marriott then outlined his plan for the investigation of "radio shadows," and the carrying out of this work was systematically begun. The attendance was thirty-seven.

THE USE OF MULTI-PHASE RADIO TRANSMITTERS*

By

William C. Woodland

(Engineer, Packard Electric Company)

The production of radio frequency current from currents of commercial frequency is a matter of great interest and usefulness. The fact that a pure sine wave is not necessary or even desirable for radio purposes makes possible a great many ways of dividing up audio frequency current so as to duplicate the results obtained by radio frequency generators.

For example, it is possible to divide up audio frequency 3-phase currents into any number of intermediate phases, all of which have equal wave peaks occurring successively. By placing a separate radio transmitter in each phase, it would seem possible to operate at almost any desired frequency.

Several advantages of multiphase current over single phase current may be pointed out.

1. In an 8-phase, 60-cycle equipment the tone of the spark would be equivalent to that given by a 480-cycle generator. The condensers, however, would operate on 60-cycle current, which would allow the use of much larger capacity and a lower voltage for the same total amount of energy. The large condensers discharging at a comparatively low voltage would give short thick sparks of low resistance, resulting in an improved efficiency. The lower voltage would increase the life of both the condensers and the transformers.

2. The reliability of the equipment would be increased; because if a transformer or condenser should break down, the message might be finished on the other phases with no more trouble than a slight weakening in power; whereas on a single phase equipment, the operator would be unable even to advise his correspondent of the nature of the difficulty.

3. There would be some advantage in having to break only 58 per cent. of the corresponding single phase current at the key.

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* Presented before The Institute of Radio Engineers, New York, November 3, 1915.

4. Spare transformers and condensers could be carried at less expense than with single phase equipment.

5. I will show a little later that it is possible to operate multiphase equipments with very high power-factors and leading current so that the generator voltage does not fall on depressing the key.

6. When operating directly on 60-cycle, 3-phase current, the efficiency is certainly very much improved.

My attention was first called to the use of multiphase current for radio transmission in the early part of 1912, when it occurred to me that by dividing up a 3-phase, 60-cycle current into a sufficient number of intermediate phases, the results of the higher frequency generators might be duplicated using only commercial current.

I did not, at that time have in mind anything further than placing a separate radio transmitter of the fixed gap type in each phase, depending on it to discharge at the peak of the wave independently of the other sets.

This plan did not meet with success on account of the fact that the maximum point of the wave is not sufficiently definite to secure the phases against interference with each other. I found also that other investigators had carried the work up to this point, but that all systems had been rendered more or less inoperative because of the interference mentioned above.

All of these difficulties were overcome by the use of a rotary spark gap in each phase which had for its purpose the definite localizing of the point of discharge.

Eight (8) phase, five (5) phase, four (4) phase and three (3) phase equipments of this description have been built with entire success.

In the remainder of this article, I shall describe more minutely the 3-phase, 120-cycle, 3-kilowatt set shown in Figures 1 and 2. The three transformers are each of 1 kilowatt rating, 120 cycles, 63.5 volts primary, and 7,500 volts secondary, and are star connected on both sides. The open ends of the star primary go direct to the generator and the other ends are closed on each other thru the relay key when operating.

The condenser set consists of 9 banks, 3 condensers in series on each of the 3 phases, each bank being of 0.035 microfarad capacity, the effective capacity per phase being 0.0166 microfarad. The condenser phases are also connected in star relation to each other.

The open ends of the condenser star go direct to the 3-phase spark gap, by means of which they are successively discharged thru the helix.

While testing for phase order and rotation, the neutral of the condenser set is connected to the transformer neutral and each phase operated separately; but after the testing is done, the

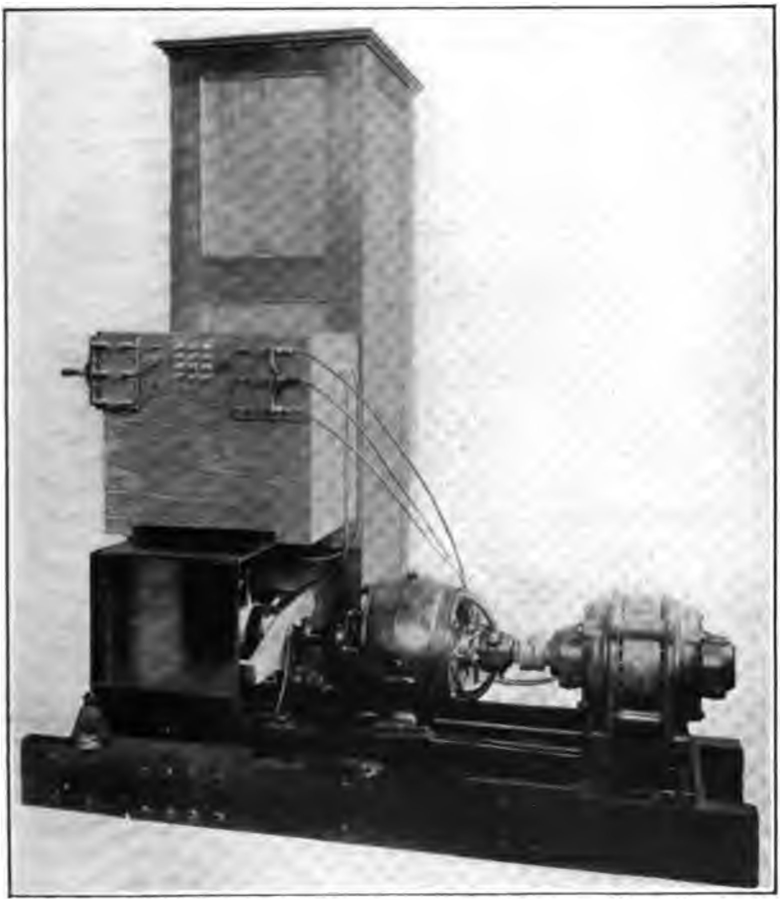


FIGURE 1—Three-Phase Transmitter

results are improved by removing the neutral connection. The shifting of the neutral serves to limit the short circuit current of the transformers and makes operation possible with much less inductance than is common.

The point above mentioned is worthy of a fuller explanation.

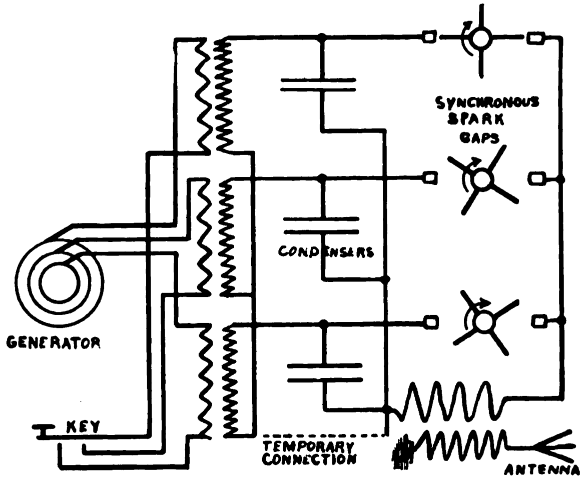


FIGURE 2—Diagram of Connections for Three-Phase Transmission

In most radio transmitters operating on spark systems, the discharge of the condenser acts as a momentary short circuit on the transformer, and means have to be provided to prevent a power arc from following the oscillatory discharge. On single phase equipments, this is usually accomplished by using a limiting inductance, which has to be capable of limiting the entire transformer output in order to be effective. On the multi-phase system, however, a much smaller inductance can be used, since the short circuit current is limited partly by the inductance of the active transformer and partly by the shifting of the neutral of the other two. The shifting of the neutral has a beneficial effect since it serves to increase the voltage on the transformer discharging next in order.

A transformer with series inductance short circuited on single phase supply draws more than double the current that it would if short circuited on a 3-phase star connection. The best results have been secured with a power factor of 90 per cent. leading; that is, the amount of inductance is less than that required for 120-cycle resonance.

The rotary spark gap which is direct driven from the generator shaft is shown in Figure 3. Considerable range of input, efficiency, and quality of tone is secured by shifting the point at which the discharge takes place. It has been found best to operate on the falling side of the wave, well over the peak, since this gives the highest efficiency combined with the best tone.

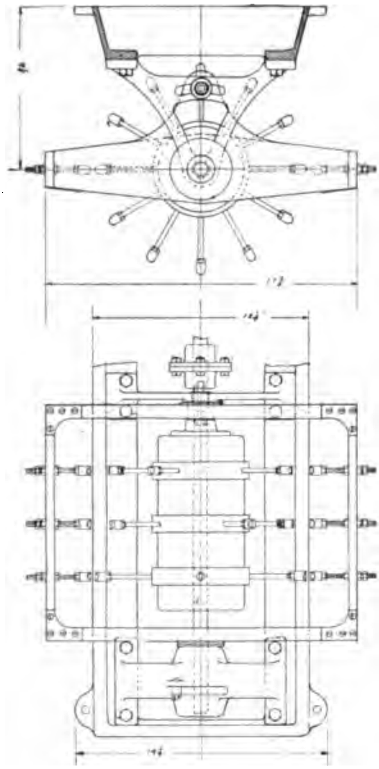


Figure 3—Three-Phase Rotary Spark Gap

Figure 4—Oil Break Relay Key

Advancing the point of discharge only increases the input without effecting the output on the aerial.

Some form of relay key is desirable since there are a minimum of 2 primary circuits to be closed simultaneously. The oil break form shown in Figure 4 has been found very satisfactory for this purpose. Complete efficiency tests are not available at this time, but the set above described has radiated 14 amperes on a 2,000-meter wave-length without exceeding its rating of 3 kilowatts.

September, 1915.

SUMMARY: The advantages of multi-phase over single-phase spark transmitters are given; some of which are higher tone, lower condenser voltages, greater reliability (because of spare phases in case of breakdown of one phase), and smaller current broken at key.

A 3-phase, 120-cycle, 3-kilowatt set is then described in detail.

CAPACITIES*

BY
FRITZ LOWENSTEIN

As the seat of energy of an electrical field is in the space outside of the charged bodies we will consider the shape and concentration of the field only, but not that of the body itself. This distinction is necessary because capacities are usually attributed to the bodies charged, whereas the energy is excluded from that space which is occupied by the body. Considering the space between two charged bodies as the only seat of energy, the expression "charged body" is best replaced by "terminal surface" of the field.

Comparing geometrically similar elements of two geometrically similar fields, the elementary capacities are proportional to lineal dimensions. (See Figure 1.)

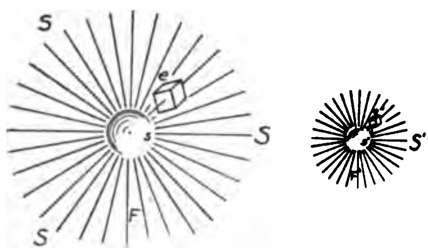


FIGURE 1

Extending this law over the entire field by the integrating process, we find that geometrically similar fields have capacities proportional to the lineal dimensions of the terminal surfaces. It is to be expected, therefore, that capacities expressed in dimensions of terminal surfaces should be of lineal dimensions.

That the capacity is by no means a function of the volume of the field or of the terminal body may be easily seen from Figure 2 where a field element is increased to double the volume by adding

* Presented before The Institute of Radio Engineers, New York, December 1, 1915.

volume in the direction of the field lines and in a direction perpendicular to the lines. In the first case the capacity has been decreased whereas in the latter case increased, altho in both cases the volumetric increase is the same.

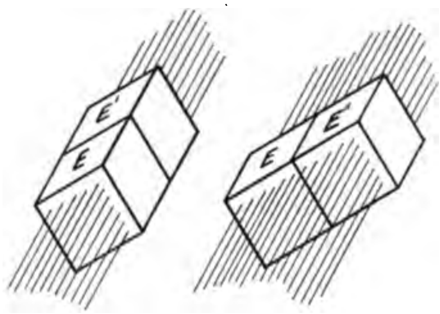


FIGURE 2

It is seen, therefore, that instead of being dependent on the volume, the capacity is rather a function of lineal dimension and therefore the maximum lineal dimension predominates.

An interesting example of this predominating lineal dimension or "maximum reach" is given by the composite capacity of two wires joining at one end under various angles, as shown in Figure 3.

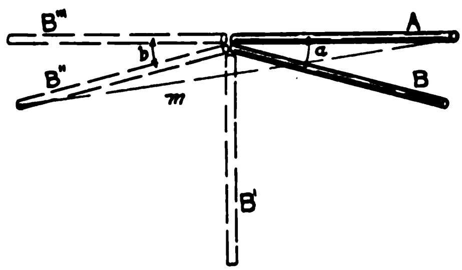


FIGURE 3

When the angle is small the composite capacity is practically the same as that of the single wire, since the addition of the second wire has not increased the maximum reach. If the second wire *B* be joined to *A* at an angle of 180 degrees, which means in straight continuation of wire *A* the total capacity has oubled, as the maximum reach now is twice that of the single wire. We notice also that by deviating wire B slightly from the traight continuation of wire A, the maximum reach of the system is not materially altered, from which one may correctly conclude that turning the wire B thru an appreciable angle b does not materially change the capacity of the system. On the other hand a great change of maximum reach is produced by variations of the angle when the two wires are approximately perpendicular, and in fact the capacity of the total structure is most sensitive to changes of angle between the two elements at about 90 degrees.

In Figure 4, I have given a table of capacities per centimeter of the greater lineal dimension of the different configurations.

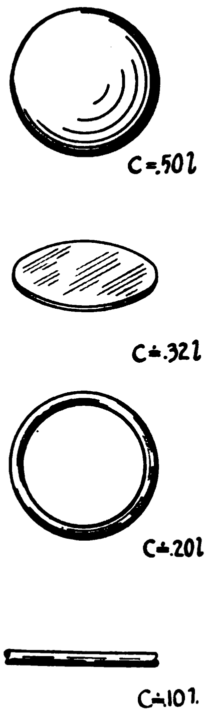


FIGURE 4

In Figure 5 the wire AB is assumed to be moved by the variable abscissae x, thereby generating a conducting sheet S.

It is instructive to follow the variation of the capacity C_x.

At $x=\theta$ the capacity is that of the wire C_{ab}; as long as x is small the capacity is practically constant because the width of the sheet is small compared to the length AB and a change of x does not involve a change of the predominating lineal dimension; however, as x increases and finally becomes greater than AB, it assumes the part of the predominating dimension, and, indeed, the graph shows the capacity then to be proportional to x.

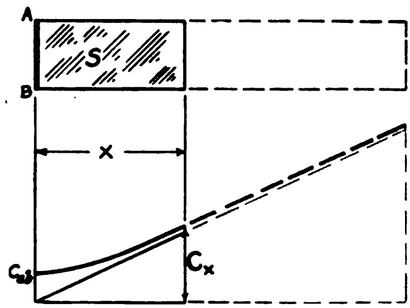


FIGURE 5

Comparing the capacities of a sphere and of a wire, it is found that the capacity of the sphere is only three or four times as great as the capacity of the wire in spite of the million times greater volume.

I have spoken of the capacities of a wire and of other bodies instead of the capacity of the field simply because I do not wish to distract attention from the familiar conceptions. Let me analyze the field shown in Figure 6, having two concentric spheres as terminal surfaces, and defining as "volumetric energy density" the energy contained in one cubic centimeter. As the energy of a field element is made up of the product of potential along the lines of force within that element and of the number of lines traversing it, the energy of a cubic centimeter of electric field is proportional to the square of the field density. Since the field density diminishes as the square of the distance from the center of field, the volumetric energy density diminishes with the fourth power of the distance from the center. The diagram to the left in Figure 6 shows the decrease of volumetric energy density.

Of greater interest than the volumetric energy density is the lineal energy density, which may be defined as the energy contained

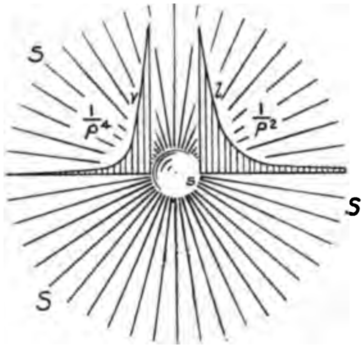


FIGURE 6

in a spherical layer of one centimeter radial thickness; and as the volume of such layer increases with the square of the distance from the center, the law follows from this fact, and from the volumetric energy density law that the lineal energy density decreases inversely as the square of the distance from the center. Such dependence is graphically shown to the right in Figure 6. The shaded surface below this curve represents the total energy of the field and it is easily seen therefrom that the maximum energy of the field is concentrated near the smaller of the two spheres.

I have taken a simple case of a field with spherical terminal surfaces to show that the concentration of energy lies near the smaller terminal surface. Similar considerations can be applied when substituting for this field radiating three-dimensionally, a field of bi-dimensional radiation (as that occurring in the case of long cylindrical terminal surfaces); where, as in this instance, the bulk of the energy of the field is to be found near the smaller one of the two terminal surfaces.

In Figure 7, I have shown a field with concentric terminal surfaces (either spherical or cylindrical), and have increased the scope of the field by reducing the size of the smaller terminal surface without, however, changing either the total number of field lines or the larger terminal surface. As the lineal energy density is very great near the smaller terminal surface, such addition of the field at that point must have materially increased the energy of the field and the change in capacity to be expected should be considerable. In fact, a considerable change in capacity of a sphere is obtained by a change of its diameter.

If, in Figure 7 the larger terminal surface alone is changed, even materially, the total energy of the field will be increased very slightly only; due to the fact, as we have seen, that the energy density near the larger terminal surface is very small. Such a small change in energy corresponds to only a small change in the capacity of the field, from which we conclude:

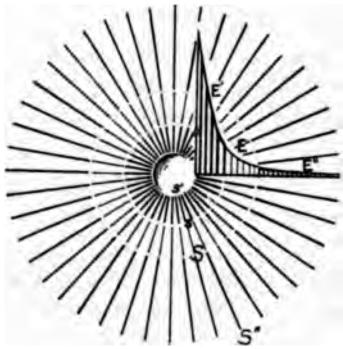


FIGURE 7

In a field having two terminal surfaces of greatly different size, a change of the smaller surface produces a great change in capacity, whereas a change of the larger terminal surface affects the capacity of the field only very slightly. The capacity of a field is, therefore, almost entirely determined by the shape of the smaller terminal surface.

That is why we may with correctness speak of the capacity of a sphere, or any other body, without mentioning the size and shape of the other terminal surface, as long as the assumption is correct that such other terminal surface is of greatly larger dimensions.

It may not be amiss to call your attention to the fact that the increase of field energy as illustrated in Figure 7 is accompanied by a decrease in capacity. This relation may easily be deduced from physical considerations, as well as from consideration of the mathematical expression for the capacity

$$C = \frac{\phi^2}{32\pi^2 W} \qquad \text{where } \phi = \text{total field lines}, \quad W = \text{energy},$$

wherein the capacity is expressed as a property of the field alone. I am tempted to introduce here the reciprocal value of capacity and apply to it the term "stiffness of the field," as an increase of energy would be followed by an increase of stiffness. I am, however, loath to mar any additional insight which may be gained from these explanations by deviation from so familiar a term as capacity.

For a better conception of the slight change of capacity caused by a considerable increase of the larger terminal surface, I refer to Figure 7, where the difference of capacity is only 1 per cent in spite of the diameter of the larger terminal surface being increased 100 per cent. It appears, therefore, that that part of the capacity of an antenna which is due to the flat top is not materially changed by its height above ground.

While considering the capacity of a flat top antenna to ground, it must have occurred to many engineers, as it did to me, that the statement to be found in many text books on electrostatics is rather misleading: "That the free capacity of a body considered alone in space must not be confounded with the capacity the body may have against another body considered as a plate condenser." This statement is quite erroneous. As the strength and direction in any point of a field is of single and definite value, only one electric field can exist in a given space at a given moment, and, therefore, only one value of capacity. It is incorrect, therefore, to distinguish between free capacity and condenser capacity. This clarifying statement is deemed advisable, or at least permissible, in view of the quoted errors.

By speaking of the capacity of the field instead of that of the body, no such erroneous thought is possible, and it is clear that by free capacity of a body is meant the capacity of the field whose smaller terminal surface is the given body and whose larger terminal surface is one of vastly greater dimensions. It is not essential that this greater terminal surface be located at infinite distance, because of the fact that even if construed as of ten times the lineal dimensions of the small surface the change caused by removing it to an infinite distance would result in a change in capacity of not more than one-tenth of 1 per cent.

At a time when I had not realized the singly determined value of a field capacity, I considered a comparison between free and plate capacity as shown in Figure 8, wherein to an upper disc (of which the free capacity is $\frac{2}{\pi}r$), was added another lower disc, thereby forming a plate condenser. The problem arose in my mind to determine the distance of separation of the two plates so that the plate capacity would equal the free capacity of the single disc. From the well-known formulas for the disc

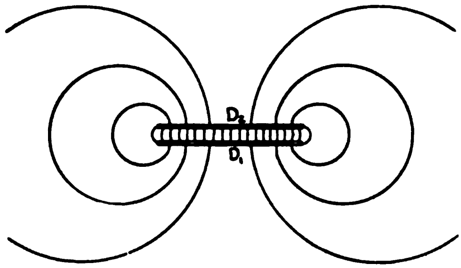


FIGURE 8

capacity and plate capacity, it would appear that the two were equal at a distance equal to $d=\frac{\pi}{8}r$, and I must confess that I had quite a struggle to decide whether in speaking of the capacity of the upper plate I would not have to add the two capacities. While such a mistake need hardly be called to the attention of the majority of engineers, I do not hesitate to make mention of it for the benefit of even the few students who might gain therefrom.

The advent of the aeroplane has opened another field, for radio communication. Whereas in the static field of an antenna, one terminal surface is artificial and the other provided by the surrounding ground, both terminal surfaces in an aeroplane outfit have to be artificial and are, therefore, open to design. The question arises in such a radio oscillator as to how much may be gained in energy for each single charge by increasing that one of the two terminal surfaces which consists of a dropped wire. The arrangement is shown in Figure 9. It is evident that

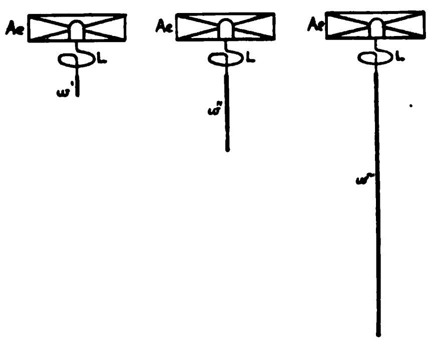


FIGURE 9

as long as the dropped wire is of smaller dimensions than the electrostatic counterpoise provided on the aeroplane, an increase in length of such dropped wire will materially increase the capacity of the field and, therefore, the energy per charge (as we may conclude by analogy from Figure 7). As soon, however, as the dropped wire is materially longer than the conductor on the aeroplane it assumes the role of the larger terminal surface of the field, and any further increase of its length will not materially contribute to an increase of electrostatic capacity nor of the energy per unit charge.

Figure 10 shows the function of the volumetric and lineal energy density in a field whose smaller terminal surface is a long cylinder. Such a field, radiating bi-dimensionally only, shows an energy concentration not so accentuated as that found in the

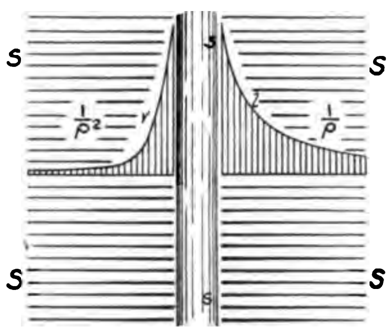


FIGURE 10

tri-dimensionally radiating field; but considering the larger terminal surface of a diameter ten times that of the smaller surface, the capacity would only be changed 1 per cent by increasing the larger terminal surface infinitely.

In all cases, therefore, where the larger terminal surface does not come closer at any point than (say) ten times the corresponding dimension of the smaller terminal surface, we need not be concerned with the actual shape of the larger terminal surface when we determine the seat of energy, the capacity and the configuration of the field lines emanating from the smaller surface. It will be seen, therefore, that from the flat top of an antenna, lines emanate almost symmetrically both upwards and downwards as though the larger terminal surface were one surrounding the antenna symmetrically on all sides, in spite of the fact that the ground is located entirely at the bottom of the antenna. This is clearly illustrated in Figure 11.

By integrating the lineal energy density of a three-dimensionally radiating field between the radius of the smaller sphere

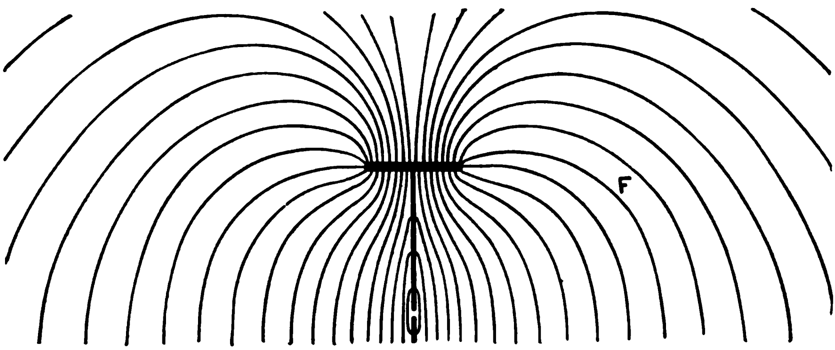


FIGURE 11

and that of the larger sphere, we can find the energy of such a field; whereby the capacity is determined. The lineal energy density follows the law of $\frac{1}{r^2}$, and its integral is proportional to $\frac{1}{r}$; and consequently the capacity of the field varies as r.

We have deduced, therefore, the capacity of a sphere from properties of the field alone, considering the sphere as a terminal surface only.

In deducing similarly the capacity of the wire from properties of the field alone, we have to start with the bi-dimensionally radiating field the lineal energy density of which follows the law $\frac{1}{r}$ as we have seen. The integral of such function is of logarithmic nature, as indeed is the capacity of the wire.

I wish to call your attention to the fact that in a sphere segments of the same projected axial length contribute equally to the capacity of the sphere, as shown in Figure 12.

If a charge were made to enter a sphere and traverse the sphere in the direction of a diameter, the sphere as a conductor would behave like a straight piece of wire of uniform lineal capacity. This fact was first recognized, to my knowledge, by

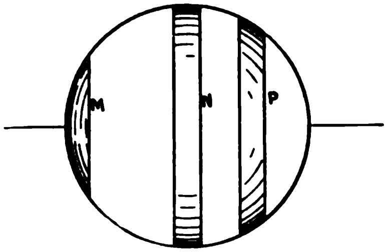


FIGURE 12

Mr. Nikola Tesla, and I expect to come back to the behavior of a sphere as a conductor of radio frequency currents at some later date.

The study of capacities of composite bodies is most instructive and conducive to a clear conception of capacity. Let, as in Figure 13, a number of small spheres of radius be so arranged as to cover completely the surface of the larger sphere, the radius R of which be 100. If each one of the

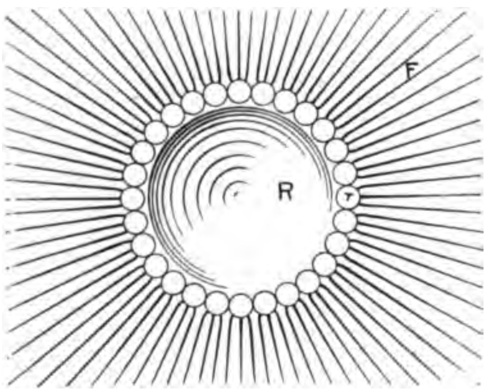


FIGURE 13

31,400 smaller spheres could be counted at its full value of capacity, the capacity of the composite body would be 31,400; as a matter of fact, however, it is not more than radius R of the larger sphere, that is 100. Indeed, the configuration of the electric field F could not have changed materially by the arrangement of the small spheres, and the capacity clearly presents itself as a property of the configuration of the field lying outside of the enveloping surface of the composite structure.

Capacity may play a part in the conduction of electricity thru liquids and gases. Let us assume a series of spheres in lineal arrangement as shown on Figure 14.

As long as the distance between the spheres is great compared to the diameter of the spheres, each sphere will retain its full capacity as given by its radius. By decreasing the distance

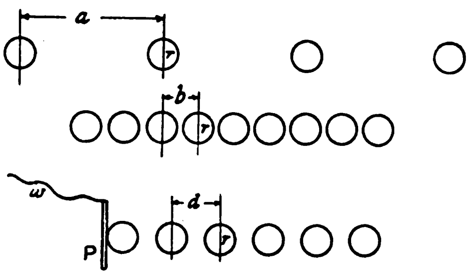


FIGURE 14

between spheres the individual capacities of the spheres decrease, because of the negative capacity coefficients. If such approximation be carried to the point of contact between the spheres, the capacity of each individual sphere would be reduced to approximately $\frac{1}{\varepsilon}$ of the original capacity. If such a row of spheres were conceived as freely movable, so as to enable each sphere to make contact with a plate P, which is kept charged to a certain potential, then the charges carried away by the spheres after contact with the plate would be proportional to the full capacity of each sphere as long as the spheres are far apart, and would be only $\frac{1}{\varepsilon} = \frac{1}{2.718}$ *th* part of such maximum charge when the spheres are in contact. As we assumed the plate P to be maintained at a certain potential by an outside source of electricity, the convection current represented by the departing charges of the spheres would vary approximately in a ratio of 2.71 to 1.

In the passage of electricity thru an electrolyte, the molecular conductivity has been found to be the same for all electrolytes, and varying only with the concentration of the solution; the molecular conductivity being approximately 2.5 times as great in the very dilute solution as in the concentrated solution.

I wish to call your attention to the striking similarity between the ratio of conductivity experimentally determined in electrolytes of small and large concentration and the ratio of conductivity of the row of spheres where the spheres are far apart or close together. I do not pretend at this moment that a

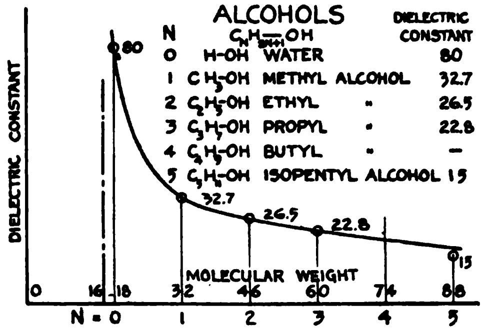


FIGURE 15

plausible modification of the theory of conduction thru electrolytes and gases can be based on such a coincidence; and in fact, assumptions would have to be made. For example, a lineal arrangement of the ions in the direction of the static field impressed on the electrolyte or on the gas must be assumed.

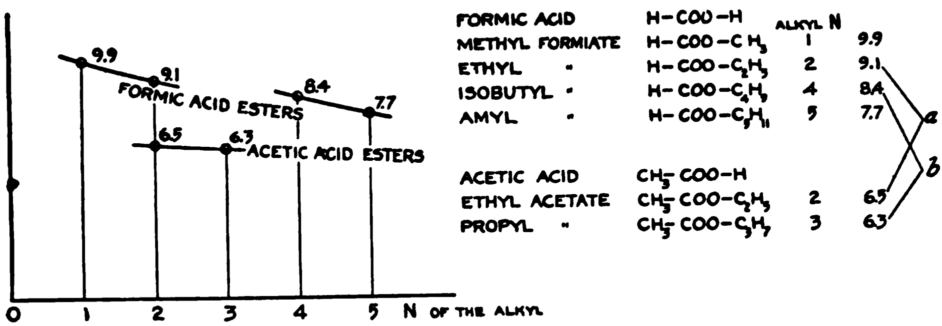


FIGURE 16

But the fact that such ratio in the case of the spheres is deduced from geometrical considerations alone, coupled with the fact that in electrolytes the same ratio follows from purely geometrical considerations, is sufficient to warrant further thought. I do not hesitate to bring this interesting coincidence to your knowledge, with the hope that other physicists may carry on investigations in the same direction. I have said that the molecular conductivity of electrolytes arose from geometrical considerations only, and I think it advisable to call your attention to the foundation of such a statement. While it is true that the conductivity of different electrolytes varies considerably, it has been found that the molecular conductivity is the same for all electrolytes. The similar behavior, of the same number of molecules, independently of the weight of the molecule, therefore reduces the phenomenon to a purely geometric basis.

SUMMARY: Considering that electrostatic energy is actually in the space surrounding a charged body, the latter is called a "terminal surface." It is shown that capacity is predominantly a function of the ***maximum*** lineal dimension of the terminal surface. The volumetric and lineal energy densities in the field are defined and studied in a number of cases. It is proven that the capacity between two terminal surfaces is greatly affected by changing the lineal dimensions of the ***smaller*** terminal surface, but not so for changes of the larger. Certain current errors in connection with "mutual capacity" are considered.

The practical applications to a radio antenna and to aeroplane counterpoises are given.

When a charge traverses a sphere, entering parallel to a diameter, the sphere behaves as a conductor of uniform lineal capacity.

Applications of the theoretical considerations are also given in connection with the conductivity of concentrated and dilute electrolytes.

DISCUSSION

John L. Hogan, Jr.: The computation of antenna capacity has been a problem of great interest to radio engineers for a good many years, and many methods have been proposed. As a rule, the mathematical solution of the problem is not only very complicated, but is likely to lead to results which are so far from accurate that the labor of the computation is not repaid. Measurement methods almost always involve the erection of the antenna itself or of a substantially accurate copy of it, and so are impracticable for many uses.

In design of station apparatus, whether for sending or receiving, it is convenient to be able to determine antenna capacity quickly and easily, and for ordinary purposes an accuracy of approximately 5 per cent is all that is required. In fact, even rougher approximations than this will often satisfy the requirements, since possible variations can usually be compensated for in the selection of instruments. I have not seen published any simple method of computing antenna capacity which can be solved quickly and easily, and yet which will give results within a reasonably close value of the actual measurements; however, there is such a method in practical use.

For a number of years I have been collecting data which interconnects the geometric and electrical constants of a great many practical radio telegraph aerial systems, including those of both ship and shore stations. From these I have been able to secure an approximate relation between the area of the aerial system and its effective capacity. For flap top antennas of the usual form, the capacity is almost directly proportional to the area plus a constant, and amounts to something like 0.00024 microfarads per thousand square feet plus 0.0004. The linear relation, of course, is not sufficiently accurate for close calculation, but may be used in securing a first approximation of capacity of medium sized antennas. For more accurate results the capacity C, and area A, are related by the following expression

$$C = p\,A^q$$

where p and q are constants depending on the type of antenna. An expression of this same form is useful for pre-determination of capacity of umbrella antennas; for this purpose the exponent q varies with the number of wires in the umbrella and the area is measured as the surface of the cone which has the rib wires as its elements. It is not ordinarily necessary to take into account the height of the antenna from the ground, for as a rule this distance is too great to affect the result to any marked extent. Closeness of grounded towers, etc., may have to be compensated for, and if a large fan instead of a single wire is used as a lead-in, its part of the capacity must also be added.

If I had realized that the paper and discussion tonight would approach this matter of the practical pre-determination of antenna capacities, I would have prepared some further quantitative data in that connection. I hope that at some future time I may present to the Institute complete information as to this method of computing capacities.

A NULL METHOD OF MEASURING ENERGY CONSUMPTION IN A COMPLEX CIRCUIT*

By

Alfred S. Kuhn

(Engineer, Fritz Lowenstein Company)

In April, 1914, the problem arose of measuring the power consumed in a chamber in which an electrochemical action took place. The chamber was comparatively complex, having glass and gas parts; and the chemical action would have been seriously affected had terminals been introduced into the apparatus for determining the constants of the different parts. It was also determined that the properties of the device varied considerably, but to an unknown extent, with slight changes of applied voltage and of gas pressure. Furthermore, as part of the energy supplied had to pass thru the glass, the use of direct currents was precluded.

Attempts to build a wattmeter showed that much additional apparatus would be required to carry out the measurements by the wattmeter method. When the problem rose again about a year later, the following arrangement was adopted.

Across the secondary of the power transformer, two circuits were placed in parallel. One circuit contains the reaction chamber in series with a coil. The other circuit consists of a variable capacity (a variable inductance sometimes), a variable resistance, and a coil identical with the coil in the first circuit. To each of the aforesaid coils is coupled a secondary coil, the two secondary coils being identical. The coupling between the primary and the secondary of one circuit is identical with that of the other. The arrangement of circuits is shown in the illustration.

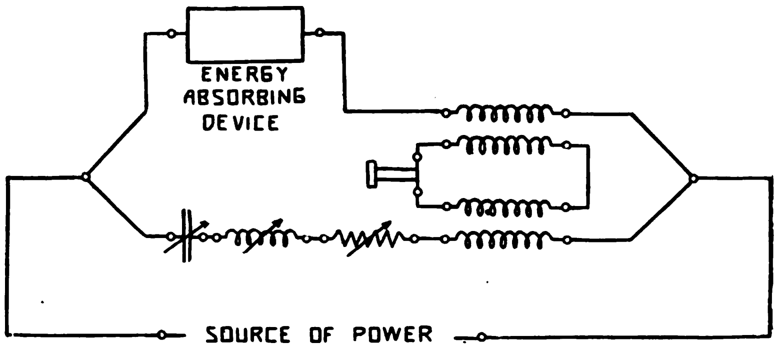
If then the two secondary coils be connected in series with a telephone or other suitable indicator, there will be no indication of energy in the indicator when the currents in the two circuits are equal and in phase. When such is the case, the power consumption in the two circuits is the same. The power in the artificial circuit may easily be measured and equated to the power in the complex circuit.

* Communicated to the Editor, September 28, 1915.

It may be mentioned that it is not necessary to use *both* capacity and inductance in the balancing circuit. In addition to the equivalent resistance, there need be only an equivalent reactance factor:

$$\omega L - \frac{1}{\omega C}.$$

This reactance may be made up of capacity, inductance, or both.



In measuring the power in the balancing circuit, wattmeters of the ordinary types may not be used if the power factors are very low. Thus, in the experiments mentioned power factors as low as 5 per cent arose. As a rule, a high power factor wattmeter may be used in such case provided current potential transformers are available. This device is doubly useful when working at high potentials and low power factors.

SUMMARY: A null method of measuring power absorption at low power factors in complex circuits at any frequency is described.

THE DARIEN RADIO STATION OF THE U. S. NAVY (PANAMA CANAL ZONE)*

By

Lieut. R. S. Crenshaw

(U. S. N.)

Darien Radio Station of the U. S. Navy

In its system of radio communication for the Canal Zone, the Navy has maintained the high standard set by the Canal in general in having thoroly modern equipment. The layout comprises one coastal station at each end of the Canal for ship to shore work, and one high power station for long distance work.

The Darien radio station is located just twenty-five miles

* Received for publication December 18, 1915. This paper has been censored by the Navy censors.

south of Colon on the Panama Railroad. The railroad forms the east boundary of the reservation which contains eighty-seven and one half acres. The southwest boundary is the Canal itself from which runs a channel twenty feet deep and seventy-five wide into the center of the station plot. Those who may have visited the Canal before the water was allowed to rise may remember this site as being adjacent (to the southeastward) to the old town of San Pablo. On the "relocation" of the railroad, the nearest stop was Caimito Cabin at the North end of the dumps; and the site was known variously as the San Pablo or Caimito radio, or simply as *Radio*, in the early stages. Finally the Department assigned the name Darien when, by request, the Governor named the railroad stop here Darien. Until the name was well known, mail frequently went stray to the southernmost province of the Panama Republic, which is known as the Darien section.

Work was started on construction in December, 1913, by the Quartermaster Department of the Panama Canal. The site was far removed from any of the Canal Zone towns and was mostly jungle. A spur from the railroad was put in, laborers' barracks built; and, as it is unsafe to load cement in the open due to the sudden rains, a cement shed was erected close to the spur. With a hoisting engine on the hill, all material was hauled up a narrow gauge road in De Cauville dump cars. This narrow gauge road continued around the station site for the delivery of material for the buildings and towers. The small locomotives, cars and track were relics of the French construction days. Water for the station was pumped to a tank on a hill by a Worthingston pump, which obtained its steam from a boiler of an old Belgian locomotive side-tracked for that purpose. Drinking water was distilled at this pump station. This equipment supplied the station until the arrival of the electric turbine pump. The Gatun Lake water is now used and merely sterilized for drinking and cooking purposes.

The dwellings on the site are the house for the Radio Officer, cottage for the chief electrician in charge, and barracks for the operators equipped to house seventeen men. Servants' quarters are also provided in the barracks building. Rations are commuted at a dollar a day per man, and a mess is run by the operators.

All the buildings are screened, including the porches. There is such a large breeding area for mosquitoes about the site that the cost would prohibit sufficient sanitation work to keep the mosquitoes down entirely, and the means adopted are (1) to keep the screening as tight as possible; (2) each morning a sanitary inspector makes the rounds, and catches the mosquitoes inside the living quarters and office; (3) no containers are allowed to collect water; in which there may be breeding on the station site; (4) all drains are kept clear so no water stands in puddles; (5) around the edge of the water the bank is kept skinned to allow the small fish to eat the mosquito larvae (this means is remarkably effective); (6) the force of five laborers allowed the station is kept at work on the grounds to keep the jungle growth cut down as well as possible. When one case of malaria appeared, the whole station was put on a quinine diet for ten days, in order to prevent an epidemic.

The other buildings of the station, with the exception of the boat-house, are of concrete. The boat-house is of old form lumber left over from the concrete work, and corrugated iron roofing robbed from old, abandoned shacks on the site, one of which was a distillery.

The power-house is sixty feet by thirty feet, and contains the motor generators for the main transmitting set.

The main distributing and controlling switchboards are here, with the auxiliary transformers. This building also houses the machine tools, a small lathe, a drill press, milling machine, and emery grinder, and is fitted with a five-ton overhead travelling crane. All wiring is in conduit in wire trenches.

The operating building contains the arc room (where is located the main transmitting set with its auxiliary electric controlling devices), the receiving room and the office, besides a spare room for an auxiliary sending set if needed later. The arc room and the receiving room both have wire mesh embedded in their walls, floor and ceiling, in order to prevent induction from the transmitting set injuring the receivers. The building is fireproof, which is necessary on account of the action of the continuous oscillations used at such high voltage. The charging current into iron in the vicinity of any live lead heats the iron quickly. Some of the reinforcing had to be taken out of one concrete base because the current jumped to it; and one wall 19 inches (47 centimeters) away from the end of the helix heats so that the hand cannot be kept on it after a twenty-minute run. The reinforcement in this wall is merely metal lathing, but it is directly in the field of the main helix.

The contract for the towers was let to the Penn Bridge Co. which in turn sublet the fabrications to the Toledo Bridge and

Iron Works, and the erection to Mr. J. O. Childers. In all three towers there are about 1000 tons of structural steel. They are 600 feet (183 meters) high each; the feet form a triangle 150 feet (46 meters) on a side, and the tower tapers to a 10-foot (3 meters) triangle at the top. An iron ladder runs up the outside of one leg on each tower, having rest platforms about every 50 feet (15.2 meters).

When first erected there was considerable swaying in the bottom long diagonals, but these and others above were stiffened up by cross bracing, and now they are perfectly rigid. When the antenna was hoisted and adjusted to the sag which would give a pull of about 13000 pounds (5500 kilograms), the top of each tower was pulled over only 4 inches (10 centimeters) during hoisting, and settled back to 2 inches (5 centimeters) when hoisting stopped. All the bend was in the upper 200 feet (60 meters).

As mentioned earlier in this article, it was the first intention to locate the towers on the tops of the hills, but on making the actual location it was found that the thrusts (which come on to the footings at the angle of slightly over sixteen degrees from the vertical) would be nearly too parallel to the face of the hills to give solid backing for the footings. They were finally located so that all footings except one butt into the hills. In order to do this, however, the footings were put on about the 120-foot (36 meters) level instead of the 170-foot (52 meters). The surface of Gatun Lake is normally at the 85-foot (26 meters) level. The block for each footing is 16 feet (5 meters) deep and 20 feet (6 meters) square, heavily reinforced with old railroad rail. Each block filled entirely the hole excavated for it without back filling in order to have it bearing in undisturbed earth. The distance between towers is: one and two, 897 feet (273 meters); two and three, 751 feet (254 meters); one and three, 969 feet (295 meters); the antenna covering about six acres.

The Darien Towers, being farther apart, and not so directly beneath the antenna seem not to affect the capacity as at Arlington. Darien evidently has a greater effective height than Arlington.

The antenna was made at the New York Navy Yard and shipped to Darien, each wire on a separate reel and tagged to mark the points where other wire crossed. The cables are all phosphor-bronze; the outside ones being ¾-inch (1.9 centimeter) diameter, the four strain cables thru the mast, ⅜-inch (0.9 centimeter) diameter, and the sixty-six radiating wires of regular antenna wire. The first 150 feet (48 meters) of the down land of twenty-six wires is a fan and is then grouped by spacing hoops to form the rattail. Each corner is insulated with the Arlington type of Locke insulators with, however, two strings in parallel, as the strain was too near the mechanical breaking limit of the insulators. Lightning has already struck the antenna twice without damage, because of the safety gap feature of these insulators and the towers being grounded. An electric winch on each tower furnishes the power needed for handling the antenna. The feet of each tower are insulated.

The feet rest on 10 porcelain block insulators 11 inches (27.9 centimeters) high, having each three petticoats. Insulators are also placed under the yokes, which secure to the anchor bolts for taking the upward thrusts; and others are placed between the footing and the channel irons projecting from the block to take the side thrusts. However the arc "pulls" better with the towers grounded; so they are operated in that condition by being grounded thru large knife switches to the ground system of the station.

The general ground conditions of this site are excellent, since the Gatun Lake lies on three sides of it, with an arm reaching into the center of the station plot. An artificial ground was laid in addition to cover all the land as follows: 100,000 feet (30,000 meters) of annealed copper wire was laid in the shape of a grid forming rectangles about 50 feet (15 meters) on a side. All intersections were soldered; the ends of all wires, on reaching the water's edge were run 100 feet (30 meters) into the lake, and the main ground plate and the ground plate for each tower are tied into the large grid by busses reaching well out into it. This ground system is buried about 4 inches (10 centimeters) for protection.

The main transmitting set was furnished by the Federal Telegraph Company, and the arc generator is their type of the Poulsen arc.

The signalling is done by short circuiting or opening a compensating helix in series in the antenna circuit. The key for accomplishing this contains 13 pairs of points mounted on a yoke in parallel, so that each pair of points breaks only the voltage due to one turn of the auxiliary helix. This yoke is on the armature of a solenoid, the current controlling which is broken in a strong magnet field; and the key is thus positive and fast in action. The D. C. supply is protected from the radio frequency current by having air core choke coils in both positive and negative lead to the machines. The arc field spools are in the negative lead, and carbon rod protection in the power-house guards further against high voltages getting into the D. C. generators. This set gives Arlington a signal easily readable thru all but the worst electrical storms.*

The arc can be controlled entirely from the operator's seat; the main generator voltage being controlled there, circuit breakers closed or tripped, the arc struck and starting resistance short-circuited. While running, the arc is regulated to take up the wear of the carbon by foot pedals, so that the operator may not have to interrupt his sending. All circuits are electrically interlocked so that on starting, the correct sequence must be followed.

The regular receiving cabinets with tikkers, as used by the Federal Telegraph Co., were provided with the outfit.

For short wave work, an oscillating audion detector is used on one of the Federal Company's cabinets.

Only government work is handled by this station, and at present there is not enough of this to demand continuous watch so that schedules are run. The complement requires eight operators on watch, two at a time, a chief radio electrician in general charge, a hospital steward of the Navy in charge of sanitation and general health work, a yeoman (who is a clerk for the station and for the Radio Officer), and a machinist. Five laborers are employed on the grounds, which were high jungle when the station was built. The Radio Officer of the Canal Zone lives here, having his office in that of the station. With excellent telegraph and telephone service to all parts of the Isthmus, the station, tho isolated, is in close touch with the Canal Government.

When the Navy Department decided upon the kind of set and the general features of this station most of them were in the experimental stage, and the excellent results obtained at this station have been watched with keen interest and gratification.

SUMMARY: The buildings, sanitary arrangements, towers, ground connection, antenna, and some features of the transmitter and receiver of the Darien radio station of the United States Navy are described.

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* (The distance from Arlington to Darien is 1,900 miles (3,000 km.), practically due south.—EDITOR.)

FURTHER DISCUSSION ON "THE TRAINING OF THE RADIO OPERATOR"

By

M. E. Packman

(Instructor in Radio Telegraphy, Dodge Institute of Telegraphy)

(See Proceedings of the Institute of Radio Engineers, Vol. 3, Number 4, Page 311)

M. E. Packman (communicated, November 16, 1915): Mr. Hogan's proposed plan for training students in telegraph receiving thru the use of commercial receiving sets is of course a good one, provided the student is sufficiently far advanced in radio work to be capable of manipulating the instruments. However this is not likely to be the case. It invariably develops that that student who is the most adept in learning to copy telegraphic signals is the slowest in learning to operate radio apparatus. Under these conditions he would make very slow progress. As an extreme case take the new student just out of an office, a store or from a farm and imagine him successfully adjusting a modern receiver in order that he could hear signals. It has been our experience that it is highly desirable to differentiate between the code practice and the theory and operation of radio equipment. Considering this from another point of view, in a school the size of ours where we have from fifty to eighty students in the code work the greater portion of the time, the expense of this number of tuners costing from two hundred and fifty to five hundred dollars each would be entirely out of the question amounting to an outlay of twenty-five or thirty thousand dollars for receiving apparatus alone. Assuming that these tuners would be "modern" for a period of five years it is quite evident that the scheme is not feasible.

As previously mentioned, a plan somewhat similar to Mr. Hogan's arrangement is used for instruction purposes in connection with the study of receiving apparatus. Two methods which we use are shown in Figure 1 and Figure 2. In Figure 1, a receiver is connected to a standard antenna, in the ground lead of which is placed a coil of several turns wound on a rectangular frame. In this last mentioned coil are induced oscillations from one or more wave meters excited with buzzers and operated from an omnigraph or otherwise. By adjusting these wave meters to different wave lengths it is possible to obtain any degree of interference desired and the student can obtain all the practice in making adjustments of his tuner to prevent the

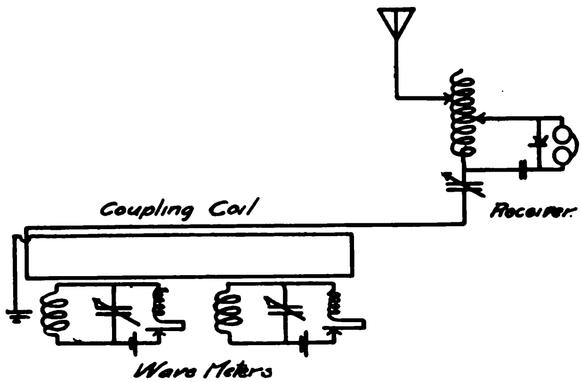

FIGURE 1

interference that is required for any condition of working. This arrangement is probably preferable to the two arrangements shown inasmuch as natural strays or induction are obtained without special arrangements. Actual radio signals can also be received at the same time. In Figure 2 an ordinary

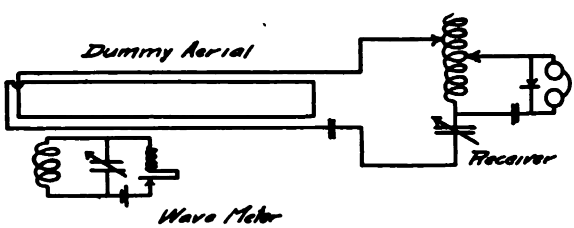

FIGURE 2

dummy aerial system is shown, in which the condenser has the same capacity as an ordinary aerial and the inductance of the loop is of such value that the dummy has any fundamental desired.

One method of producing artificial strays that I have used n in Figure 3. *A* and *B* are leads to a power line. *R* is a current regulating rheostat, P a potentiometer arrangement and I an electrolytic cell or interrupter. With proper adjustment of the interrupter and the rheostat, a very irregular current flows thru the cell. Connections to the receiving code circuits are taken from the potentiometer, condensers being interposed. With this arrangement an almost perfect imitation of strays

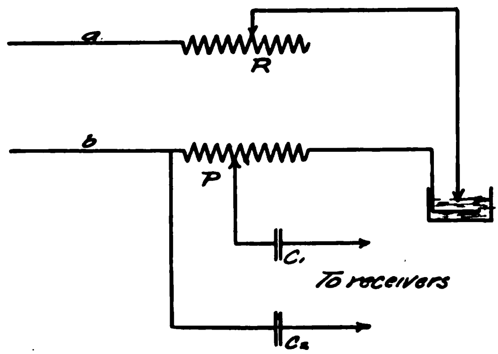

FIGURE 3

of any desired strength can be obtained. This arrangement also permits of the generation of highly damped oscillations which can be induced in any antenna system. This method of producing artificial strays has an advantage over the scheme of using an irregular notched wheel-interrupter in that there is no regular sequence of characters such as would be obtained with a contact moving over the same surface revolution after revolution. Under normal conditions in the region of the Great Lakes there are sufficient strays to render such arrangements somewhat superfluous and hence they are not used in our work to any extent. I might also answer Mr. Bucher's remark regarding the necessity of artificial interference at this time by stating that in this locality conditions are very different from those in the vicinity of New York. Except at night, there is very little interference from commercial stations and I have found it desirable in our case to produce artificial interference as outlined.

I note that both Mr. Sarnoff and Mr. Bucher take exception to my statement that a deplorable condition exists in some branches of commercial radio service. Many improvements have been made during the last few years but there are still some localities in which many changes will eventually be made before the service can be comparable with that in other quarters. Their remarks are apparently based upon a close observation of conditions in and around New York, where, of necessity, the service has been advanced to its highest efficiency. I am more or less familiar with Mr. Bucher's school and the methods that he uses there, and know that the training he is giving his students in the theoretical and practical work is thoro and adequate to the demands of good service. In many respects the work is similar to that in our institution which is, of course, quite natural. There are, however, some differences especially in the code work since only fairly well advanced students are admitted whereas in our school students having no knowledge of telegraphy or radio apparatus as well as advanced students are enrolled under conditions suitable to all. Here in New York the value of the skilled operator is appreciated and in this particular district as well as possibly some others, efforts are made to obtain such men for the service, but this cannot be said of all districts. It has frequently been remarked to me, "We don't want our men to know too much about the apparatus"; and it has frequently been suggested to me by officials of a commercial company that technical training beyond that necessary for a man to slip by the government examination is not necessary and is, in fact, undesirable. When such a condition as this exists it is not likely that well trained men will be earnestly sought for operators' positions on steamships or in land stations. It is often the case that men who have had experience, no matter how poor telegraphers they may be or how limited their knowledge of radio apparatus, are placed in charge of ship or land stations when men more fit in every way are available. This I term a deplorable condition.

Mr. Sarnoff is quite right in his statements that experience is requisite for the highest efficiency, but the experience that a man acquires in a year or two on a ship fitted with a set of antiquated apparatus where he handles possibly one or two messages a week is not comparable with the experience that he obtains in a good school where he has modern apparatus to work with and every facility for mastering the technicalities of the radio service. Under the present methods of examinations for operators' licenses, it is a very simple matter for any telegrapher to secure a license, and provided he is in the employ of a commercial company during three months of the last six months of the life of his certificate, he will be issued a renewal license without examination thus making it necessary for him to be actually engaged in radio service only three months in two years. During the life of his license he has no reason to endeavor to make himself more proficient or even to maintain whatever proficiency he may have had at one time. On the other hand, if operators were selected with care, and were examined from time to time by their employers as is done by other commercial institutions, there would be every inducement for them to make themselves as proficient as possible.

The plan followed by the Marconi Company of sending all new men out for a number of trips as second operator is, of course, a good one and one that would be expected. In many cases this is not possible hence it is the function of the radio telegraph school to meet the conditions of commercial service as nearly as possible. To facilitate this and gain the desired end, there must be the closest harmony between the commercial companies and those schools upon whom they depend for their operators. Every facility for properly training these men should be extended to such institutions, both as regards new apparatus and traffic methods.

THE IMPEDANCES, ANGULAR VELOCITIES AND FREQUENCIES OF OSCILLATING-CURRENT CIRCUITS*

By

A. E. Kennelly

INTRODUCTION

It is the object of this paper to disclose a simple yet powerful proposition, recently discovered by the writer, which applies to transient currents, charges, discharges, or temporary disturbances, in electric circuits. This proposition, which is believed to be new, may be briefly stated in the following terms: The impedance of any closed circuit or group of circuits, to free oscillations, is zero.† The angular velocity, or velocities, of the oscillations are such as will bring about this condition.

Simple Resistanceless Oscillating-Current Circuits

The simplest type of oscillating-current circuit consists of a condenser in series with a reactor of negligible resistance, as indicated in Figure 1. It is known that the angular velocity of the free oscillations in this simple resistanceless circuit is such as

$$p_c = i^2 z_c = \mp i^2 j\sqrt{\frac{l}{c}} \qquad p_l = i^2 z_l = \pm i^2 j\sqrt{\frac{l}{c}} \qquad \text{watts} \angle$$

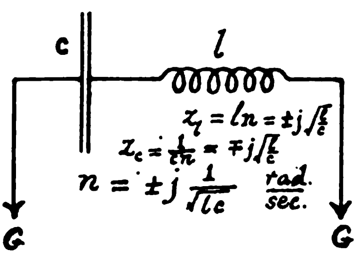

Figure 1—Simple Resistanceless Oscillating-Current Circuit of Capacitance and Inductance

* Presented before the Institute of Radio Engineers, New York, November 3, 1915. Manuscript received 30th August, 1915.

† See, however, a paper by George A. Campbell in "Trans. A. I. E. E.," April, 1911; on "Cisoidal Oscillations," Vol. XXX, part II, page 902, to which paper the attention of the present writer has been called since this paper was set in type.

(The notation used in this paper is tabulated at the end of the paper.—Editor.)

makes the total reactance zero.* Thus, if l is the inductance of the reactor in henrys, c the capacitance of the condenser in farads, ω the angular velocity of free oscillations thru the circuit, in radians per second, and $j = \sqrt{-1}$; then the reactance of the reactor at this velocity will be $jl\omega$ ohms, that of the condenser $\frac{1}{jc\omega}$ ohms, and the total reaction of the circuit will be $jl\omega + \frac{1}{jc\omega}$ ohms. Equating this total reactance to zero, we obtain:

$$l.j\omega + \frac{1}{c.j\omega} = 0 \qquad \dagger \text{ohms} \angle \quad (1)$$

$$(j\omega)^2 lc + 1 = 0 \qquad \text{numeric} \angle \quad (2)$$

$$(j\omega)^2 = -\frac{1}{lc} \qquad \left(\frac{\text{radians}}{\text{second}}\right)^2 \angle \quad (3)$$

$$j\omega = \pm j \frac{1}{\sqrt{lc}} \qquad \frac{\text{radians}}{\text{second}} \angle \quad (4)$$

Selecting the positive sign, the reactance of the reactor is:

$$X_{\cdot l} = j\sqrt{\frac{l}{c}} = jl\omega \qquad \text{ohms} \angle \quad (5)$$

And that of the condenser is:

$$X_c = -j\sqrt{\frac{l}{c}} = \frac{1}{jc\omega} \qquad \text{ohms} \angle \quad (6)$$

Thus, a condenser of 0.01 microfarad ($c = 10^{-8}$), is connected in series with a reactor of 0.01 henry or 10 millihenrys ($l = 10^{-2}$). The angular velocity of the free oscillations of this circuit is $\omega = 10^5$ radians per second. The reactance of the reactor will then be $j1000$ ohms, and that of the condenser $-j\,1000$ ohms, making the total reactance zero.

It is furthermore known that if we count time t in seconds from a suitable epoch, either the instantaneous voltage of the condenser, or the instantaneous current in the reactor, may be represented by the instantaneous projection Op, of a vector OP, Figure 2, on a straight line of reference $X'OX$, the vector revolving with the angular velocity ω radians per second, and therefore describing in time t, a circular angle $XOP = \varepsilon^{j\omega t}$ radians; where ε is the Napierian base 2.71828. . . . Knowing the angular velocity ω, we can thus predict the electrical condition of the system at any assigned subsequent instant.

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*Bibliography (6) page 375.

† The angle sign $\angle$ attached to the unit of an equation indicates a complex quantity or "plane vector." By this means the use of special vector symbols in the equations is dispensed with. They are to be interpreted vectorially, or treated as complex quantities.

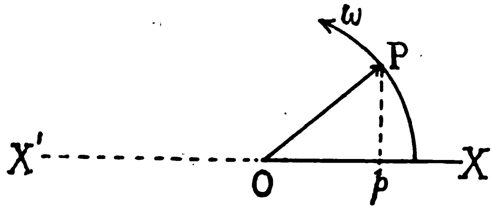

FIGURE 2—Vector Revolving with Angular Velocity ω. Its Projection Op indicates either the instantaneous voltage or the instantaneous current in System of Figure 1, according to the position of the Reference Line $X'OX$

The coefficient $j\omega$ of t, in the angular expression of the characteristic radius vector $\epsilon^{\pm j\omega t}$, is thus the characteristic angular velocity for the oscillations of the system. The imaginary quantity $j\omega$ indicates oscillation, and is significant of angular velocity in a circle. The double sign indicates that either direction of rotation is possible, and that their sum* is equivalent to a sinusoidal quantity. But suppose that the coefficient of t in the general case is denoted by n; so that n is a generalized angular velocity, which may be either real, imaginary or complex; and so that e^{nt} is the characteristic angular exponential of the system at time t seconds, which determines the voltage or current then existing. Moreover, let us suppose that the impedance offered by an inductance l henrys to an electrical discharge of angular velocity n, is ln ohms, in general a complex quantity; while the impedance offered by a capacitance c farads to the same is $\frac{1}{cn}$ ohms. Then, for the case already considered of a simple resistanceless circuit containing a condenser and reactor in series, if the total impedance is to be zero, we have:

$$ln + \frac{1}{cn} = 0 \qquad \text{ohms} \angle \quad (7)$$

or

$$n^2 . lc = -1 \qquad \text{numeric} \angle \quad (8)$$

and

$$n = \pm\sqrt{\frac{-1}{lc}} = \pm j \frac{1}{\sqrt{lc}} \qquad \frac{\text{radians}}{\text{second}} \angle \quad (9)$$

But this is precisely the coefficient of t which we have found to exist in the simple resistanceless oscillating-current circuit.

According to the above assumptions, therefore, which we shall proceed to justify, an inductance of l henrys offers an impedance of ln ohms, to any generalized alternating current of generalized angular velocity n radians per second, of the type

* The sum of two oppositely directed rotations having the same frequency is well known to be a sinusoidally varying quantity in a straight line.

$a + j\,\omega$, where a is a real quantity, which may be regarded as a hyperbolic angular velocity, or uniform angular velocity in a hyperbola, expressible in hyperbolic radians per second;* while $j\,\omega$ is an "imaginary" angular velocity, which may be regarded as a circular angular velocity, or uniform angular velocity in a circle, expressible in circular radians per second. Similarly, the above assumptions lead to the conclusion that a capacitance of c farads, offers an impedance of $\frac{1}{c\,n}$ ohms to any generalized angular velocity of n radians per second. A pure resistance of r ohms, with negligible inductance or capacity, continues to offer an impedance of r ohms to oscillations of any angular velocity.

It will be noticed that in the case of any simple and sustained alternating current of angular velocity $j\,\omega$ circular radians per second, the assumption above stated reduces to the well known proposition that the impedance of an inductance l henrys is $j\,l\,\omega$ ohms; while that of a capacitance c farads is $\frac{1}{j\,c\,\omega}$ ohms. These impedances, whose sum is, in general, finite, then obey all the laws of direct-current resistances, following the rules of complex quantities. This proposition concerning sustained oscillations was discovered by the writer in 1893, and was first published by him in that year,† forming the basis of our ordinary complex algebra of the alternating-current circuit, in general use at the present day.

The new proposition, here presented, may be looked upon as an extension of the writer's earlier proposition, from sustained alternating currents or oscillations, to unsustained oscillations. From an algebraic standpoint, the value of n is extended from the pure imaginary quantity $j\,\omega$, to the complex quantity $a + j\,\omega$. Altho in engineering practice, sustained oscillations, or simple alternating currents, form the rule, and unsustained oscillations, or transients, form the exception; yet, from a physical point

* (It can be readily shown that, just as

$$\varepsilon^{j\omega t} = \cos \omega t + j \sin \omega t$$

so also

$$\varepsilon^{a t} = \cosh a t + \sinh a t.$$

There is therefore an analogy between the hyperbolic cosine (*cosh*) and the hyperbolic sine (*sinh*) and their corresponding trigonometric functions. Tables and charts permitting the ready use of the hyperbolic functions in engineering calculations have been published by Professor Kennelly, "Tables of Complex Hyperbolic and Circular Functions" and "Chart Atlas of Complex Hyperbolic and Circular Functions," Harvard University Press, Cambridge, Mass., 1914. —Editor.)

† Bibliography (2).

of view, the unsustained oscillations of complex angular velocity may be regarded as the general case, and the proposition of 1893, for sustained oscillations, reduces to a mere particular instance of the new proposition.

UNSUSTAINED OSCILLATIONS OF REAL ANGULAR VELOCITY

Condenser in Simple Circuit with Non-Inductive Resistance

Let a condenser of capacitance c farads, be connected in a simple circuit with resistance r ohms and negligible inductance as indicated in Figure 3. Let n be the angular velocity of the

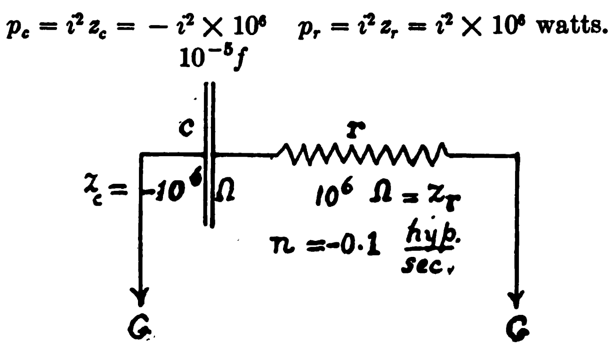

Figure 3—Simple Circuit of Capacitance and Resistance

discharge, as yet undetermined. Then, by assumption, $\frac{1}{cn}$ will be the impedance of the condenser, to this angular velocity, in ohms. The remaining impedance in the discharging circuit will be the resistance r ohms. The total impedance in the circuit will then be $\frac{1}{cn} + r$ ohms. But, by assumption, this total impedance of the circuit during free discharge must be zero: or

$$\frac{1}{cn} + r = 0 \qquad \text{ohms} \quad (10)$$

whence

$$n = -\frac{1}{cr} \qquad \text{hyps. per second} \quad (11)$$

That is, the angular velocity n of discharge is a negative real quantity, which may be regarded, therefore, as a negative angular velocity in a hyperbola, expressible in hyperbolic radians per second or, by abbreviation, in hyps. per sec. The angle described in t seconds of discharge will then be nt or $\frac{-t}{cr}$ hyps., which is the well known exponent of the discharge factor of the system, such that if U is the initial potential difference at condenser terminals, in volts, just before closing the circuit, the potential difference u remaining after t seconds, is

$$u = U\varepsilon^{nt} = U\varepsilon^{-\frac{t}{cr}} \qquad \text{volts} \quad (12)$$

As an example, we may consider a condenser of ten microfarads ($c = 10^{-5}$), charged to a potential difference of 500 volts ($U = 500$), and allowed to discharge thru a non-inductive resistance of one megohm ($r = 10^6$). Then $n = -0.1$ hyp. per second. The hyperbolic angle described during 5 seconds would be -0.5 hyp. and the voltage across the condenser, remaining at that time, would be $500\varepsilon^{-0.5} = 303.3$ volts, the instantaneous current 0.3033 milliampere, and the instantaneous dissipated power 0.092 watt. The impedance offered by the condenser during discharge is -10^6 ohms, or one megohm negative. We may regard a negative resistance $(-r)$ in a discharge element as involving a dissipative *absorption* of power *into* the circuit of $-i^2r$ watts, under an instantaneous current strength of i amperes. This is just equal to the dissipative *liberation* of power *out of* the circuit of $+i^2r$ watts, in heat or in electromagnetic radiation. In any discharging oscillation system, the instantaneous sum of the negative or absorbed powers, and the positive or liberated powers is zero; or $\sum i_n^2 z_n = 0$ watts; where i_n is the instantaneous current in the oscillation impedance z_n of branch n. In general, the instantaneous power $i_n^2 z_n$ is a complex quantity. The real component is dissipative power. The imaginary component is reactive or non-dissipative power; i. e., the power of storing energy. In the case considered, the instantaneous power absorbed into the circuit from the dielectric of the condenser is -0.092 watt, and the instantaneous thermally liberated power in the resistance r is $+0.092$ watt.

The same conditions apply to the sustained oscillations of alternating-current circuits; but with $a = 0$, negative real components of impedance and power do not present themselves.

Reactor in Simple Circuit with Non-Condensive Resistance

If a reactor of inductance l henrys, (Figure 4) is connected in a simple discharge circuit of total resistance r ohms, with negligible side-capacitance, then, according to assumption, the impedance of the reactor to any generalized angular velocity n, will be ln ohms, and the impedance of the remainder of the circuit will be r ohms; so that the total discharge impedance

$$p_l = i^2 z_l = -i^2 \times 10^2 \qquad p_r = i^2 z_r = i^2 \times 10^2 \text{ watts.}$$

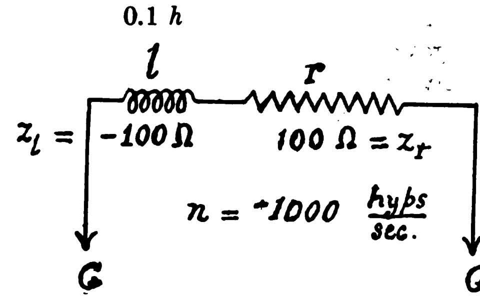

FIGURE 4—Simple Circuit of Inductance and Resistance

will be $ln + r$ ohms. If now the value of n so adjusts itself in free discharge that this sum total reduces to zero,

$$ln + r = 0 \qquad \text{ohms } \angle \quad (12a)$$

whence

$$n = -\frac{r}{l} \qquad \frac{\text{hyps.}}{\text{sec.}} \quad (13)$$

The angle described in t seconds of discharge will be nt or $-\frac{rt}{l}$ hyps. which is the well known exponent of the discharge factor $\varepsilon^{-\frac{rt}{l}}$ of the system, such that if I is the initial current in the reactor just before discharge, the current remaining after t seconds is

$$i = I\varepsilon^{nt} = I\varepsilon^{-\frac{rt}{l}} \qquad \text{amperes} \quad (14)$$

Thus, if a reactor of inductance 100 millihenrys ($l = 0.1$) discharges thru a total resistance of 100 ohms ($r = 100$), with an initial current of 2 amperes, the angular velocity of discharge will be $n = -1000$ hyps. per second. The hyperbolic angle described in 0.5 millisecond will be -0.5 hyp. and the current flowing at that instant will be $2\varepsilon^{-0.5} = 1.213$ amperes. The discharge impedances of the reactor, assumed resistanceless, will be $ln = -100$ ohms. The instantaneous rate of dissipative energy absorption from the reactor's magnetic field into the circuit is $-100i^2 = -147.2$ watts, and the corresponding rate of dissipative energy liberation from the circuit in the resistor r is $+100i^2 = +147.2$ watts. This power is affected with a damping factor $\varepsilon^{-\frac{2rt}{l}}$. There is no reactive power component and therefore no oscillatory storage of energy.

We have seen that our proposition leads to the deduction that the discharges of either condensers or reactors, thru simple resistances, involve real negative values of n; or what may be represented as uniform angular velocities in a hyperbola. We may consider briefly the geometry of this representation.

Let the rectangular coördinate axes OX, OY, Figure 5, be

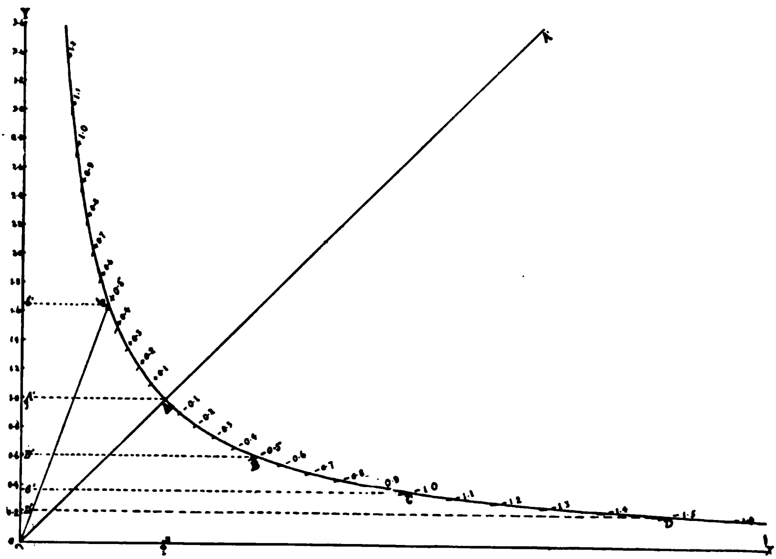

Figure 5—Rectangular Hyperbola with Successive Equal Increments of Hyperbolic Angle and Their Exponential Projections

the asymptotes of the rectangular hyperbola $c\ b\ A\ B\ C\ D$. Let this hyperbola have the axis OAA' and pass through the point A, whose coordinates are $x = 1, y = 1$. Then let a radius vector Ob, starting from the position OA, move with center O over the curve in the positive direction, so as to include an assigned hyperbolic angle θ, defined by the area of the sector OAb, these angles being marked off in Figure 5 along the curve. The ordinate Ob' of the extremity of the vector Ob, is known to be ε^{θ} units in length. In the case presented, $\theta = 0.5$ hyp., and $O\,b' = \varepsilon^{0.5} = 1.649$. In this sense, the value of ε^{θ} may be said to define the hyperbolic angle θ. Similarly, if starting from the initial position OA, the radius vector moves over the curve in a negative or clockwise direction, so as to occupy successively the positions $OB, OC, O\,D$, which include respectively $-\,0.5$, $-\,1.0$, and $-\,1.5$

hyps., the corresponding ordinates OB', OC', OD', measure $\varepsilon^{-0.5}$, $\varepsilon^{-1.0}$, and $\varepsilon^{-1.5}$ units. An exponential $\varepsilon^{-\theta}$, in this sense defines projectively a negative hyperbolic angle $-\theta$. As, therefore, a radius vector OB moves over the hyperbola with uniform hyperbolic velocity $+n$ hyps. per second, describing equal areas in equal times, the ordinate of the moving extremity of the radius vector follows the exponential ε^{+nt} units of length. Conversely, the expression ε^{-nt} may be interpreted geometrically as defining a radius vector which moves over a hyperbola with a uniform negative hyperbolic angular velocity $-n$ hyps. per second. The quantity $-n$ is often called a "damping constant" and the expression ε^{-nt} a damping factor, or damping coefficient; but the conception of n as an angular velocity seems better adapted for our present purposes.

Circular Angles and their Exponential Symbols

It is well known that the exponential quantity $\varepsilon^{j\beta}$ defines the position B (Figure 6) of a point situated in a plane, and on a

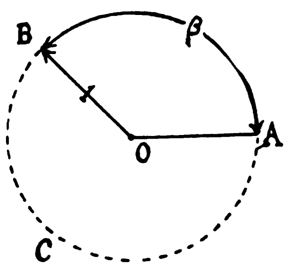

Figure 6—Representation of the Exponential $\varepsilon^{j\beta}$

circle $A\ B\ C$ of unit radius, which is at a distance of β units of length from A, the initial point on the circle. It therefore likewise defines the number of circular radians β, which the radius vector OB has described in its rotation or rotations, about the center O, from the initial position $O\ A$. The exponential $\varepsilon^{j\beta}$ thus defines the total circular radians described in passing from OA to OB. Consequently the exponential ε^{jnt} may be interpreted geometrically as defining a radius vector which moves over a circle of unit radius with a uniform circular angular velocity n circular radians per second.

Complex Angular Velocities and Their Exponential Symbols

Since $\varepsilon^{(\pm a \pm j\omega)t} = \varepsilon^{\pm at} \times \varepsilon^{\pm j\omega t}$, it follows that the exponential $\varepsilon^{(\pm a \pm j\omega)t}$ may be interpreted as representing the product of two angles, each increasing uniformly with time, one a plus or minus hyperbolic angle, and the other a plus or minus circular angle. Or, we may interpret it as a radius vector rotating in a plane with uniform circular angular velocity $\pm\omega$ circular radians per second, the radius vector at the same time changing in length, in projective accordance with uniform hyperbolic angular velocity $\pm a$ hyperbolic radians per second. The path of such a moving point is known to be an equiangular spiral. Such angular velocity may be described as complex, being defined by the complex quantity $(\pm a \pm j\omega)$ radians per second.

Proof of Oscillation Impedance Theorems

The following considerations will probably suffice to establish the three propositions: (1) that the oscillation impedance of an inductance l henrys is $l\,n$ ohms; (2) that the oscillation impedance of a capacitance c farads is $1/(cn)$ ohms; (3) that the total oscillation impedance of a circuit or path to free oscillation is zero.

(1) Let pure inductance of l henrys, assumed devoid of either resistance or capacitance, carry an instantaneous oscillating current of i amperes, which obeys the law

$$i = I\,\varepsilon^{nt} \qquad \text{amperes} \angle \quad (15)$$

where I is an initial current at time $t = 0$ and $n = -(a \pm j\omega)$, a generalized angular velocity. Then the instantaneous back emf. of self induction, opposed to the current, is:

$$-e_l = -l\frac{di}{dt} = -I\,n\,\varepsilon^{nt} = -l\,n\,i \qquad \text{volts} \angle \quad (16)$$

The instantaneous driving emf. which is therefore necessary to overcome this back emf. is:

$$e_l = l\,n\,i \qquad \text{volts} \angle \quad (17)$$

The instantaneous apparent resistance offered by the inductance at its terminals is then:

$$z_l = \frac{e_l}{i} = l\,n \qquad \text{ohms} \angle \quad (18)$$

(2) Let a pure capacitance of c farads carry the same instantaneous oscillating current i, above considered under (1). Then the instantaneous back emf. of condensance, opposing the current, is:

$$-e_c = -\frac{1}{c}\int i\,dt = -\frac{1}{cn}I\,\varepsilon^{nt} = -\frac{i}{cn} \qquad \text{volts} \angle \quad (19)$$

The instantaneous driving emf. to overcome this is:

$$e_c = \frac{i}{c\,n} \qquad \text{volts} \angle \quad (20)$$

The instantaneous apparent resistance of the capacitance is then:

$$z_c = \frac{e_c}{i} = \frac{1}{c\,n} \qquad \text{ohms} \angle \quad (21)$$

(3) At any instant the total driving emf. of disturbed energy must be equal to the total back emf. in the circuit, including $r\,i$ drop in the circuit. Otherwise the unsatisfied driving emf. would create a greater current than actually flows at the instant considered. That is, at each and every instant,

$$i\,.\,\Sigma z = 0 \qquad \text{volts} \angle \quad (22)$$

where $i\,.\,\Sigma z$ signifies the vector sum of all the oscillation impedances drops in the circuit, including simple ohmic drops of the type $i\,r$ volts.

Dividing (22) by i, we obtain:

$$\Sigma z = 0 \qquad \text{volts} \angle \quad (23)$$

Or the sum of all the oscillation impedances in the path of the current i is zero at every instant.

In any closed loop of an oscillating-current system, the total instantaneous emf., including $i\,r$ drops, must be zero; or, if z_n is the oscillation impedance of conductor n carrying instantaneous current i_n and forming part of a closed loop, $\Sigma i_n z_r = 0$ volts. In the case of sustained oscillations, or impressed alternating currents ($a = 0$), this reduces to the extended complex or two-dimensional form of Kirchoff's loop-voltage law* first published by Steinmetz in 1893.

Case of a Simple Circuit of Capacitance, Inductance and Resistance

We may now proceed to consider more complicated cases of unsustained oscillations. In Figure 7, the reactor of inductance l henrys is in series with a total resistance of r ohms (including that within the reactor) and the condenser of capacitance c farads. If n is the generalized angular velocity of unsustained oscillation, the condenser will have an impedance of $1/(c\,n)$ ohms, and the reactor an impedance of $l\,n$ ohms. Equating the total impedance to zero, we have:

$$\frac{1}{c\,n} + r + l\,n = 0 \qquad \text{ohms} \angle \quad (24)$$

* Bibliography (3).

or $$n^2.cl + ncr + 1 = 0 \qquad \text{numeric } \angle \qquad (25)$$

whence $$n = -\frac{r}{2l} \pm \sqrt{\left(\frac{r}{2l}\right)^2 - \frac{1}{cl}} \qquad \frac{\text{hyp. radians}}{\text{second}} \qquad (26)$$

or $$n = -\frac{r}{2l} \pm j\sqrt{\frac{1}{cl} - \left(\frac{r}{2l}\right)^2} \qquad \frac{\text{radians}}{\text{second}} \angle \qquad (27)$$

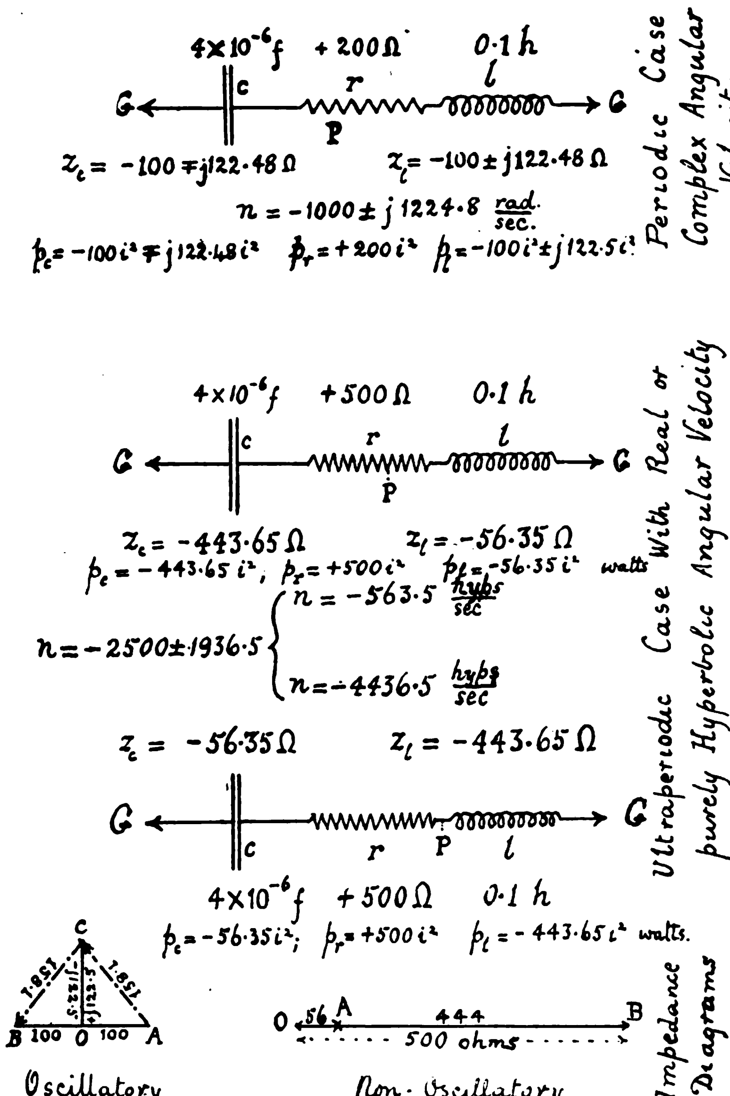

FIGURE 7—Condenser and Resistive Reactor. Periodic and Ultraperiodic Cases

according as $\left(\frac{r}{2l}\right)^2$ is greater or less than $\frac{1}{cl}$. In the former case, the discharge is ultraperiodic, and the angular velocity wholly hyperbolic.† In the latter case, the discharge is periodic, and the angular velocity is partly hyperbolic and partly circular. In the intermediate condition, with $\frac{r}{2} = \sqrt{\frac{l}{c}}$, the discharge is aperiodic. If we denote $r/(2l)$ by a, and $\sqrt{\frac{1}{cl} - \left(\frac{r}{2l}\right)^2}$ by ω, then, in the periodic case,

$$n = -a \pm j\,\omega \qquad \frac{\text{radians}}{\text{second}} \angle \tag{28}$$

As before, the dual sign of the circular angular velocity $j\,\omega$ indicates that either direction of rotation may be adopted, and the sum of two opposite rotations gives rise to a sinusoidal quantity.

The quantity of electricity, q coulombs, in the condenser at any instant t seconds from the beginning of the discharge is known to be:

$$q = A\,\varepsilon^{(-a+j\omega)t} + B\,\varepsilon^{(-a-j\omega)t} \qquad \text{coulombs} \tag{29}$$

where A and B are arbitrary constants depending upon the initial conditions. This general and well known result was first published in 1853* by Lord Kelvin, from an analysis based on energy relations. It is evident that the impedance equation (24) leads directly to the angular velocity of discharge. The oscillation frequency is $f = \frac{\omega}{2\pi}$ cycles per second, and the damping constant $-a$.

As an example, let $c = 4 \times 10^{-6}$ farad, $l = 0.1$ henry, and $r = 200$ ohms. Then $n = -1000 \pm j\,1224.75$ radians per second, and the frequency of oscillation is $f = \frac{1224.75}{2\pi} = 194.92$ cycles per second, accompanied by a damping constant of 1000; or a damping factor of ε^{-1000t}. The discharge impedance, or oscillation impedance of the reactor, assumed resistanceless, is $-100 \pm j\,122.48$ ohms, and that of the condenser $1/(c\,n) = -100 \mp j\,122.48$ ohms. If the resistance $r = 0$; or were entirely removed from the circuit, the angular velocity would, by (9), become sustained

† Bibliography (5).
* Bibliography (1).

at $n = \pm j\,1581.1$ radians per second, the impedance of the reactor $\pm j\,158.11$ ohms, and that of the condenser $\mp j\,158.11$ ohms. The frequency would then be sustained at $1581.1/(2\pi)$ or 251.65 cycles per second. If this frequency were sustained by an independent alternator or impressing source, only the upper signs would be applicable under international notation; i. e., the reactor's impedance would be $+j\,158.11$ and the condenser's impedance $-j\,158.11$ ohms. The dual signs presenting themselves in the solutions of free oscillations may be attributed to the absence of an independent source of impressed current. Either the condenser or the reactor may become the source of discharges, and either direction of current the direction of reference. With this understanding, the dual signs of imaginary (circular) angular velocities need give rise to no ambiguity or uncertainty.

If the resistance r of the circuit were increased to say 500 ohms, then (26) would apply, and $n = -2500 \pm 1936.5$ radians per second $= -563.5$ or -4436.5 hyps. per second. There are thus two hyperbolic angular velocities present, and two damping factors, $\varepsilon^{-563.5t}$ and $\varepsilon^{-4436.5t}$. The impedance of the reactor to the lower angular velocity is shown in Figure 7, to be -56.35 ohms, and that of the condenser -443.65. At the higher velocity, these values interchange, the reactor taking -443.65, and the condenser -56.35 ohms. In the complete analysis of this ultraperiodic case, it is optional either to assign a certain share of the discharge to each independent hyperbolic angular velocity: or to combine them into the single hyperbolic angular velocity 1936.5 hyps. per second, associated with the damping factor ε^{-2500t}. The results in either case are the same.*

Combination of Condensers and Reactors in Series Circuit

If a circuit contains a plurality of condensers in simple series with a plurality of reactors and resistances, the angular velocity of disturbance in the circuit is readily found.

Let C_1, C_2, C_3 . . . be the respective capacitances in the circuit (farads).

Let l_1, l_2, l_3 . . . be the respective inductances in the circuit (henrys).

Let r_1, r_2, r_3 . . . be the respective resistances in the circuit (ohms).

Let the capacitance-reciprocals, or elastances, of the condensers be found, $s_1 = 1/c_1$, $s_2 = 1/c_2$, $s_3 = 1/c_3$ · · · These may be ex-

* See Bibliography (6), Page 411, for a more detailed analysis of this case.

pressed in darafs. Then the total elastance of the circuit is $S = s_1 + s_2 + s_3 + \cdots$ darafs. The total inductance is $L = l_1 + l_2 + l_3 + \cdots$ henrys, and the total resistance $R = r_1 + r_2 + r_3 + \cdots$ ohms. Then the oscillation impedance of the total elastance is $\frac{S}{n}$ ohms, that of the total inductance Ln ohms, and of the total resistance R ohms.

Consequently
$$\frac{S}{n} + Ln + R = 0 \qquad \text{ohms} \angle \quad (30)$$

whence, as in (27),
$$n = -\frac{R}{2L} \pm j\sqrt{\frac{S}{L} - \left(\frac{R}{2L}\right)^2} \qquad \frac{\text{radians}}{\text{second}} \angle \quad (31)$$

assuming that the resistance R is less than $2\sqrt{LS}$ ohms; i. e., that the disturbance is oscillatory; otherwise the roots of (30) are real, as in (26).

As an example let $S = 25$ darafs; $L = 0.1$ henry; $R = 200$ ohms; then $n = -1000 \pm j\,1224.75$ radians per second.

Oscillation Angular Velocity of Resistanceless Discharging Elements in Parallel

The simplest case of discharging elements in parallel, producing oscillations, is perhaps that indicated in Figure 8. A discharging element may be defined as an element capable of containing electromagnetic energy, and therefore capable of having the amount of its energy content disturbed. A discharging element may therefore be a reactor of inductance l henrys, which may contain magnetic energy of $li^2/2$ joules, when traversed by a current of i amperes. It may also be a condenser, of capacitance c farads, which may contain electric energy of $ce^2/2$ joules, when charged to a potential difference of e volts. The oscillations here considered may be those accompanying either an increase, or a decrease of energy in any element; i. e. accompanying either charge or discharge; but discharge is the easier phenomenon to analyze; because in charge, a final steady state has ordinarily to be superposed upon that transient state of disturbance which is the immediate subject of discussion. We may, therefore, confine our discussion to cases of discharge, with the understanding that the results apply also, with reversal of currents and powers, to cases of charge, if the subsequent steady state is independently superposed.

In Figure 8, let a number of condensers of capacitances $c_1, c_2, c_3 \ldots$ farads, respectively, be connected in parallel to common bus-bars BB', bb'. Let any number of reactors be also

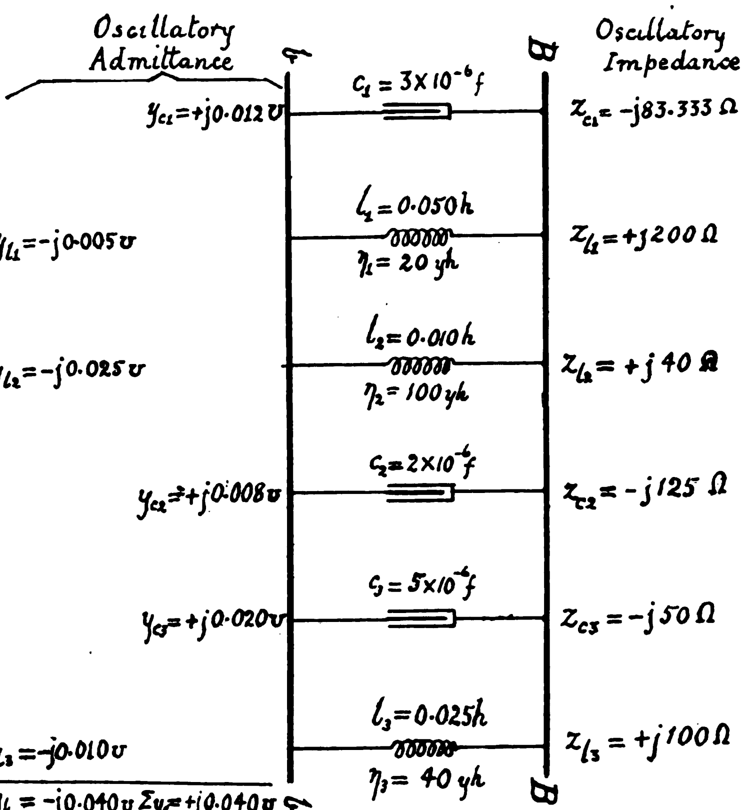

Figure 8—Parallel Connection of a Number of Discharge Elements, with Negligible Resistance

connected in parallel to the same bars, and let the resistance of each and all the elements of the circuit be negligibly small. When reactors of inductances l_1, l_2, l_3 . . . henrys are connected in parallel, it is convenient to use the reciprocals of these values $\gamma_1 = 1/l_1$, $\gamma_2 = 1/l_2$, $\gamma_3 = 1/l_3$, . . . for arithmetical purposes. These reciprocal inductances may be called *ductances*, for want of a better term. A ductance may be expressed in (henrys)$^{-1}$; or, as Karapetoff suggested, say, in yrnehs. An inductance of 0.1 henry is therefore a ductance of 10 yrnehs. The total ductance of a number of ductances in parallel is then their numerical sum, just as the total capacitance of a number of capacitances in parallel is their numerical sum.

It is evident that the system of Figure 8, assumed resistanceless, is equivalent to a single condenser of capacitance $C = c_1 +$

$c_2 + c_3 +$ farads, in simple series with a single ductance $E = \gamma_1 + \gamma_2 + \gamma_3 +$ yrnehs. The case of Figure 8 thus reduces to that of Figure 1, with C in place of c farads, and $1/E$ in place of l henrys. The discharge impedance of the combined condenser is then $1/(Cn)$ ohms, and that of the combined ductance n/E ohms. Consequently

$$\frac{1}{Cn} + \frac{n}{E} = 0 \qquad \text{ohms} \angle \quad (32)$$

and

$$n^2 C + E = 0 \qquad \text{yrnehs} \quad (33)$$

or

$$n = \pm j\sqrt{\frac{E}{C}} \qquad \text{radians/sec.} \angle \quad (34)$$

which result is in agreement with (9), and is almost self-evident in view of (9). Our proposition states, however, that the discharge impedance of the system to any element must be zero. Consider the element c_1 as having its energy suddenly disturbed, and as discharging thru the rest of the system. The discharge impedance of c_1 is $1/(c_1 n)$ ohms. That of the remaining condensers is $1/(C - c_1)n$ ohms, and that of the ductances n/E ohms, as already considered. The remaining condensers are in parallel with the ductances, and their joint impedance must be taken in relation to c_1; so that

$$\frac{1}{c_1 n} + \frac{\frac{n}{E} \times \frac{1}{n(C-c_1)}}{\frac{n}{E} + \frac{1}{n(C-c_1)}} = 0 \qquad \text{ohms} \angle \quad (35)$$

from which

$$n^2 = -\frac{E}{C} \qquad \left(\frac{\text{radians}}{\text{sec.}}\right)^2 \angle \quad (36)$$

or

$$n = \pm j\sqrt{\frac{E}{C}} \qquad \frac{\text{radians}}{\text{sec.}} \angle \quad (37)$$

This is the same result as was reached in (34). It means that the angular velocity of discharge oscillations is the same in each individual condenser as in the system as a whole; so that there is one and only one oscillation frequency $f = n/(j\,2\pi)$ cycles per second. Moreover, this frequency is such that the impedance of the system is zero, taking each condenser in turn as the main path of discharge.

Similarly, taking any one ductance, say γ_1, as the main path of discharge, this element only having its energy suddenly disturbed; then its impedance is n/γ_1 ohms, and that of the remaining ductances, in parallel, $n/(E - \gamma_1)$ ohms.

Consequently

$$\frac{n}{\gamma_1}+\frac{\dfrac{n}{E-\gamma_1}\times\dfrac{1}{nC}}{\dfrac{n}{E-\gamma_1}+\dfrac{1}{nC}}=0 \qquad \text{ohms} \angle \quad (38)$$

from which

$$n^2 = -\frac{E}{C} \qquad \left(\frac{\text{radians}}{\text{second}}\right)^2 \angle \quad (39)$$

or

$$n = \pm j\sqrt{\frac{E}{C}} \qquad \frac{\text{radians}}{\text{second}} \angle \quad (40)$$

again the same result as in (34) and (37). There is thus one and the same oscillation frequency in all branches of the system. If resistances are injected into the various branches, this simple relation is destroyed, and altho the same principles and method of procedure apply, the result is usually an equation of the nth degree, for n discharging elements, giving on solution, n roots, every one root corresponding to the angular velocity of each discharge element, considered in turn as the main path. The number of distinct oscillation frequencies is, however, usually distinctly less than n; because each pair of conjugate complex roots gives rise to but a single oscillatory frequency.

If we take as an example the following values:— $c_1=3\times 10^{-6}$, $c_2=2\times 10^{-6}$, $c_3=5\times 10^{-6}$, $C=10^{-5}$, $\gamma_1=20$, $\gamma_2=100$, $\gamma_3=40$, $E=160$; then $n=\pm j\,4000$ circular radians per second, and the oscillation frequency $f=4000/(2\pi)=636.6$ cycles per second. The impedances and admittances of the various elements at this frequency are indicated in Figure 8, just as if the frequency were independently sustained in an alternating-current circuit. It may be observed that the total admittance of the branches of the system is zero, and this we shall find to be a general law, whether resistances are present in the various branches, or not.

Condenser, Reactor, and Resistance, in Star Connection

We may next consider the case represented in Figure 9, of a condenser, reactor and non-inductive resistance in star connection; or, what is of course the same, connected in parallel between bus-bars. Here we have two discharge elements and an inert or energyless resistance leak, all in parallel. It is optional to consider either discharge element as the main path and the two others, in joint connection, closed on it. Let c be the capacitance in farads, l the inductance in henrys containing a resistance of r ohms, and g the conductance of the leak in

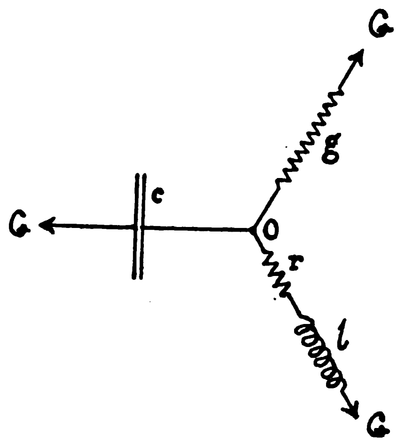

FIGURE 9—Fleming's Case of a Condenser c Shunted by a Leak g and Discharging thru a Reactor l

mhos. Then taking the condenser as the main path, we obtain:—

$$\frac{1}{cn} + \frac{1}{g + \frac{1}{r + ln}} = 0 \qquad \text{ohms} \angle \quad (41)$$

whence $n^2 cl + n(cr + gl) + (1 + gr) = 0$ numeric $\angle$ (42)

$$\text{and } n = -\left(\frac{r}{2l} + \frac{g}{2c}\right) \pm j\sqrt{\frac{1+gr}{cl} - \left(\frac{r}{2l} + \frac{g}{2c}\right)^2} \quad \frac{\text{radians}}{\text{second}} \angle \quad (43)$$

$$= -\left(\frac{r}{2l} + \frac{g}{2c}\right) \pm j\sqrt{\frac{1}{cl} - \left(\frac{r}{2l} - \frac{g}{2c}\right)^2} \qquad \text{"} \quad \angle \quad (44)$$

Formula (44) was derived by Fleming, from a different method, in 1913.* If we prefer to take the reactor as the main path of discharge: then

$$ln + r + \frac{1}{cn + g} = 0 \qquad \text{ohms} \angle \quad (45)$$

whence $n^2 cl + n(cr + gl) + (1 + gr) = 0$ numeric $\angle$ (46)

which is identical with (42) and therefore leads to the same result.

In view of the practical instance cited by Fleming, no example of this case needs to be discussed arithmetically.

Two Resistive Reactors and a Condenser, in Star or Parallel Connection

In the case presented in Figure 10, we have three discharge elements in parallel, two of them reactors of l_1 and l_2 henrys,

* Bibliography (7).

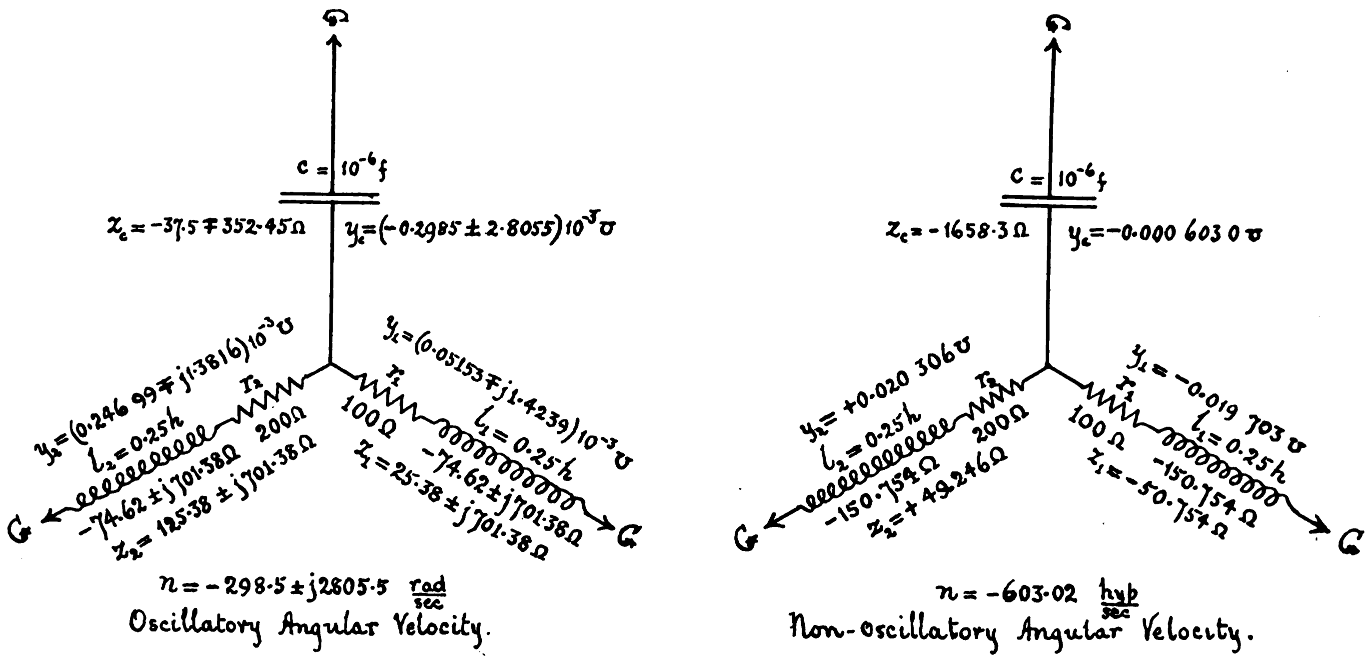

FIGURE 10—Case of Condenser Connected to Pair of Unlike Reactors in Parallel

r_1 and r_2 ohms respectively, and the third a condenser of capacitance c farads.. It is a matter of indifference which element we take as the main path of discharge; but taking the condenser element, we have:

$$\frac{1}{cn} + \frac{(r_1 + l_1 n) \times (r_2 + l_2 n)}{(r_1 + l_1 n) + (r_2 + l_2 n)} = 0 \qquad \text{ohms} \angle \quad (47)$$

whence

$$n^3 + n^2 \left(\frac{r_1}{l_1} + \frac{r_2}{l_2}\right) + n \left(\frac{1}{c\,l_1} + \frac{1}{c\,l_2} + \frac{r_1 r_2}{l_1 l_2}\right) + \frac{r_1 + r_2}{c\,l_1 l_2} = 0 \qquad \left(\frac{\text{radians}}{\text{second}}\right)^3 \angle \quad (48)$$

a cubic equation having one real and two complex roots. There are thus three values of the angular velocity n, which, substituted in (47), will enable that equation to hold. The real value may be regarded as pertaining to the discharge from one reactor thru the other, and thru the resistance $r_1 + r_2$, in their circuit. The two conjugate complex values may be regarded as pertaining to the discharge from the condenser into a certain single resistance and inductance, equivalent to the pair of parallel reactors.

As an example, we may take $c = 10^{-6}$ farad, $r_1 = 100$ ohms, $l_1 = 0.25$ henry, $r_2 = 200$ ohms, $l_2 = 0.25$ henry. Then (48) becomes

$$n^3 + 1200\,n^2 + 8.32 \times 10^6 n + 4.8 \times 10^9 = 0 \qquad \left(\frac{\text{radians}}{\text{second}}\right)^3 \quad (49)$$

This equation may be solved by the use of an auxiliary hyperbolic angle in the well known manner; but it is easy to find the roots by first plotting the value of (49), as ordinates, against arbitrarily selected values of n, as abscissas, in the regular way, as indicated in Figure 11, which shows that the graph passes through the zero line of ordinates near $n = -600$. A few more arithmetical trials, close to this value of n, will give a more nearly correct value of -603.02. This is the numerical value of the real root. Dividing (49) by $(n + 603.02)$, we obtain as the quotient:

$$n^2 + 596.98\,n + 7.96 \times 10^6 = 0 \qquad \left(\frac{\text{radians}}{\text{second}}\right)^2 \angle \quad (50)$$

an ordinary quadratic equation, of which the solution is:—

$$n = -298.49 \pm j\,2805.5 \qquad \frac{\text{radians}}{\text{second}} \angle \quad (51)$$

This equation gives the two remaining complex roots of (49).

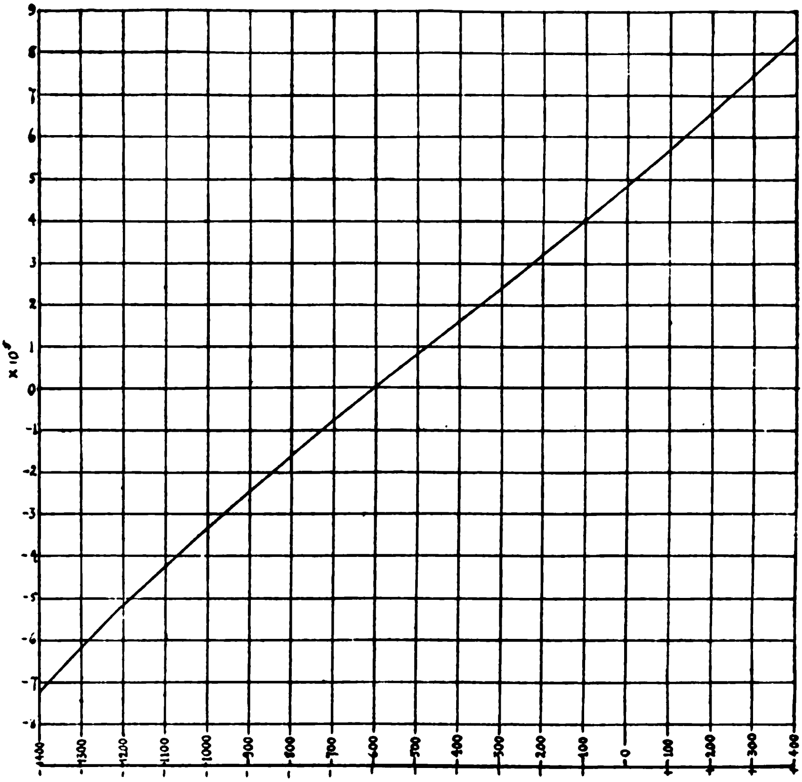

Figure 11—Graph of Expression $n^3 + 1200\,n^2 + 8.32 \times 10^6\,n + 4.8 \times 10^9$ Between the Values $n = -1400$ and $n = +400$

If the condenser were disconnected from the system, leaving the two reactors connected thru $r_1 + r_2 = 300$ ohms, we find, by (13), that their free angular velocity would be $n = -600$ hyps. per second; so that the presence of the condenser merely modifies this to -603.02. Again, if the two resistances were removed ($r_1 = r_2 = 0$), leaving the condenser in series with the reactor, we find, by (9), that the free angular velocity would be $n = \pm j$ 2828.4 radians per second. The presence of the resistances reduces this to $-298.49 \pm j$ 2805.5.

The system of Figure 10 in the example considered, thus dissipates disturbance energy in two different modes. One is a non-oscillatory discharge of -603.02 hyps. per second, or accompanied by a damping factor of $\varepsilon^{-603.02t}$. This is the discharge between the two reactors, slightly modified by the presence of the condenser. The other is an oscillatory discharge of angular velocity $-298.49 \pm j\,2805.5$ radians per second, having a frequency of $2805.5/(2\pi) = 446.5$ cycles per second, accompanied by a damping constant of 298.49, or a damping factor of $\varepsilon^{-298.49t}$. This damped oscillatory discharge is between the condenser and the joint ductance, as modified by the presence of the resistances.

It is shown in Figure 10 that at $n = -603.02$, the condenser has an impedance of -1658.3 ohms, one reactor -50.754 ohms, and the other $+49.246$. Taking the admittances, or reciprocals of these quantities, the condenser has -0.60302 millimhos, one reactor -19.703 millimhos, and the other $+20.306$. The sum of these admittances is zero.

Similarly, at $n = -298.49 \pm j\,2805.5$ radians per second, Figure 10 shows that the sum of the three branch admittances is zero. We may proceed to establish this proposition generally.

The Sum of the Oscillation Admittances About any Branch Point is Zero

In Figure 12, a number of branches, to ground or common connection, meet at the point O Each branch may contain a

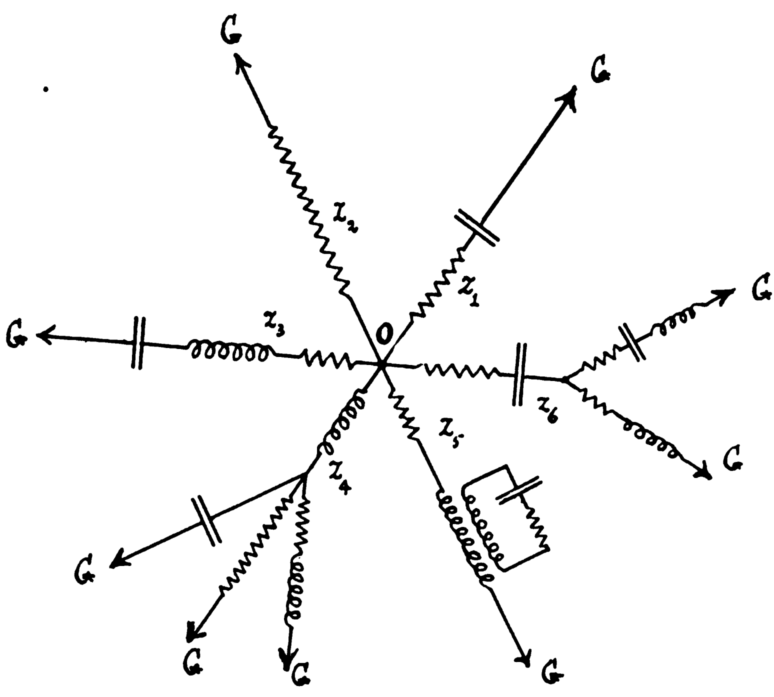

Figure 12—Group of Oscillatory Impedances meeting at a Knot Point O plurality of discharge elements, or resistances, or sub-branches. Then let the discharge impedance of these branches be z_1, z_2, z_3 . . . etc., each being formed on the understanding that an inductance has ln ohms, and a capacitance $1/(c\,n)$ ohms, n being the subsequently determined generalized angular velocity. Then, in order that the total discharge impedance in the path of any one branch, say z_1, shall be zero, we must have:

$$z_1 + \frac{1}{\frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4} + \cdot\ \cdot} = 0 \qquad \text{ohms} \angle \quad (52)$$

or if $y_1 = 1/z_1$, $y_2 = 1/z_2$, $y_3 = 1/z_3$, $y_4 = 1/z_4$. . are the respective discharge admittances,

$$z_1 + \frac{1}{y_2 + y_3 + y_4 + \cdot\ \cdot} = 0 \qquad \text{ohms} \angle \quad (53)$$

whence $$y_2 + y_3 + y_4 + \quad = -y_1 \qquad \text{mhos} \angle \quad (54)$$

or $$y_1 + y_2 + y_3 + y_4 + \quad = 0 \qquad \text{''} \quad (55)$$

or $$\Sigma y = 0 \qquad \text{''} \quad (56)$$

This relation must hold for each and all values of n which may satisfy (52). It applies not merely to a subdivided circuit: but also to a single undivided circuit; such as that in Figure 7, if any available point P be selected as a branch point of two branches. Knot-point cases* may often be solved advantageously by using this rule.

A number of less simple oscillating-current networks have been worked out by the methods here presented, and checked by independent means. No discrepancies have yet been found.

INDUCTIVELY COUPLED CIRCUITS

If two circuits are inductively coupled by a mutual inductance μ henrys, as in Figure 13, the primary having constants c_1, l_1, r_1, and the secondary c_2, l_2, r_2, it was shown by the writer in 1893† that the impedance z'_{12} ohms of the primary circuit to sustained oscillations in the presence of the closed secondary circuit, is:

$$z'_{12} = z'_1 - \frac{(\mu j \omega)^2}{z'_2} \qquad \text{ohms} \angle \quad (57)$$

*Thus, the case presented in Figure 10, with equation (47), may be stated as follows:—

$$c\,n + \frac{1}{r_1 + l_1 n} + \frac{1}{r_2 + l_2 n} = 0 \qquad \text{mhos} \angle \quad (56a)$$

which reduces to (48).

†Bibliography (4).

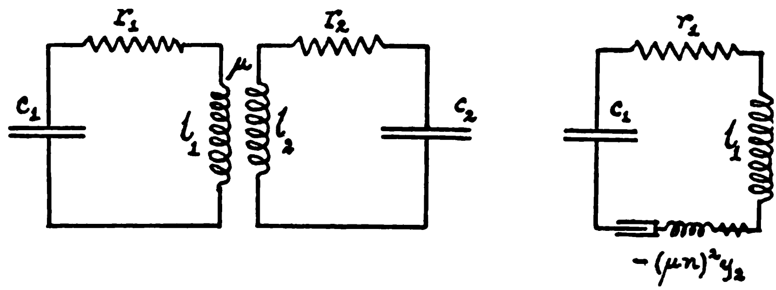

FIGURE 13—Pair of Inductively Connected Oscillatory Circuits and the Equivalent Single Primary Circuit

where z'_1 is the impedance of the primary circuit with the secondary circuit open, z'_2 the impedance of the secondary circuit with the primary circuit open (ohms $\angle$), and ω is the angular velocity of the impressed alternating current, in radians per second. Knowing the impedance of the primary circuit from (57), the current in that circuit to any impressed alternating emf. is immediately obtained. The emf. induced in the secondary circuit is then found by multiplying the primary current with $-\mu j \omega$ ohms.

The corresponding rule for free oscillations of generalized angular velocity n radians per second is:

$$z_{12} = z_1 - \frac{(\mu n)^2}{z_2} = z_1 - \mu^2 n^2 y_2 \qquad \text{ohms} \angle \quad (58)$$

so that, considering the primary circuit as the discharging circuit, with zero oscillation impedance, z_{12} must be equated to zero; or:

$$z_1 - \frac{(\mu n)^2}{z_2} = 0 \qquad \text{ohms} \angle \quad (59)$$

is the condition for determining n, it being understood that z_1 is the impedance of the primary circuit to angular velocity n, when the secondary circuit is open, and z_2 the impedance of the secondary circuit to angular velocity n, when the primary circuit is open.

The proposition may be proved as follows:

Let the instantaneous oscillating current i_1 in the primary circuit follow the law

$$i_1 = I_1 \varepsilon^{nt} \qquad \text{amperes} \angle \quad (60)$$

where n is a generalized or complex angular velocity, and I_1 the initial value of the primary current when $t = 0$. Then the instantaneous induced emf. in the secondary circuit will be

$$e_2 = -\mu \frac{d i_1}{d t} = -\mu n I_1 \varepsilon^{nt} = -\mu n i_1 \qquad \text{volts} \angle \quad (61)$$

and if the oscillation impedance of the secondary circuit is z_2 ohms, the instantaneous secondary current strength will be

$$i_2 = \frac{e_2}{z_2} = -\frac{\mu n i_1}{z_2} = -\mu n i_1 y_2 \qquad \text{amperes} \angle \quad (62)$$

The instantaneous emf. induced in the primary circuit by the rate of change in secondary current will be

$$-e_1 = -\mu \frac{d i_2}{d t} = \mu^2 n^2 i_1 y_2 \qquad \text{volts} \angle \quad (63)$$

The instantaneous driving emf. in the primary circuit, needed to overcome $-e_1$ will be

$$e_1 = -\mu^2 n^2 i_1 y_2 \qquad \text{volts} \angle \quad (64)$$

The instantaneous impedance in the primary circuit due to the reaction of the secondary will be

$$z_{12} = \frac{e_1}{i_1} = -\mu^2 n^2 y_2 = -\frac{\mu^2 n^2}{z_2} \qquad \text{ohms} \angle \quad (65)$$

a result which includes (57), when $\alpha = 0$ and $n = j\omega$.

General Considerations

An equation in n of the second degree can be satisfied either by two real roots ($-\alpha_1$, $-\alpha_2$, non-oscillating angular velocities) or by a pair of conjugate complex roots, of the type ($-\alpha \pm j\omega$), entailing one oscillation frequency. An equation in n of the third degree, or of any odd degree, indicates at least one real root $-\alpha_1$, which can ordinarily be evaluated in the manner exemplified by Figure 11. The remaining two roots, if conjugate, represent one oscillation frequency. Similarly, an equation in n, with lowest terms, and of the fourth degree, may indicate the presence of two independent oscillation frequencies, and an equation of the sixth degree, three oscillation frequencies. An inspection of the oscillation system connection-diagram may help in forming a judgment as to the number of independent oscillation frequencies present.

In view of the close analogy which exists between the arithmetics of electric oscillations in oscillatory-current circuits, and of small mechanical oscillations in mechanically vibrating systems,* it is evident that the rules above discussed apply, in general, also to mechanical oscillation-systems, provided the

* Bibliography (8).

elastic forces are proportional to the corresponding displacements and the frictional forces to the first powers of the velocities.*

In the case represented in Figure 13, the full expression of (59) yields an equation of the fourth degree in n, with two pairs of conjugate complex roots, corresponding to two oscillation frequencies and damping constants. The complete solution of this fourth-degree equation is, in general, very tedious; but full results for engineering purposes may be obtained by abbreviated methods. The detailed discussion of oscillation frequencies in mutually coupled circuits calls however, for a separate paper, and need not be continued here.

Conclusions

(1) The oscillation impedance of a circuit traversed by free electric oscillations is zero.

(2) The oscillation impedance of a pure resistance is equal to its ohmic resistance.

(3) The oscillation impedance of a capacitance c farads, to angular velocity n, is $1/(c\,n)$ ohms $\angle$. In other words, its oscillation admittance is $c\,n$ mhos $\angle$.

(4) The oscillation impedance of an inductance of l henrys, to angular velocity n, is $l\,n$ ohms $\angle$.

(5) The oscillation impedances of the elements of a circuit or system of circuits follow the laws of resistances in such circuits when traversed by continuous currents, subject to the rules of complex quantities, or of plane-vector arithmetic.

(6) The impedance of a circuit, or system of circuits, to sustained oscillations, or impressed alternating currents, is a particular case under the general laws above stated, ($a = 0$, $\Sigma' z \neq 0$).

(7) Any free oscillation in a circuit, or system of circuits, selects such an angular velocity, n radians per second, as will reduce its total impedance to zero.

(8) A generalized angular velocity n is a complex quantity containing a real and an imaginary component. The real component is the damping constant, and may be regarded as the projection of a hyperbolic angular velocity. The imaginary component is a circular angular velocity, of 2π times the oscillation frequency. Its projection, on an axis of reference, gives a sinusoidal quantity.

* Since the printing of this paper, the author's attention has been directed to a statement by Mr. H. W. Nichols, which indicates that certain mathematical propositions concerning mechanical oscillations, bearing closely on this matter, are already known to physicists.

(9) The sum of the oscillation admittances of the branches of a multiple oscillation circuit, at a knot point, is zero.

(10) The oscillation angular velocities of mutually coupled circuits can be expressed in terms of their mutual impedances.

(11) The sum of the instantaneous oscillation-impedance drops ($\Sigma i_n z_n$) around any closed loop in an oscillation system is zero. In the case of sustained oscillations, i. e., alternating currents, with $a = 0$, this reduces to Steinmetz's extension of Kirchoff's law into two dimensions.

(12) The instantaneous power of discharge in an oscillation impedance z is $i^2 z$ watts $\angle$, the phase angle of the instantaneous current i being taken as zero. Negative power values signify powers absorbed into the circuit. Positive values signify powers liberated out of the circuit. Real components signify dissipative powers. Imaginary components signify non-dissipative and transformed or reactive powers. The same conditions apply to the sustained oscillations in alternating-current circuits, except that with $a = 0$, negative real components do not present themselves.

(13) The total instantaneous discharge power in an oscillation system ($\Sigma i_n^2 z_n$) is zero.

(14) The share of oscillating current which a discharging element delivers to any one of a group of oscillation admittances in parallel is proportional to the oscillation admittance of that path, computed according to the rules of complex quantities or plane vectors.

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LIST OF SYMBOLS EMPLOYED

A, B	Integration constants of initial electric quantity (coulombs).
a	Projected hyperbolic angular velocity; (hyps. per sec.) or damping constant.
β	Circular angle described by a rotating unit radius vector (radians).
$C_1 = c_1 + c_2 + c_3 +$	Sum of capacitances in parallel (farads).
c	Capacitance of a condenser (farads).
c_1, c_2, c_3	Capacitances of individual condensers (farads).
d	Sign of differentiation.
$E = \gamma_1 + \gamma_2 + \gamma_3 +$	Sum of pure ductances in parallel (yrnehs).
$\gamma = 1/l$	Ductance, or reciprocal of a pure inductance (yrnehs).
$\gamma_1, \gamma_2, \gamma_3$	Ductances of individual pure reactors (yrnehs).
e	Instantaneous emf. of a discharging element (volts $\angle$).
$-e_1, e_2$	Instantaneous primary and secondary induced emfs. (volts $\angle$).
$-e_c$	Instantaneous back emf. of a capacitance (volts $\angle$).
$-e_l$	Instantaneous back emf. of an inductance (volts $\angle$).
$\varepsilon = 2.71828 \ldots$	The Napierian base (numeric).
$f = \omega/2\pi$	Oscillation frequency (cycles per second). In drawings, the symbol for a farad.
G	In drawings, the symbol for a ground connection, assumed perfect.

g Conductance of a leak (mhos).
h In drawings, the symbol for a henry.
θ Hyperbolic angle (hyperbolic radian or hyp.).
I Initial current strength (amperes $\angle$).
I_1 Initial primary current strength (amperes $\angle$).
i Instantaneous current strength (amperes $\angle$).
i_1, i_2 Instantaneous primary and secondary currents (amperes $\angle$).
$j = \sqrt{-1}$ Sign of "imaginary" quantity.
$L = l_1 + l_2 + l_3 +$ Sum of individual inductances in series (henrys).
l An inductance (henrys).
l_1, l_2, l_3 Individual inductances (henrys).
μ Mutual inductance between two circuits (henrys).
$n = -(a \pm j\omega)$ A generalized complex angular velocity (radians per second $\angle$).
p_c, p_l, p_r Instantaneous powers in a capacitance, inductance, or resistance (watts $\angle$).
$\pi = 3.14159\ldots$ (Numeric).
q Quantity of electricity in a condenser (coulombs).
$R = r_1 + r_2 + r_3 +$ Sum of a number of pure resistances in series (ohms).
r A pure resistance (ohms).
$S = s_1 + s_2 + s_3 +$ The sum of individual elastances in series (darafs).
$s = 1/c$ Elastance of a condenser of capacitance c (darafs).
s_1, s_2, s_3 Elastances of individual condenser (darafs).
Σ Sign of summation.
t Time elapsed from an epoch, or original condition (seconds).
U Initial difference of potential across a discharging condenser (volts).
u Instantaneous difference of potential across a condenser (volts).
$X_l = l\omega$ Reactance of an inductance to sustained angular velocity ω (ohms).

$X_c = 1/(c\,\omega)$ Reactance of a capacitance to sustained angular velocity ω (ohms).

x, y Rectangular Cartesian coordinate of a point in a plane (cm).

$y = 1/z$ Admittance of an impedance z (mhos $\angle$).

$y\,h$ In drawings, a symbol for yrnehs.

y_2 Admittance of a secondary oscillation circuit (mhos $\angle$).

y_1, y_2, y_3 Individual admittances in parallel (mhos $\angle$).

z An oscillation impedance (ohms $\angle$).

z_1, z_2, z_3 Individual oscillation impedances (ohms $\angle$).

z_1, z'_1 Oscillation impedance of a primary circuit with the secondary open or removed (ohms $\angle$).

z_2, z'_2 Oscillation impedance of a secondary circuit with the primary open or removed (ohms $\angle$).

z_{12}, z'_{12} Oscillation impedance of a primary circuit with the secondary closed or present (ohms $\angle$).

z_c Oscillation impedance of a capacitance c (ohms $\angle$).

z_l Oscillation impedance of an inductance l (ohms $\angle$).

z_r Oscillation impedance of a resistance r (ohms).

Ω In drawings a symbol for ohms.

℧ In drawings a symbol for mhos.

ω Circular angular velocity (radians per second).

$\angle$ Angle sign appended to a unit, indicating the existence of a complex quantity or plane vector.

SUMMARY: Corresponding to the usual angular velocity (2π times the frequency) of an alternating current is the generalized angular velocity of an oscillating current. The generalized velocity is a complex quantity; the real portion determining the damping constant, the imaginary portion the frequency of the current. The author shows that the oscillation impedances of resistances, inductances and capacities are formed in the same way from generalized angular velocities as from the usual angular velocity. The oscillation impedance of any circuit or system of circuits is found by the usual law of resistances for continuous currents, due regard being paid to the rules of complex quantities. It is then shown that free oscillations of any system of circuits select such angular velocities as to reduce the total oscillation impedance to zero. A number of cases of parallel and series oscillating circuits are treated by this method with much simplicity. The total oscillation admittance at a knot point is shown to be zero, as also is the sum of the instan-

taneous oscillation-impedance drops around a closed loop. The instantaneous discharge power in any oscillation impedance is readily derived and shown to be zero in a pure oscillation system. The problem of coupled circuits is given a preliminary treatment by these methods.

DISCUSSION

J. A. Fleming (communicated): In reference to the paper of Dr. A. E. Kennelly, I may mention that for about ten years past I have been accustomed to give to my students in lectures a proposition which is very nearly identical with the one which forms the basis of his paper.

I have usually put the matter as follows:

If a circuit has capacity (C) and inductance (L) in series and is submitted to a simple periodic E. M. F. having a frequency n and *pulsation* or angular velocity $p = 2\pi n$, then the circuit is non-inductive for a frequency equal to the natural frequency of the circuit.

I have also been accustomed to employ the idea of a complex angular velocity or complex pulsation P in connection with damped oscillations.

In teaching the elements of alternating current theory, the students are, of course, taught that the quantity Lp, called the reactance, is of the dimensions of a resistance and can be measured in ohms and that the product of this quantity and the current (I) is called the reactance voltage LpI. They also learn that the quantity $\frac{1}{Cp}$ which I have always called the "captance" is a quantity of the dimensions of a resistance and that the product of captance and current $\frac{I}{Cp}$ is of the dimensions of an E. M. F. and is measured in volts.

Hence for a circuit of ohmic resistance R and reactance Lp and captance $\frac{1}{Cp}$ the resultant vector impedance is

$$R + j\left(Lp - \frac{1}{Cp}\right)$$

and the size of this vector is

$$\sqrt{R^2 + \left(Lp - \frac{1}{Cp}\right)^2}$$

Accordingly, if the frequency is such that $Lp - \frac{1}{Cp} = 0$, the circuit is non-inductive. But the natural frequency is given by the condition $p = \frac{1}{\sqrt{LC}}$, which is identical with the above condition for non-inductivity. I think that this equality is, however, only exact if we can neglect the resistance of the circuit in comparison with its reactance.

A generalised proof of the proposition may be obtained as follows: Maxwell showed in his "Treatise on Electricity and Magnetism," Vol. II, that the equations which Lagrange established for the dynamics of mechanical systems could be applied to electrokinetic systems with certain modifications in the meaning of the symbols.

The Maxwell-Lagrange equations are as follows: If T is the conserved energy of the system and H is the rate of dissipation of energy in the system, and if x is any current in any mesh or circuit in which the impressed electromotive force is E, then we have

$$\frac{d}{dt} \cdot \left(\frac{dT}{dx}\right)+\frac{1}{2}\frac{dH}{dx}=E \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

If the system has an electromotive impulse given to it, and if it is then left to itself, there is then no impressed E. M. F., and, therefore, the time of free oscillation must be given by solving the above equation for p or n when the left hand side is equated to zero. Hence the free period is obtained from the equation:

$$\frac{d}{dt}\left(\frac{dT}{dx}\right)+\frac{1}{2}\frac{dH}{dx}=0 \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Now the dissipation function, H is a quadratic function of the currents. If R is the resistance of the circuit then $H=Rx^2$ and $\frac{1}{2}\frac{dH}{dx}=Rx\cdot$ Hence for a single circuit of resistance R, the equation (1) takes the form

$$\frac{d}{dt}\left(\frac{dT}{dx}\right)+Rx=E \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Hence the condition for the circuit being non-inductive is

$$\frac{d}{dt}\left(\frac{dT}{dx}\right)=0 \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

If R is small, the condition (4) is nearly the same as the equation (2) which determines the free frequency. In other words, if the resistance is small or negligible compared with the reactance, then the circuit is non-inductive for a frequency equal to its natural frequency of oscillation.

Let us take as an example the simple case of a condenser of capacity C in series with a coil of resistance R and inductance L. Let n be the frequency of the impressed E. M. F., and $p=2\pi n$. Then the energy function T is

$$T=\frac{1}{2}Lx^2+\frac{1}{2}\frac{q^2}{C} \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where x is the current at any instant and q is the charge in the condenser at the same instant. But $x=\frac{dq}{dt}$ because the coil is in series with the condenser. Moreover x is independent of q. Hence we have

$$\frac{d}{dt}\left(\frac{dT}{dx}\right)=L\frac{dx}{dt}+\frac{q}{C} \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

and $$H=Rx^2$$

therefore $$\frac{1}{2}\frac{dH}{dx}=Rx. \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Equation (1) becomes then

$$L\frac{dx}{dt}+\frac{q}{C}+Rx=E \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

The circuit is therefore non-inductive if

$$L\frac{dx}{dt}+\frac{q}{C}=0$$

or if $$L\frac{d^2x}{dt^2}+\frac{x}{C}=0 \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Also the frequency is derived from the equation

$$L\frac{d^2x}{dt^2}+R\frac{dx}{dt}+\frac{x}{C}=0 \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

for a solution of the above is $x=\varepsilon^{m}$ where m is found from

$$m^2+\frac{R}{L}m+\frac{1}{LC}=0 \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

or $$m=-\frac{R}{2L}\pm j\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$$

$$=\alpha+j\beta.$$

This gives the solution of (10) in the form

$$x=\varepsilon^{-\alpha t}\left\{A\cos\beta x+B\sin\beta x\right\}$$

from which it follows that the frequency is

$$n=\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}} \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

or if R is negligible

$$n=\frac{1}{2\pi\sqrt{LC}} \quad \text{or} \quad p^2=\frac{1}{LC} \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

If, however, x is a simple periodic current, then $\frac{d^2x}{dt^2}=-p^2x$, and equation (9) becomes $p^2L=\frac{1}{C}$ or $p^2=\frac{1}{LC}$.

Hence if the applied frequency is the same as the *free* frequency the circuit is non-inductive.

The mechanical equivalent of this proposition is as follows: A mechanical system having inertia and elastic constraints acts as an inertia-less system towards applied impulses having a frequency equal to its free oscillations.

This can be proved as above shown from Lagrange's equations. This is merely another form of the principle of resonance, viz., that very large displacements can be produced by infinitely small impulses, if these latter are applied at intervals exactly equal to the free frequency of oscillation of the system.

I may say also that I have long been accustomed to make use of the conception of a complex pulsation or angular velocity, and to explain to students that the expression ε^{jPt} represents:

(1) An undamped simple periodic oscillation,

(2) A dead beat motion, or

(3) A damped oscillatory motion,

according as P is a real, imaginary, or complex quantity; and have used this idea to calculate the frequency of the oscillations in a condenser circuit having in it inductance and a leaky condenser.

(See my "Wireless Telegraphists' Pocket-Book of Notes, Formulae and Calculations."—The Wireless Press, Ltd., London, page 165.)

Problems connected with damped oscillations in circuits having resistance are easily treated by assuming that the currents and voltages are proportional to the real part of the expression ε^{jPt},where $P=(p+j\,\alpha)$ and $p=2\pi n$ and $\alpha=n\,\delta$, where δ is the decrement per complete period.

It is worth noticing that the expression ε^{jPt} is an operator which when applied to a vector $\alpha+j\,\beta$ causes that vector to rotate thru an angle $p\,t$ and at the same time to shrink in size in the ratio of $1:\varepsilon^{-\alpha t}$.

The free extremity of such a vector pivoted at one end therefore describes a logarithmic spiral and the projection of the free end on a line moving uniformly parallel to itself gives us the decrescent wavy curve which graphically denotes a damped oscillation.

Arthur G. Webster (communicated): Professor Kennelly's paper is, like all of his contributions, extremely clear and helpful to the student. He is one of a group of engineering teachers who have the gift of making difficult things seem easy, and of presenting clear directions for the solution of problems. I am sorry to say, however, that these gentlemen do not always make it plain as to what is new, and what is merely put in a new way. Since in the first paragraph of his paper Professor Kennelly states that he believes his proposition to be new, and later that a particular case was discovered by him in 1893, I take the liberty of making a few historical observations: The proposition with which this paper is concerned appears explicitly in a paper by Heaviside (to whom the notion of impedance is due) published in 1884, and is found in Vol. I of his "Electrical Papers" (page 415, equation 137). The whole matter is treated in his paper on "Resistance and Conductance Operators," published in 1887, and found in Vol. II, page 355, in an extremely general manner.

I am free to say that it has always seemed to me that it is not enough to know that a thing is so, but that one should know why it is so, and that rather than put off the student with these apparently simple methods it would be better to advise him to learn the small amount regarding differential equations that would enable him to understand how these problems are really to be handled. The reason for the simplicity of all these matters is that the differential equations involved are all *linear* with constant coefficients, and as has been known since the time of Cauchy all such equations may be solved by exponential functions, since these functions preserve their form on differentiation. That is the reason for the appearance of the function ε^{nt}. Now, since it seems to be thought by many persons in this country that the application of the complex variable or rotating vector to the study of alternating currents was invented by Mr. C. P. Steinmetz, I will say a few words on that matter. The formula $\varepsilon^{j\theta} = sin\,\theta + j\,cos\,\theta$ was given by Euler about a century and a half ago. When the complex diagram was introduced by Argand in about 1800, or by Wessels a little earlier, it became evident that the real part of $\varepsilon^{j\omega t}$, that is $cos\,\omega t$, could be used to represent any harmonically varying quantity. I find such use by Cauchy in 1821 in optics, where we have oscillations, and all thru his works he makes great use of this imaginary exponential. But lest it be said that this is not electricity, I may say that Helmholtz, in a paper on the telephone, published in 1878, uses the imaginary exponential to represent sustained vibrations. But the method of solving all cases of oscillations in nets containing any sort of apparatus is given by Maxwell in his great paper on "A Dynamical Theory of the Electromagnetic Field," published in 1864, in which he applies the method of Lagrange to electric circuits. Now Lagrange solved all these problems in his "Mécanique Analytique," published in 1788, and the equations can be seen on page 375 of Vol. I. So this is really the age of the theorem.

Without going into Lagrange's method, let us apply the simple method of action and reaction to the problem of two coupled circuits. We know that the electromotive force necessary to fill a condenser c with a charge q is $q\,c$. But if the charge comes from a current I, we have $q=\int I\,dt$. Also the back electromotive-force from a coil of inductance L is $L\dfrac{dI}{dt}$, and if there is an influencing current I_2 the effect of that is $M\dfrac{dI_2}{dt}$. If there is resistance, we need to overcome the electromotive force $R\,I$. Putting all together, we need

$$E_1=\frac{1}{C_1}\int I_1dt+R_1I_1+L_1\frac{dI_1}{dt}+M\frac{dI_2}{dt}.$$

Now treating the other circuit in the same way, we need

$$E_2=\frac{1}{C_2}\int I_2\,dt+R_2\,I_2+M\frac{dI_1}{dt}+L_1\frac{dI_2}{dt}.$$

Suppose that there is no electromotive force impressed in the second circuit, $E_2=0$, and that there is a simple harmonic E. M. F. in the first. This can be represented by the real part of $E\,\varepsilon^{j\omega t}$ and, for very simple reasons, we may take as a solution the real parts of $I_1=A\,\varepsilon^{j\omega t}, I_2=B\,\varepsilon^{j\omega t}$. Now since differentiation of the exponential multiplies it by $j\,\omega$ and integration divides it by the same, we find, dividing out the $\varepsilon^{j\omega t}$,

$$E=\left(\frac{1}{j\,\omega\,C_1}+R_1+j\,\omega\,L_1\right)A+j\,\omega\,M B$$

$$0=j\,\omega\,M\,A+\left(\frac{1}{j\,\omega\,C_2}+R_2+j\,\omega\,L_2\right)B.$$

So that our problem of calculus has disappeared, and we have merely algebra. We have now arrived at the state treated by Professor Kennelly. If we eliminate B we get

$$\frac{E}{A}=\frac{1}{j\,\omega\,C_1}+R_1+j\,\omega\,L_1-\frac{j^2\,\omega^2\,M^2}{\dfrac{1}{j\,\omega\,C_2}+R_2+j\omega L_2}=Z$$

which is the impedance of the first circuit, influenced by the second. Now if there is no impressed E. M. F. in the first circuit either, we have $E=0$, and since A is not zero, $Z=0$, which is Heaviside's equation. The method is perfectly general, and is described in my book on "Electricity and Magnetism," § 241, as well known at the time of publication, 1897. The method of elimination of the constants A, B is perfectly general, and leads to Lagrange's equation for the periods, as described by Kennelly.

From such considerations as arise from the definition of impedance by the equation $E=Z I$, it follows at once that impedances in series are additive, and in parallel their reciprocals are additive, and this gives the application of Kirchhoff's two laws, which is far older than Steinmetz.

With regard to Professor Pupin's experimental method, it is exactly the application of this theorem, that is, if we neglect the resistances. If not, the values of n given by the period equation can never be purely imaginary, and thus the impedance for the actual current can not be exactly zero, but as Professor Pupin says the difference is very small in practise. I have made a remark on this in my book, page 499.

In conclusion, I cannot let pass the opportunity to criticise Professor Kennelly's practise of printing the units in the margin, as I consider it a cardinal principle in writing formulae that the formula should be true whatever the units. Fancy writing

$$s=v\,t \qquad\qquad \text{feet}$$

when the formula is equally true for miles, centimeters, or what not, all the units being properly taken.

Joseph G. Coffin (communicated): I have read Professor Kennelly's article with great interest, and also with surprise. Professor Kennelly has no doubt discovered a simple yet powerful proposition, but I am surprised that he has not discovered that it is not new and that it is known to many of us. I shall merely refer to Perry's "Calculus for Engineers," (1897), pages 231–261, a work not mentioned in his article and with which he seems to be unfamiliar. I refer especially to pages 236 and 237 where may be found the words:

"In any network of conductors we can say exactly what is the actual resistance (for steady currents) between any point A and another point B if we know all the resistances r_1, r_2, etc., of all the branches. Now if each of these branches has self-induction l_1, etc., and capacity k_1, etc., what we have to do is to substitute $r_1+l_1\theta+\dfrac{1}{k_1\theta}$ instead of r_1 in the mathematical expressions, and we have the resistance right for currents that are not steady."

The following is a direct demonstration of the theorem, based upon Perry's statement and well-known mathematical results; which may be of interest to others as the standpoint is somewhat different.

In any network of linear conductors in which steady currents are circulating, two laws, well known as Kirchhoff's laws hold.

1. At any point where two or more conductors meet (branch point), the sum of the currents all taken as flowing into (or out of) that point is zero. This means merely that there is no accumulation of electricity at any such point.

2. Around any (closed) circuit of the network taken at random the sum of the Ri drops is equal to the sum of the impressed E. M. F.'s in that circuit.

These two rules of great importance are expressed by the equations

$$\Sigma i=0 \tag{1}$$

$$\Sigma Ri=\Sigma e \tag{2}$$

For variable currents these equations still hold at any instant, and hence at all instants, provided there is added to the ohmic drop, Ri, the back E. M. F.'s due to self-induction, $L\dfrac{di}{dt}$, and the back E. M. F.'s due to charges on condensers, $\dfrac{\int idt}{C}$, and the back E. M. F.'s due to mutual induction, $M\dfrac{di}{dt}$.

For example, the integro-differential equation for the current in a single circuit containing R, L and C in series is

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int i\,dt=e \tag{3}$$

where e, any given function of the tune, is the impressed E. M. F. on the circuit. In all cases of any importance, e is the exponential function $E\varepsilon^{mt}$; as it is well known that $e=0$, $e=E\varepsilon^{mt}$, $e=E\sin\omega t$ and $e=E\varepsilon^{-at}\sin\omega t$ are obtained by making $E=0$, $m=n$, $m=\pm j\omega$ and $m=-a\pm j\omega$ respectively.

The important thing then is the solution of equation (3) when $e=E\varepsilon^{mt}$ where m can be real, purely imaginary or complex. Taking the case of free action, let $e=0$. Equation (3) becomes

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=0$$

or $$\left\{L\frac{d}{dt}+R+\frac{1}{C}\int(\ )\,dt\right\}i=0 \tag{4}$$

It is well known that, assuming

$$i=e^{nt}, \tag{5}$$

where n is as yet an undetermined constant, real, purely imaginary or complex, we obtain

$$Ln+R+\frac{1}{Cn}=0 \tag{6}$$

as the condition that n must satisfy to make (5) a solution.

If one wishes to call the expression on the left of (6), the impedance of the circuit, there is certainly no objection.

To understand Perry's symbolic resistances, consider the equation for the current thru an ohmic resistance R; it is

$$Ri=e. \tag{7}$$

One may say that R operating on i gives the E. M. F.; or that $\frac{1}{R}$ operating on e gives the current.

The equation for the current in a circuit containing an inductance L is

$$L\frac{di}{dt}=e. \tag{8}$$

Note: Let us abbreviate, as does Perry, by letting

$$\frac{d}{dt}(\ )\equiv\theta(\ ). \tag{9}$$

Equation (8) is then

$$L\theta i=e. \tag{10}$$

This means that differentiating i with respect to the time and multiplying by L gives a result always equal to the impressed E. M. F. e.

By comparison with equation (7), one may say that $L\theta$ operating on i is a result equal to the impressed E. M. F. $L\theta$ is therefore analogous to a resistance and may be called the symbolic resistance due to L.

The equation for the current in a circuit containing a capacity C is given by

$$i=C\frac{de}{dt}=C\theta e \tag{11}$$

or by $$q=\int i\,dt=Ce. \tag{12}$$

So that we see not only that symbolically

$$\frac{1}{C\theta} i = e, \tag{13}$$

but that $\frac{1}{\theta}$ means integration with respect to the time. One may say that $\frac{1}{C\theta}$ is the symbolic resistance due to C. $\frac{1}{\theta}$ is called the inverse operator to θ.

From this point of view, (4) may be written

$$L\theta i + Ri + \frac{i}{C\theta} = e$$

or

$$\left(L\theta + R + \frac{1}{C\theta}\right) i = e. \tag{14}$$

One may say finally that

$$L\theta + R + \frac{1}{C\theta}$$

is the symbolic resistance due to L, R and C in series.*

Considering equations (1) and (2), it is now seen that they become for varying currents

$$\Sigma i = 0 \tag{15}$$

and

$$\Sigma \left(L\theta + R + \frac{1}{C\theta}\right) i = \Sigma e, \tag{16}$$

putting aside the consideration of mutual induction for the moment.

Equations (15) and (16), the generalized Kirchhoff equations, which are well known, give the solution in the following manner. Deduce the resistance around any chosen circuit or between any two points of the network, in terms of the resistances R_1, R_2, etc., of the separate branches as if there were merely ohmic resistances; in the resulting expression, replace R_1, R_2, etc., by the symbolic resistances

$$L_1 \frac{d}{dt} + R_1 + \frac{1}{C_1}\int (\ \ ) \, dt$$

or by

$$L_1\theta + R_1 + \frac{1}{C_1\theta}, \text{ etc.}$$

It is easily seen that the symbol θ always obeys the following laws in combination with i (to the first power only), and with a and b (constants).

$$(\theta + d)\, i = \theta i + d i$$

$$\theta\,(i_1 + i_2) = \theta i_1 + \theta i_2 \qquad \textit{Distributive Law.}$$

$$\theta a i = a \theta i \qquad \textit{Commutative Law.}$$

* See Perry, bottom of page 236.

and if $\theta\,\theta\, i$ is written $\theta^2 i = \frac{d^2 i}{dt^2}$, etc., it is easily seen that

$$\theta^n\, \theta^m\, i = \theta^{n+m}\, i \qquad \textit{Index Law.}$$

When any quantity obeys these laws, that quantity enters into expressions in combination with *i's* and constants exactly as do ordinary algebraic quantities.

For instance, it is easy to see that

$$(\theta + a)\,(\theta + b)\, i = [\theta^2 + (a+b)\,\theta + a\,b]\, i$$

or that $(\theta + a)\,(\theta + b)$ is the same thing as

$$(\theta + b)\,(\theta + a), \text{ etc.}$$

It follows from this that the expression obtained when the *R's* are replaced by the $\left(L\,\theta + R + \frac{1}{C\,\theta}\right)$'s may be simplified, and the result will be

$$f_1\,(L\,R\,C\,\theta)\, i = f_2\,(L\,R\,C\,\theta)\, e, \tag{17}$$

where f_1 and f_2 are polynomials in θ with constant coefficients.

In the case of free action $e=0$ and (17) becomes

$$(d_o\,\theta^m + d_1\,\theta^{m-1} + \cdot\;\cdot\;\cdot\;\cdot\;\cdot\; + d_{m-1}\,\theta + d_m)\, i = 0 \tag{18}$$

It is well known since the time of Euler that the solution of (17) is made up of two parts: (**a**) the solution when $e=0$, i. e. of (18), plus (**b**) a particular solution of (17). The first part is the natural or free action of the system left to itself and the second is the forced part.

At present, consider the free action of the system, i. e. of (18).

The well-known method is to assume a solution of the form

$$i = \varepsilon^{n\,t} \tag{19}$$

where n is as yet an undetermined constant which may be real, purely imaginary or complex.

There results on substitution of (19) in (18) since

$$\theta\, i = \theta\, \varepsilon^{n\,t} = n\, \varepsilon^{n\,t} = n\, i$$
$$\theta^m\, i = n^m\, i,$$

that is $\theta^m \equiv n^m$ when applied to an exponential function.

$$(a_o\, n^m + a_1\, n^{m-1} + \cdot\;\cdot\;\cdot\;\cdot\;\cdot\; + a_{m-1}\, n + a_m)\, i = 0 \tag{20}$$

which shows that if n has for its value any of the m solutions of the *algebraic* equation

$$a_o\, n^m + \cdot\;\cdot\;\cdot\;\cdot\;\cdot\; + a_m = 0 \tag{21}$$

then $$i = A_1\, \varepsilon^{n_1 t} + A_2\, \varepsilon^{n_2 t} + \qquad + A_m\, \varepsilon^{n_m t} \tag{22}$$

satisfies (18) because each term on the right satisfies it separately.

This is the general solution as it contains m arbitrary constants. In general, there will be real roots either positive or negative and imaginary roots either pure or complex, the latter always occurring in conjugate pairs.

The A's in (22) are arbitrary constants. In any particular case, they are determined by the initial or final state, of the system, which are supposed to be known. As a matter of fact, it can be shown that the real roots in any electrical case are always negative, and that the real parts of the imaginary roots are always negative.

For every real root $(-n_1)$ (excepting equal roots), there is a solution of the form

$$i_1 = I_1 \varepsilon^{-n_1 t} \tag{23}$$

For each pair of conjugate complex roots (always in pairs),

$$n_2 = -a + j\,\omega \qquad (j \equiv \sqrt{-1})$$
$$n_3 = -a - j\,\omega$$

there is a solution

$$i_2 = A_2\,\varepsilon^{(-a+j\omega)t} + A_3\,\varepsilon^{(-a-j\omega)t},$$

which by means of Euler's equation,

$$\varepsilon^{\pm jx} = \cos x \pm j \sin x$$

reduces to

$$i = I\,\varepsilon^{-at} \sin(\omega t + \phi) \tag{24}$$

where I and ϕ are constants determined by the initial conditions.

Solution (23) is a decaying current and (24) a damped harmonic oscillation of period $\frac{2\pi}{\omega}$; (24) is a sustained harmonic oscillation if $a=0$; i. e. if n_2 and n_3 are purely imaginary.

Now equation (21) is the result of placing the impedances,

$$Ln + R + \frac{1}{Cn}, \qquad \text{etc.,}$$

treated as resistances in the circuit of the network equal to zero; hence the theorem. The whole thing is a case of Kirchhoff's Laws, and properties of the exponential function.

Consider the case exemplified in Figure 12 of Professor Kennelly's paper. The points G are in reality a single point, so that the problem is one in which there are a number of circuits connected in parallel between O and G.

By Kirchhoff's law, the sum of the currents entering O is zero

$$\Sigma i = 0$$

Also the $R i$ drop in each branch is the same hence

$$R_1 i_1 = R_2 i_2 = R_3 i_3, \qquad \text{etc.}$$

If we call $\frac{I}{R}$, the admittance, $=A$, we may write

$$\frac{i_1}{A_1} = \frac{i_2}{A_2} = \quad . \quad . \quad . \quad . \quad . = \frac{\Sigma i}{\Sigma A}.$$

But as $\Sigma i = 0$, then must

$$\Sigma A = 0;$$

which is Professor Kennelly's statement.
In the case of two mutually inductive circuits, Figure 13, the equations of Kirchhoff give

$$R_1 i_1 + M\theta i_2 = e,$$

$$M\theta i_1 + R_2 i_2 = 0,$$

where the R's stand for $L\theta + R + \frac{1}{C\theta} = Z$

From the second equation

$$i_2 = -\frac{M\theta}{R_2}.$$

Substituting in the first

$$\left(R_1 - \frac{M^2\theta^2}{R_2}\right) i = e.$$

If $e = 0$, we obtain

$$R_1 - \frac{M^2\theta^2}{R_2} = 0$$

or

$$Z_1 - \frac{M^2 n^2}{Z_2} = 0 \qquad \text{(Preceding article, Equation 59)}$$

as the condition to be fulfilled by θ, i. e. n, for the free action of the primary of such a system.

These methods of solution have been familiar to me for at least sixteen years, and are the result of applying the matter in Perry's "Calculus" to these problems.

Professor Perry was interested more particularly in the result when $e = E \sin \omega t$, as free action has become of more importance only since radio work has become important.

But he has distinctly stated the case for free action in the cited pages of his book. It is an extremely powerful method and the most satisfactory one for the solution not only of the free action in any network but for the case where the network has alternating currents impressed upon it.

H. W. Nichols (communicated October 22, 1915): We owe most of our practical methods of treating electrical circuit problems to Heaviside, who originated in 1887 ("Phil. Mag.," December, 1887, and "Electrical Papers," Volume II, pages 355 to 374 and elsewhere) as a special case the so-called "complex method" of treating alternating current problems, later popularized by Steinmetz and others. Unfortunately his papers are hard to read, a remark on which his own characteristic comment was that "they were even harder to write."

It is, however, a fact which has been discovered by many later investigators that most apparently new methods in circuit problems are really all in the book, and this method just described is no exception, being very clearly stated in the paper above cited (page 371 ff. of the "Papers").

It is also in common use by some telephone engineers, and we find it again stated explicity in a paper by G. A. Campbell (A. I. E. E., April, 1911, page 902) in the words:

> "The characteristic feature of free oscillations is that, thruout the part of the network over which the oscillation extends, the driving point impedance is equal to zero. This follows from the fact that as the driving point impedance is equal to the impressed electromotive force divided by the current, it vanishes when the electromotive force vanishes, provided the current does not vanish. The criterion for free oscillations is therefore $A=0$.
>
> "The solution of this equation contains all the possible values of the time coefficient p. Each possible oscillation is aperiodic or not, according as p is purely imaginary or not; p cannot be real for any actual system, since energy must be dissipated in any oscillation which may occur in such a system."

One has only to read the Heaviside paper of 1887 and the ones referred to in it to see displayed the whole theory; but since engineers do not usually read Heaviside, it is doubtless of service to have him interpreted, and for that reason we are indebted to Dr. Kennelly for his interesting paper.

A. E. Kennelly (communicated): In dealing with the impedance of the simple alternating-current circuit with sustained oscillations of a frequency imposed by the generator, the impedance can only occupy one half of the plane; namely, that on the positive real side of the axis of imaginaries. That is, the impedance can only be $+R \pm jX$ ohms, where R is an essentially positive resistance, and jX a reactance either plus or minus. It is known, however, that by the aid either of electromagnetic induction (transformer action), or of the virtual impedance of a synchronous E. M. F., such as that of a synchronous motor, it is possible to invade the other half of the impedance plane, and to secure, in the steady state of operation, a representation of* negative resistance $(-R)$. As a consequence, however, of the reasoning set forth in the paper here presented, the negative half of the impedance plane is assigned to the impedance of inductances and capacitances during oscillations; so that the whole of the impedance plane comes into service in dealing with the simple alternating-current circuit, one half in the sustained oscillations, and the other half in unsustained or transient oscillations.

Since the paper was communicated, a new book by Professor Fleming ("The Wireless Telegraphists' Pocket-book of Notes, Formulae and Calculations") has reached this country. The date of the book's going to press is, however, earlier (May, 1915). The book contains passages relating to oscillatory frequencies bearing closely on the matters presented† in the paper; so that in tracing the history of the development of the propositions presented in the paper, the publications of Professor Fleming, including that cited in the Bibliography must certainly be taken into account.

Without belittling any scientific authority, or disparaging any kind of scientific work, it should be pointed out that the knowledge which requires determinants, differential equations, and a maze of symbols for its expression is not the kind of knowledge which can be readily apprehended and applied by the engineer. Propositions may be stated in such broad general terms as to possess no appreciable meaning in particular. Thus, the law of the conservation of energy may in a certain sense cover and include all future discoveries in physics. If the propositions set forth in the paper have been made known in prior publications, it is amazing how little use has been made of them up to this time.

The statements cited by Mr. Joseph G. Coffin from Professor Perry's admirable "Calculus for Engineers" are directed to forced or sustained oscillations; altho we can now see that they may also be used in connection with free or transient oscillations.

* D. C. and J. P. Jackson's "Alternating Currents and Alternating-Current Machinery," Macmillan Co., 1913 (page 238).

† See the section on "Time Period of Electric Oscillations in Circuits Having Inductance Resistance and Capacity" (page 120), and the chapter (VII) on "High-Frequency Cymometer Measurements" (page 166).

On page 238, the remark is made "In all this we are thinking only of the forced vibrations of the system," and on page 239—"We are now studying this latter part, the forced part only. In most practical engineering problems the exponential terms rapidly disappear."

In regard to Professor Webster's criticism on the use of marginal units, it is undeniable that in pure mathematics, dynamics and physics, such units constitute needless limitations to the equations, perhaps hampering rather than helping the reader. But in mathematics applied to engineering, the insertion of the units greatly assists the reader. In this paper, most of the equations are complex or plane-vector equations; but a few are simple scalar equations. The danger of confusing vectors and scalars is avoided by the use of the marginal unit. What engineer has not wasted ill-spared hours, over technical papers, in striving to discover the formula units connoted but concealed by the writers?