SECTION 2

ADVANCED PRACTICAL RADIO ENGINEERING

TECHNICAL ASSIGNMENT

COMPLEX NOTATION; PART I; OPERATOR J

232E

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COMPLEX NOTATION; PART 1; OPERATOR J

INTRODUCTION.—In previous assignments the solution of alternating current circuits was analyzed on the basis of current and voltage. The methods outlined, although somewhat cumbersome, were considered necessary for a thorough understanding of the basic theory. The student has now had sufficient practice in these methods to make the assimmilation of complex notation a simple matter. With an understanding of the complex number the solution of such circuits is simplified. Since this method of notation is used extensively in the more advanced engineering texts the student will find a knowledge of complex notation a great aid to future study.

The student has observed the growing complexity of the AC problem when the familiar laws of geometry and trigonometry alone are used. It remained for the late Dr. Charles P. Steinmetz of the General Electric Company to foresee the use of the complex number to simplify the involved calculations of a-c circuits. He noted that the solution of certain differential equations suggested the use of the complex operator j in expressing the approximate sine and cosine waveforms of commercial alternating currents.

The use of complex notation simplifies the solution of a-c problems and is employed by practically all modern engineers. It will be evident to the student that this assignment offers nothing new in alternating current theory; that which is new is the compact form in which the voltages, currents, or impedances are expressed and the ready application of the mathematical laws to this notation and the more direct solutions thus made possible.

THE REAL NUMBER.—The earliest mathematicians knew only concrete positive numbers. With the further discovery of the meaning of the negative number and the invention of zero the real number system was complete. The development of the Arabic number system added to the completeness of the system. The real number system is shown graphically in Fig. 1.

It is evident that this system can be extended indefinitely to include all numbers, positive and negative, belonging to the FIELD OF REAL NUMBERS. In Fig. 1 any given number is greater than any number to the left of it and less than any other number to the right. Thus +8 is greater than +4 but less than +9 but +4 is greater than +2 but less than +7. Similarly -4 is greater than -8 but less than -2. This is true in the algebraic sense. Any number has two values, its numerical value and its algebraic value. The numerical value is often called the absolute value and is determined without reference to sign. The algebraic value is the signed value meaning the sign (+ or -) must be included as part of the number. Thus $|15|$ means the absolute value is 15 and may represent either + 15 or -15 in the algebraic sense.

The field of real numbers may be broadly classed as all positive and negative numbers including zero. These two classes may be further subdivided into integral (whole) and fractional. Thus 2, 6, 8 are integral values while 1/2, .5 and 1.78 are fractional numbers. Integral and fractional numbers may be either positive or negative. A further subdivision of the system is rational and irrational numbers. A rational number is one whose value can be exactly expressed as the quotient of two integers. 3, $\sqrt{16}$, -10, and 3.2 are rational numbers as they could be expressed as 3/1, ±4/1, -10/1, and 32/10 respectively. A number whose value cannot be expressed exactly as the quotient of two integers is irrational. Many physical and electrical constants and all expressions of roots that cannot be stated exactly are irrational. For example, $\sqrt{2}$, π, $\pi/2$, $\sqrt[3]{16}$, and ω are irrational numbers. From the above definitions it is evident that all positive and negative numbers, whether integral or fractional, rational or irrational, belong to the field of REAL numbers as shown in Fig. 1.

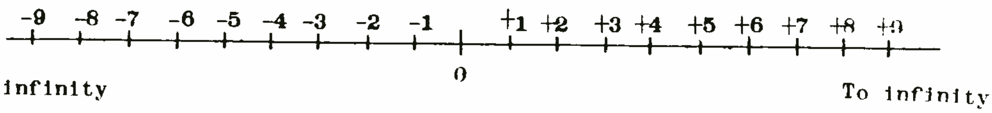

Fig. 1.—Graphical representation of real number system.

EXERCISES

1. Arrange the following in ascending order of their *absolute* magnitude: -18, 2π 7.2, $-\pi$, 1.78 $-\sqrt{11}$, $10^{1.3}$, log 75.

2. Arrange the numbers in Problem 1 in the descending order of their *algebraic* magnitude.

3. Indicate whether the following are integral or fractional: 10^{-2}, π, 1.36, $\sqrt{6}$, 3/2, -4.

4. Indicate whether the following are rational or irrational: $\sqrt{2\pi}$, $\sqrt{64}$, $\sqrt[3]{64}$, $\sqrt[6]{64}$, $\sqrt[3]{27}$, $\sqrt[5]{125}$.

5. Which of the following are (a) integral, (b) fractional, (c) rational, (d) irrational (e) positive, (f) negative: -16, 3/2, $-\sqrt{2}$, $\sqrt{A^2}$ (A is any real number), 19.26, $\sqrt{3^2 + 4^2}$, $\sqrt{18/27}$.

THE IMAGINARY NUMBER.—The solution of certain quadratic equations yields a result that has no meaning where real numbers are considered. For example the equation $X^2 + 4 = 0$. Transposing, $X^2 = -4$ and extracting square root of both sides $X = \pm\sqrt{-4}$. But there is no real number which when multiplied by itself (squared) will give -4.

Evidently the number $\sqrt{-4}$ cannot belong to the real number system. When this kind of number first appeared in the solution of quadratics the solution was rejected because it had no "real" significance. This suggested the number belonged to an "imaginary" system hence the name imaginary number. The name is to be regretted because, as will be shown, it is not imaginary and has a very definite significance. The general definition of the imaginary number is: *the indicated EVEN root of a negative number.*

Some indication of the meaning of the imaginary number appeared in the writings of early 18th century mathematicians but it remained for Argand, a Frenchman, and Wessel, a Norwegian, to explain their meaning. The graphic representation of imaginary numbers on a complex plane is due to them. About the same time, Gauss, a German mathematician, published a paper proving that every algebraic equation has a root of the form A + Bi where $i = \sqrt{-1}$ which showed the general nature of the complex number. Gauss placed the study and use of the complex number on a scientific basis. The number $A + B\sqrt{-1}$ will be a subject of considerable significance in what follows.

When the significance of the imaginary number was understood, it was only a short time until the real number system was extended to include the imaginaries.

To show the reasonableness and consistency of the interpretation of the real and imaginary numbers, the following geometric and algebraic argument is presented. In electrical work "i" is used as a symbol for current so in this discussion the letter "j" is used to represent the $\sqrt{-1}$.

Fig. 2 shows the familiar plane of reference used in plotting rectangular coordinates. YOY' is the axis of imaginary numbers while XOX' is the axis of real numbers. All values to the right of 0 are positive, all values to the left negative. Since the axis XOX' can be extended indefinitely any real number can be plotted on it if the proper scale is selected. The distance OX will be taken as the unit of the real number system. If the unit length OX is rotated 180°

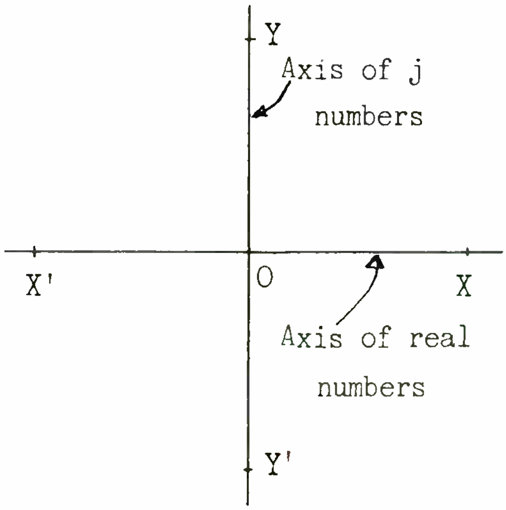

Fig. 2.—Plane used to plot rectangular coordinates.

counter-clockwise (conventional positive rotation) it will occupy the position OX' and will now represent the real number -1. The geometric operation of rotating OX 180° is equivalent to multiplying OX by -1. If a second geometric rotation of 180° is again performed on OX' it will return to the original position OX. This is equivalent to the algebraic multiplication of OX' by -1. *Therefore rotation through 180° is the geometric operator which produces the same result as the algebraic multiplying operator -1.*

The operation of rotation through 180° may be regarded as the result of two equal rotations of 90° performed in succession on OX. THE EQUIVALENT ALGEBRAIC OPERATOR TO PRODUCE THE SAME RESULT MUST BE MULTIPLICATION BY A NUMBER WHICH USED TWICE WILL PRODUCE -1. THE

ONLY POSSIBLE NUMBER WHICH WHEN MULTIPLIED BY ITSELF EQUALS -1 IS $\sqrt{-1}$. Thus $\sqrt{-1} \cdot \sqrt{-1} = -1$. Therefore the operation on OX by the algebraic operator $\sqrt{-1}$ will produce the same result as the operation of rotating OX 90° geometrically. Therefore the $\sqrt{-1}$ is an OPERATOR which rotates any vector 90° in a positive (counter-clockwise) direction. $\sqrt{-1} = i = j$, hence the name "operator j."

In Fig. 2 YOY' is the axis of the imaginary numbers because any real number rotated 90° will fall on this axis. Hence all j numbers may be represented on this axis. The positive j numbers are plotted above zero while the negative j numbers will be plotted below zero. The $\sqrt{-1}$ or j is taken as the unit of the imaginary number system. Both systems are plotted at right angles and have only the point ZERO in common. It will be seen that there is nothing mysterious or imaginary about the j number and a solution of the quadratic equation $X^2 + 4 = 0$ is now possible.

$$X^2 + 4 = 0$$

$$X^2 = -4$$

(Considering only the plus root.) $X = \sqrt{-4}$

$$X = \sqrt{4} \cdot \sqrt{-1}$$

$$X = 2\sqrt{-1}$$

Since $j = \sqrt{-1}$ then $X = 2j$

Since it is general practice to write the j preceding the number then $X = j2$ and this may be plotted in Fig. 2 as 2 units above zero on the vertical axis YOY'. A REAL NEGATIVE NUMBER CANNOT HAVE A REAL EVEN ROOT so all j numbers are at once defined. Thus $\sqrt{-9} = 3\sqrt{-1} = j3$, $\sqrt[2]{-16} = \sqrt[2]{16} \cdot \sqrt[2]{-1} = j4$, $\sqrt[6]{-64} = j2$ etc. ODD ROOTS OF NEGATIVE NUMBERS ARE REAL AND NEGATIVE, hence these numbers cannot be classed as j numbers. For example $\sqrt[3]{-27} = -3$, $\sqrt[5]{-32} = -2$, etc.

EXERCISES

Classify the following as belonging to the real number or j number system:

6. -16, $\sqrt{10}$, $-\sqrt{16}$, $3\sqrt{-6}$, $21\sqrt{2}$.

7. 16, π, $-27^{-\frac{1}{3}}$, $-27^{-\frac{1}{2}}$

   $(-27)^{-\frac{1}{2}}$

8. $\sqrt{A^2 - B^2}$ (Where $A > B$ and both are real numbers.)

9. $\sqrt{A^2 - B^2}$ (Where $A < B$.)

10. $\sqrt{A^2 - B^2}$ (Where $A = B$.)

11. $j8$, $j^2 15$, $j^3 20$, $\sqrt{j^2}$,

    $5 + j^2 3$.

PLOTTING REAL AND J NUMBERS.— With the real number axis (X axis) representing graphically the real number system, the position of any real number, positive or negative, may be readily plotted. The real numbers $+4$, π, -3, and $-1/2$ are represented on the real axis of Fig. 3. In exactly the same way the j numbers $-j$, $j4$, $j\sqrt{2}$, $-j2.5$ and $j\pi$ may be plotted (Fig. 3) on the Y or vertical axis. Other numbers would occupy corresponding positions depending on their value. It is usual, in practice, to use the same length for the unit of the j number system as for the real number system but this is not absolutely essential.

THE COMPLEX NUMBER.—The solution of the quadratic equation $I^2 = \underline{P/R}$ yields the positive root $I = \sqrt{P/R}$. As long as P and R are positive, I will be positive and real. For example, 160 watts dissipated in a 10 ohm resistance, $I = \sqrt{160/10}$ or 4 amperes. The solution yields a positive *pure real* number.

The solution of the quadratic equation $X^2 = -4$ gives the principal root $X = \sqrt{-4}$ or j2 where $\sqrt{-4} = \sqrt{-1}\,\sqrt{4} = j\sqrt{4} = j2$. This answer is mathematically a positive *pure j* number, CONSIDERING ONLY THE POSITIVE ROOT.

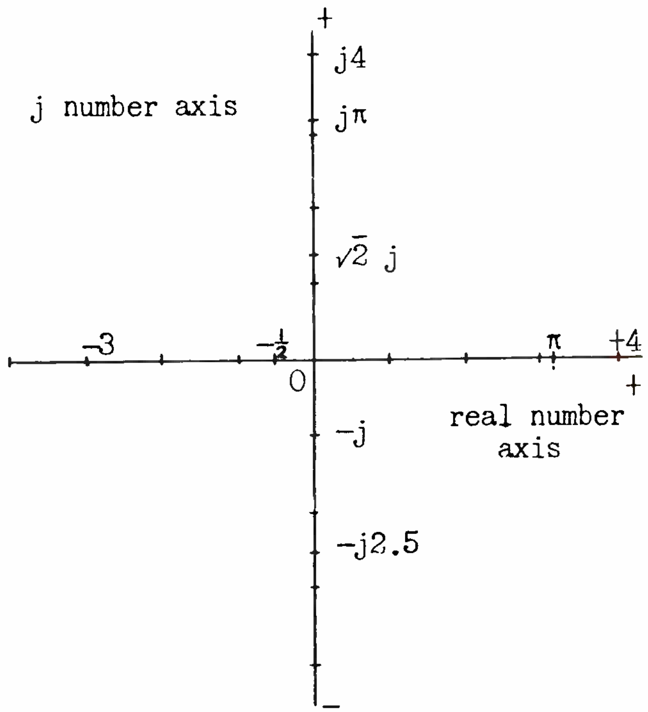

Fig. 3.—Plot of real and j numbers on the axis.

The solution of the equation $X^2 + 2X + 5$ gives the roots $X = -1 + j2$ and $X = -1 - j2$. This solution is readily effected by the formula for the solution of a quadratic equation of the form $aX^2 + bX + c = 0$, where a, b, and c are real numbers (See Algebra assignments). The solution of the equation $X^2 + 2X + 5$ is then obtained as follows:

$$a = 1,\ b = 2,\ c = 5$$

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$X = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$$

$$X = \frac{-2 \pm \sqrt{-16}}{2}$$

$$X = \frac{-2 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$X = -\frac{2}{2} \pm \frac{4\sqrt{-1}}{2}$$

$$X = -1 \pm 2\sqrt{-1}$$

$$X = -1 + j2 \text{ or } -1 - j2$$

It is important to note that the roots of $X^2 + 2X + 5$ are a combination of a real and a j number. Such a number is called a complex number and is the general expression for any number. If the j part of the expression for a number is zero, then what is left is the familiar real number. Thus the real number 10 can be written as 10 + j0 or -16 = -16 + j0. It is evident there is no necessity for writing the j0 part of such a number. With the real part of the number equal to zero, the complex number becomes a pure imaginary such as -j4 or +j3 which could have been written as 0 - j4 and 0 + j3.

To plot the complex number on the Argand diagram, it is only necessary to find the point whose coordinates are the real part and the j part. The complex numbers 3 +j4, -3 + j5, -2 -j5 and 5 - j3 are plotted as examples on the rectangular coordinate plane of reference in Fig. 4.

EXERCISES

Plot the following as rectangular coordinates:

12. 3 - j6, 5 + j5, 16 + j0, $-\sqrt{16}$

13. j8, -2 - j13, -12 - j12, -3 + j10

14. -1 - j10, -6, 10 + j^2, 15 + j, 17 - j^3

Find the roots of the following and express as complex numbers:

15. $3X^2 + X - 17 = 0$

16. $X^2 - 9X + 8 = 0$

17. $7X^2 + 2X + 12 = 10$

18. $4X^2 + 6X = 24$

19. $X^2 - X - 1 = 0$

20. $X^2 + 2X = 2X - 3X^2 - 64$

APPLICATION OF THE OPERATOR J.—The student has observed that the j axis is at right angles to the real number axis and that the complex number is nothing more than a method of stating the rectangular coordinates of a point. He has seen the effect on a real number of the rotational operator j. For example, the real number 10 when operated on (that is, multiplied by) by j becomes j10 and is now at right angles to its former position on the Argand diagram. Operation by j a second time produces $j \cdot j10$ or $j^2 10$. Since $j = \sqrt{-1}$ then $j^2 = (\sqrt{-1})^2$ or -1 and hence $j^2 10 = -1 \cdot 10$ or -10. These two operations by j have produced a 180° rotation. A third operation will produce -j10 again a 90° rotation and a fourth operation $-j^2 10 = -(\sqrt{-1})^2 10 = -(-1)\ 10 = 1 \times 10$ or 10. This fourth operation brings the number to its original position. The evident conclusion is that +j is a rotational operator producing a 90° counter-clockwise rotation on any number on which it operates. By a similar process it can be shown that -j is an operator producing a CLOCKWISE rotation of 90° on any number.

This rotational property is not confined to the use of a pure real or pure j number. It affects complex numbers in the same way. The complex number 3 + j4 is shown graphically in Fig. 5. The effect of operating on 3 + j4 is also shown. The complex number -4 + j3 is the result of multiplying 3 + j4 by j. The operation is performed as follows: $j(3 + j4) = j3 + j^2 4 = j3 + (-1)4 = j3 - 4 = -4 + j3$. Operation on 3 + j4 by -j would produce a 90 degree CLOCKWISE rotation producing the complex number 4 - j3 by the same process. It will be seen that the operator j produces a 90° counter-clockwise rotation while the operator -j produces a 90° clockwise or -90° rotation.

The cyclic variations of successive operations with the operator j shows that $j = \sqrt{-1}$, $j^2 = -1$, $j^3 = -\sqrt{-1}$ and $j^4 = +1$, etc. The effect of any number of operations may be concisely stated: $j^{4n+a} = j^a$ where n is the number of complete cycles and the "a" part of the exponent is the remainder of uncompleted operations for any value of exponent of j. Thus the effect of 14 operations by j or j^{14} is $j^{4(3)+2} = j^2 = -1$ which means that the number has made three complete rotations plus one additional 180° rotation. Hence 14 operations by j is the same as 2 operations or multiplication of the original number by -1. In the expression j^{4n+a}, the value of n is found by dividing the exponent of j by 4 and using the REMAINDER as the new exponent of j. By similar

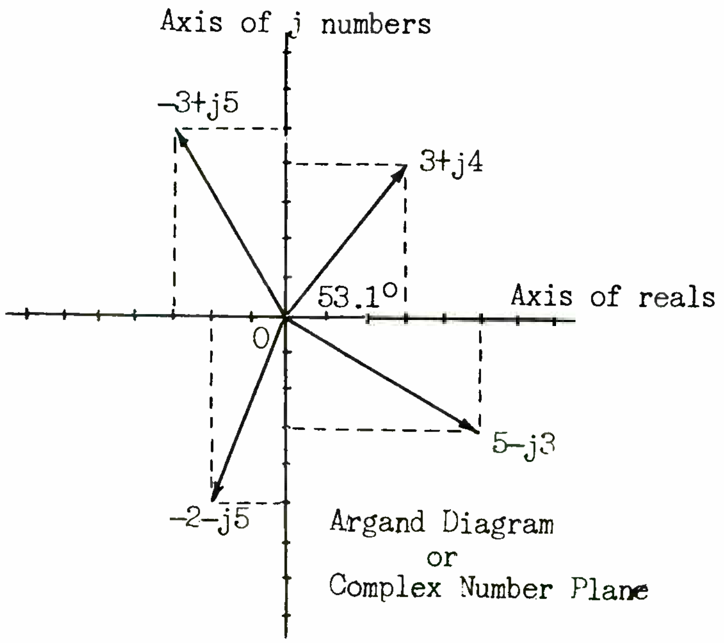

Fig. 4.—Plotting of complex numbers, Argand Diagram.

reasoning $j^7 = j^3 = -j$, $j^9 = j$; etc.
The following powers of j are of great importance and should be committed to memory:

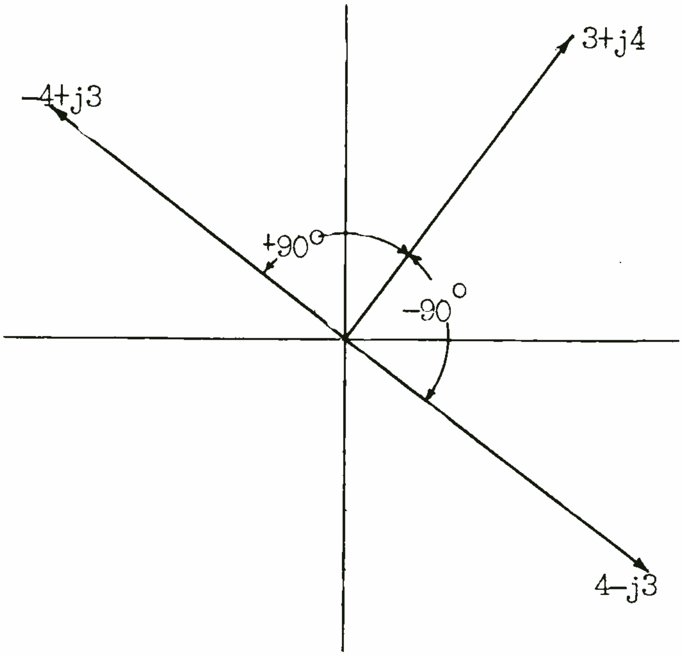

Fig. 5.—Operating on 3 + j4, multiplication by j.

$$j = \sqrt{-1}$$

$$j^2 = -1$$

$$j^3 = -\sqrt{-1}$$

$$j^4 = 1$$

USE OF OPERATOR J IN VECTOR ANALYSIS.—A vector has been defined as a quantity having both magnitude and direction. It has also been shown that a vector can be completely defined in either rectangular or polar coordinates. (See Vector Analysis). The rectangular coordinate system gives the abscissa and ordinate values of the end point of the vector, where the initial point is understood to be located at the point of origin. In polar coordinates the distance of the end point of the vector from the point of origin is taken as the magnitude of the vector. The direction is given by the angle between the vector and the horizontal axis to the right of the vertical bisector or Y axis. In this case positive angles are measured in a counter-clockwise direction from OX (Fig. 2) while negative angles are measured in a clockwise direction. Polar coordinates are readily converted to rectangular coordinates by the rules of trigonometry. Thus a vector inclined 53.1° to the positive real axis and 5 units long would be exactly defined in terms of the real and j axis coordinates which are 3 and j4. (See Fig. 4). In this case the real part is the adjacent side of a right triangle having a hypotenuse 5 units long. Adjacent side equals hypotenuse times cosine of the angle or

5 Cos 53.1 = 3 units. The j part is the opposite side of the same right triangle or 5 Sin 53.1 or j4 units (j4 units because the ordinate is plotted on the j axis). Obviously complex notation is a very convenient method to use with vectors.

Specifically, voltages and currents, as treated in electrical work, are time vectors. The effect of reactance is such that it causes a 90° relationship between it and the effects of resistance. The component of line current through a resistance is always in phase with the voltage across it. This resistive drop component of voltage is taken as the real part of the complex expression for the voltage in a series circuit. The current through any reactance either leads or lags the applied voltage by 90° depending on the character of the reactance. The component of the line voltage across the reactance is taken as the j part of the complex number. For example, if there exists in a series circuit a resistive drop of 30 volts and an inductive drop of 40 volts, then the vector expression for the voltage is E = 30 + j40 volts. If the reactive drop had been capacitive instead of inductive only the sign of the j term would be changed or E = 30 - j40 volts. The line voltage is the vector sum of these voltages or $\sqrt{30^2 + 40^2} = 50$ volts. In the first case the line current would lag the line voltage, in the second I would lead E. The vector diagram for the two cases is shown in Fig. 6.

The same notation may be applied to current. In the parallel circuit the applied voltage is the same across all branches and the line current is the *vector* sum of the branch currents. In a two branch circuit in which one branch contains only resistance and the other only inductance the line current may be

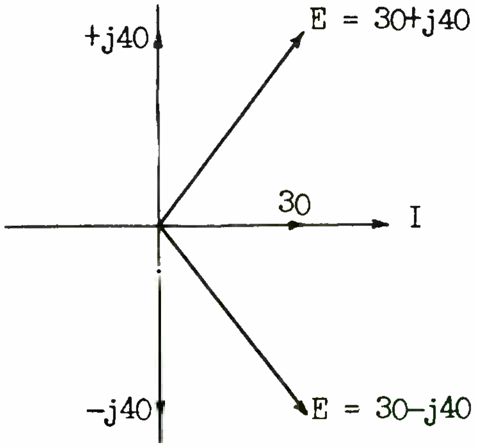

Fig. 6.—Plot of I leading or lagging line voltage (I reference).

expressed as the vector sum of the two branch currents. The line voltage is taken along the axis of reals and the two currents are plotted in reference to it as shown in Fig. 7.

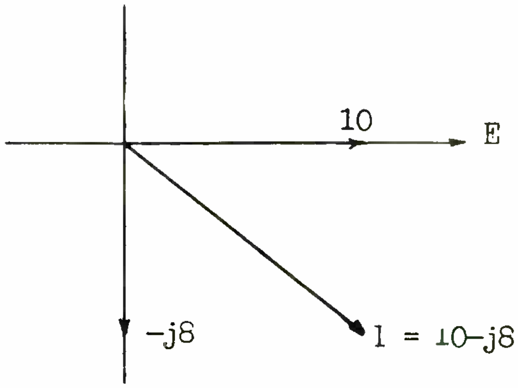

Fig. 7.—Plot of I lagging line voltage (E reference).

The resistive branch current is 10 amperes and is in phase with the line voltage and will therefore fall on the real axis. The inductive branch current is 90° lagging and so is taken along the -j axis. This current is 8 amperes and is written as -j8. The complex expression for the two currents is I = 10 - j8 amperes and the line current is the vector sum of the branch currents is $\sqrt{10^2 + 8^2}$ = 12.8 amperes. The line current lags the line voltage.

Summarizing, a resistive component of voltage or current is seen to be the real part of the complex expression for E or I and the reactive component is seen to be the j part of the expression. It must not be assumed that the line voltage or line current will always fall on the positive real axis. In the cases cited, they have been so taken for ease of explanation.

EXERCISES

Determine the value of the following in terms of j less than j^4

21. j^5, j^8, j^{16}, j^{35}, j^{111}.

22. Locate the vector 5 - j7 on the complex plane. State the complex number representing the vector for each of three successive operations by j.

23. A certain voltage is the vector sum of 55 volts resistive component and 88.7 volts inductive component. The line current is taken along the real axis. Write this voltage in the complex form and draw a vector diagram to show the relation of E and I. Any value of I may be assumed.

24. A voltage is found to be 19 - j5 volts while the current is found to be 8 - j8 amperes. Locate these on the complex plane. Is the circuit inductive or capacitive?.

25. A technician reported the voltage in a circuit as 10 - j4 volts and the current resulting from that voltage as 1 + j10 amperes. Plot E and I and show that the values stated give an impossible result.

26. The voltage is given as 173.2 + j100 volts and the current as 5 + j8.66 amperes.

(a) Plot E and I and determine angle of lead or lag.

(b) Determine magnitude of E and I.

27. In a certain problem the current is expressed as 15 Cos 40° + j15 Sin 40°. Reduce this expression to the explicit rectangular form and plot.

28. i = 10 [Cos (-10°) + j Sin (-10°)]. Reduce this expression to the explicit rectangular form and plot.

29. The current is of unit value and taken along the real axis. This unit current flows through a 10 ohm resistance in series with a 200 millihenry inductance. The frequency of the supply voltage is 20 cycles per second. Write the expression for the supply voltage in the rectangular form.

30. In a series circuit containing L, C, and R the resistance drop is 60 volts, the inductive reactive drop 48 volts and the capacity reactive drop 80 volts. Write the complex expression for line E and plot on the complex plane.

IMPEDANCE IN THE COMPLEX FORM.—The impedance of a circuit is the combined opposition to current flow of reactance and resistance. The calculation of impedance when R and X are known has been explained in earlier assignments. Determination of angle of lead or lag was also covered in these same assignments. The complex notation employed for expressing current or voltage vectors can also be applied to impedance with equal facility. However impedance is not a vector but rather an operator. That is, impedance so operates on the voltage applied to a circuit to allow a current of definite magnitude and phase difference angle in respect to the applied voltage. This difference does not impair the usefulness of complex notation in connection with impedance.

Since resistance acts on the voltage or current to cause a phase difference angle of zero degrees (in phase), the resistive component of the impedance will be taken along the axis of real numbers. Similarly since reactance causes a 90° phase difference angle between E and I, this component will be taken along the j axis. INDUCTIVE REACTANCE IS ALWAYS CONSIDERED POSITIVE AND CAPACITIVE REACTANCE NEGATIVE. Thus, in complex notation a circuit which has resistance and inductive reactance will be written $Z = R + jX$ ohms while for a circuit with capacitive reactance it will be written $Z = R - jX$ ohms. It is evident from the above that the general form for impedance in rectangular coordinates is $Z = R + j(X_L - X_c)$ where X_L and X_c are the magnitudes of the reactances. For example, consider a circuit of 24 ohms resistance and 19 ohms of inductive reactance. This is written $Z = 24 + j19$ ohms. Another circuit has 8 ohms of resistance and 30 ohms of capacitive reactance. This is expressed $Z = 8 - j30$ ohms. In a circuit containing both inductive and capacitive reactance, the usual conventions are followed to determine the predominating reactance by taking the *algebraic* sum of the reactances. A coil, capacitor, and resistance are connected in series, $X_L = 36$ ohms, $X_c = 22$ ohms, and $R = 16$ ohms. The impedance would be written $Z = 16 + j\ (36 - 22)$ or $Z = 16 + j14$ ohms. If $X_c = 36$ ohms and $X_L = 22$ ohms then $Z = 16 + j(22 - 36) = 16 + j(-14) = 16 - j14$ ohms.

When the inductance or capacity is given together with the frequency of the applied voltage, the reactance is computed in the usual manner. For a coil of 2 henries connected across a 60 cycle voltage source $X_L = 2\pi FL$ or 754 ohms. A 2 µF condenser in the same circuit would offer a reactance of $X_c = 1/2\pi FC$ or 1325 ohms. Since inductive reactance is associated with a plus sign and capacitive reactance with a minus sign the reactance would be written as $+j754$ ohms and $-j1325$ ohms. If the coil and condenser were in series with a 100 ohm resistance the total impedance would be in complex notation $Z = 100 + j(754 - 1325) = 100 + j(-571) = 100 - j571$ ohms.

Reflection on what has been discussed so far in this assignment will reveal that complex notation is simply a system for writing the X and Y components of vectors. It will be recalled that the resultant of several vectors was calculated by determining the horizontal (X) and vertical (Y) component of each vector, adding these components in the proper algebraic form and then determining the vector which is the resultant of the total vertical and horizontal components. This last step was made by applying the Right Triangle Theorem. In the system of writing vectors by the real and j axis components, the last operation mentioned above is not taken when the result is to be left in the rectangular coordinate form, that is, the vector is considered to be completely defined in terms of its X and Y components. Just why this is done will be evident from the discussion of addition and subtraction of complex numbers to follow.

EXERCISES

31. A series circuit contains 80 μH inductance and .003 μF capacity. R = 120 ohms and F = 400 KC/s. Write the impedance in complex form.

32. Write the complex expression for the impedance of the circuit in Problem 31 at resonance.

33. Write the complex expression for the impedance of the circuit in Problem 31 at 100 KC/s below resonance. R is assumed to be constant.

34. Write the complex expression for the impedance of the circuit of Problem 31 at 50 KC/s above resonance. R is assumed to be constant.

ADDITION OF COMPLEX NUMBERS.-It has been shown that voltages, currents and impedances can be written in the complex form. It is often necessary to combine factors of the above by addition or subtraction. In earlier assignments this was done by geometric addition. The same fundamental idea is preserved with complex numbers. If the voltages, currents or impedances are expressed in rectangular coordinates their addition or subtraction is relatively simple. If not in rectangular coordinate form it will be necessary to put the expressions in that form before carrying out the process of addition or subtraction.

In earlier work of adding or subtracting quantities representing voltage, current or impedance it was necessary to find the X and Y axis components. From what has been shown it is evident the complex expressions for these quantities are already expressed as the geometric sum of the X (real) and Y (j) components. Thus all that is necessary is to add the real and j parts separately and express the sum in the same form. Algebraic addition is used meaning the sign of the terms must always be considered. It must be remembered that ONLY LIKE THINGS CAN BE ADDED. Never attempt to add voltage to current, current to impedance, or some other improper combination.

Two voltages 30 + j40 and 25 + j25 are added as follows

$$\begin{array}{r} 30 + j40 \\ \underline{25 + j25} \\ 55 + j65 \end{array}$$

Write the expressions in the usual manner for algebraic addition. Add the real and j parts separately taking cognizance of the sign of each term.

The sum is 55 + j65 volts. The results of this addition are shown graphically in Fig. 8. Note that the result is the familiar parallelogram of forces. Other examples are

Add

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10 - j14
 5 + j11
16 + j4
 3 - j8
34 - j7
```

Add

```
 3 - j16.4
10 - j9.0
14 + j0.0
 0 + j26.5
27 + j1.1
```

Where one or the other component is zero, it is not necessary to write

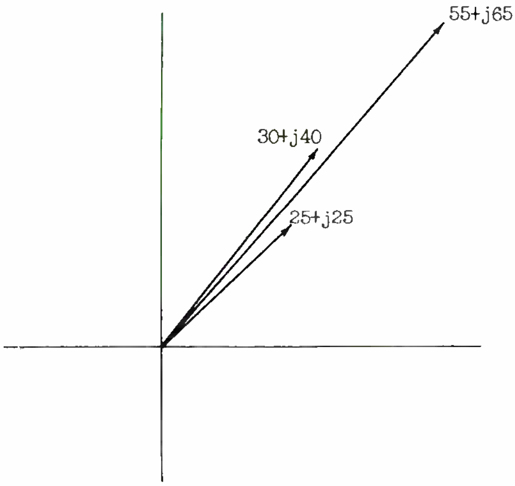


Fig. 8.—Addition of vectors.

that part as zero but by so doing the chances of error are reduced. Normally 0 + j26.5 is written simple j26.5.

RULE: ADD THE REAL PARTS OF EACH VECTOR WITH DUE REGARD TO SIGN OF EACH TERM AND WRITE THE SUM AS THE REAL PART OF THE ANSWER. ADD THE J PARTS OF EACH VECTOR WITH DUE REGARD TO SIGN OF EACH TERM AND WRITE THE SUM AS THE J PART OF THE ANSWER.

EXERCISES

Add

35.
```
  4 + j9
 -7 + j8
    - j10
 18 - j22
```

36.
```
  7 - j91
 27 + j42
  0 - j7
-20 + j51
```

37.
```
  0 - j4.3
7.9 - j26.1
 43 - j72
  8 + j37
```

38.
```
 17 - j26.1
 49 + j18.3
-27 + j29
 14 - j31
```

39.
```
111 - j1356
 79 + j899
167 - j742
  0 + j641
```

40.
```
  12 - j.76
13.8 - j7.89
  72 + j64.1
  37 + j19.4
```

SUBTRACTION OF COMPLEX NUMBERS.—Two complex numbers may be subtracted by subtracting the real parts and the j parts separately. Algebraic rules for subtraction are followed. For example subtract 10 + j40 from 60 + j30. Write the two numbers in the usual way for subtraction.

$$\begin{array}{r} 60 + j30 \\ \underline{10 + j40} \\ 50 - j10 \end{array}$$

Subtract the real and j parts separately. The result of this subtraction is shown geometrically in Fig. 9. To subtract vectors geometrically rotate the vector represented

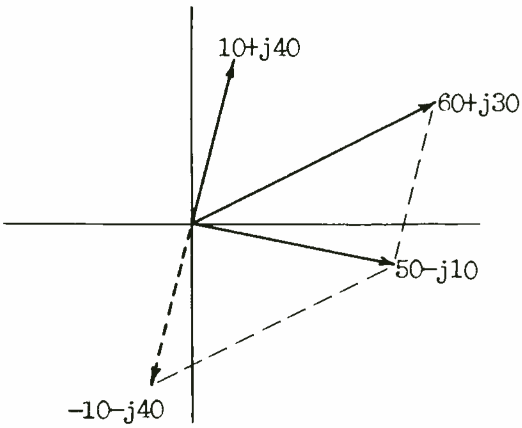


Fig. 9.—Subtraction of vectors.

by the subtrahend 180° and then add to the minuend. To reverse the vector 10 + j40 it is necessary to multiply by j^2 or -1, This gives -10 - j40 as shown by the dotted line vector in Fig. 9. Adding -10 -j40 to 60 + j30 gives 50 - j10. Again the familiar parallelogram of forces is evident. The results obtained algebraically are the same as those obtained geometrically. The complex number shows the intimate connection between algebra and geometry as applied to vectors of time or space.

EXERCISES

41. 37 - j40 from 180 + j90

42. 62 - j18.9 from 70 - j43.6

43. 73 + j48 from 169 - j65

44. -107 - j19 from 260 - j95

45. 17.9 + j74 from -8 + j19

Check the answers to the above problems by means of geometric subtraction.

MULTIPLICATION OF COMPLEX NUMBERS.—The multiplication of complex numbers involves the same fundamental algebraic ideas as multiplying any polynomial expressions. It is suggested that the student review algebraic multiplication as given in Algebra, if there is any difficulty experienced in understanding the following explanation.

The following side by side examples will show the similarity between multiplying (2X + 3Y) by (X - 2Y) and (2 + j3) by (1 - j2).

(a)

Step 1	$2X + 3Y$
Step 2	$\underline{X - 2Y}$
Step 3	$2X^2 + 3XY$
Step 4	$\underline{- 4XY - 6Y^2}$
Step 5	$2X^2 - XY - 6Y^2$

(b)

Step 1	$2 + j3$
Step 2	$\underline{1 - j2}$
Step 3	$2 + j3$
Step 4	$\underline{- j4 - j^2 6}$
Step 5	$2 - j1 - j^2 6$

Note the similarity of the steps. Step 1 consists of writing the multiplicand. Step 2 consists of writing the multiplier directly under the multiplicand in the usual manner. Step 3 is the first series of partial products obtained in (a) by multiplying (2X + 3Y) by X and in (b) by multiplying 2 + j3 by 1. Step 4 is the second series of partial products obtained in (a) by multiplying (2X + 3Y) by -2Y and in (b) by multiplying (2 + j3) by -j2. The partial products are arranged in columns of similar terms. Step 5 is the result when the partial products are added. The final step for the multiplication in (b) is the evaluation of the j^2 term. $J^2 = (\sqrt{-1})^2 = -1$ and substituting this value in the expression $2 - j1 - j^2 6$ gives $2 - j1 - (-6) = 2 - j1 + 6 = 8 - j1$ or simply 8 - j the 1 being understood in the j term. At times, when dealing with circuits in general, it is convenient to write the quantity in general terms such as $Z = R + jX_L$ for the impedance of a certain circuit and $Z = R - jX_c$ for another circuit. If the product of these impedances is desired they are multiplied in the same manner as in the previous examples.

$$
\begin{array}{llll}
R + jX_L & & & \\
\underline{R - jX_c} & & & \\
R^2 + jRX_L & & & \text{(Partial product for } R(R + jX_L)) \\
\underline{\qquad\qquad - jRX_c - j^2X_LX_c} & & & \text{(Partial product for } -jX_c(R + jX_L)) \\
R^2 + jRX_L - jRX_c - j^2X_LX_c & & & \text{(Sum of partial products)}
\end{array}
$$

$jRX_L - jRX_c$ can be factored thus $j[R(X_L - X_c)]$ and $-j^2X_LX_c = -(-1) X_LX_c = +X_LX_c$. The real terms may be written together (and usually are) so the final product would be written as $R^2 + X_LX_c + j R[(X_L - X_c)]$. The real part of this expression is $R^2 + X_LX_c$ and the j part $j R[(X_L - X_c)]$.

It does not always happen that both the real and j parts will appear in the answer. In fact, a very important application (to be shown later) is illustrated in the following problem. Find the product of 13 + j8 and 13 - j8.

$$
\begin{array}{l}
13 + j8 \\
\underline{13 - j8} \\
169 + j104 \\
\underline{\qquad - j104 - j^2 64} \\
169 + j0 \quad - j^2 64 =
\end{array}
$$

$$169 - (-1 \cdot 64) = 169 + 64 = 233$$

The product of (13 + j8) (13 - j8) yields a product entirely real.

Another case where the product of two j numbers gives a real number is the product of any two pure reactances. For example jX_L times $-jX_c$ gives $-j^2X_LX_c$ or $-(-1) X_LX_c = X_LX_c$ a real number.

In the parallel circuit one method of solution takes the form of

$$\frac{Z_1 Z_2}{Z_1 + Z_2}$$

where Z_1 and Z_2 are the individual branch impedances. Fig. 10 shows the circuit. The impedance of the inductive branch is $R + jX_L$. The capacitive branch is considered to have zero loss (Resistance) so the impedance will be pure capacitive reactance $-jX_c$. The impedance of the circuit will then be:

$$Z = \frac{(R + jX_L)\ (-jX_c)}{R + jX_L + (-jX_c)} = \frac{X_L X_c - jRX_c}{R + j\,(X_L - X_c)}$$

Both addition and multiplication were used in the above since

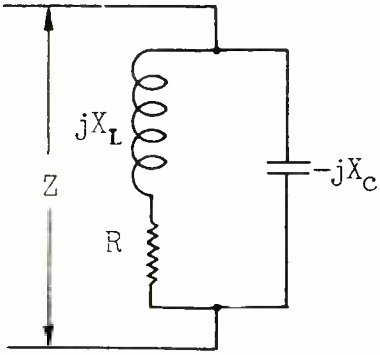


Fig. 10.—Sample parallel circuit showing notation used.

the numerator is the product of the branch impedances and the denominator their sum.

This result will be the subject of further discussion later in this assignment.

EXERCISES

46. $(10 - j9)\ (12 + j7) = ?$

47. $(13 + j22)\ (13 - j22) = ?$

48. $(29 - j16)\ (32 - j16) = ?$

49. $(7.8 + j14)\ (9.6 + j6.8) = ?$

50. $3(1 + j9)\ (2 - j7)\ (0 + j14) = ?$

DIVISION OF COMPLEX NUMBERS.— Division, the inverse of multiplication, is not quite as simple as multiplication of complex numbers. When the complex numbers are in rectangular coordinates the division is indicated in the usual fractional form. Thus to divide 3 + j4 by 2 + j3 the division is indicated by writing

$$\frac{3 + j4}{2 + j3}$$

The numerator and denominator are then multiplied by the CONJUGATE of the denominator to complete the division. The conjugate of the denominator is the denominator with the sign of the j term reversed. Thus the conjugate of 2 + j is 2 - j; of 3 + j4 is 3 - j4; of R + jX is R - jX; etc. The division of 3 + j4 by 2 + j3 is carried out as follows:

$$\frac{3 + j4}{2 + j3} \times \frac{2 - j3}{2 - j3} = \frac{(3 + j4)\ (2 - j3)}{(2 + j3)\ (2 - j3)} =$$

$$\frac{18 - j}{13}$$

The resulting fraction

$$\frac{18 - j}{13}$$

is then written as two fractions by splitting the numerator into a real part and a j part:

$$\frac{18}{13} - \frac{j}{13}$$

This can be further simplified by reducing the fractions to decimals and is the preferred method

$$\frac{18}{13} - j\frac{1}{13} = 1.38 - j.077$$

It will be seen that this is a

different kind of division than ever before encountered, except when dealing with fractions having a radical in the denominator. *The process of division is considered accomplished when the j number has been removed from the denominator.*

This method of dividing complex numbers is normally employed in practice only when dealing with general algebraic expressions involving literal quantities only. For the division of circuit values another method of writing the complex number will be used and the division accomplished in a different manner. Where literal complex numbers are involved division by the conjugate method is usual. A knowledge of this method is required to understand the derivation of complex formulas often found in advanced textbooks. Suppose it is necessary to find the reciprocal of $R + jX_L$. This is accomplished as follows:

$$\frac{1}{R + jX_L} = \frac{1}{R + jX_L} \cdot \frac{R - jX_L}{R - jX_L} =$$

$$\frac{R - jX_L}{R^2 + X_L{}^2}$$

This may be split into a real and a j part thus,

$$\frac{R}{R^2 + X_L{}^2} - j\frac{X_L}{R^2 + X_L{}^2}$$

The process outlined above for the general case may be specifically applied to any problem with numerical coefficients. Consider the parallel circuit with circuit values as shown in Fig. 11. The values shown are typical of modern tank circuit design where a moderate VA/W ratio is permissible. In this problem R is considered constant for frequencies off resonance. This is

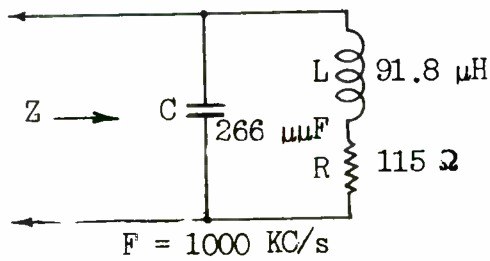


Fig. 11.—Parallel circuit with actual values given.

not strictly true in practical circuits because R is a function of frequency but will be satisfactory to demonstrate the practical application of complex numbers in a-c problems.

At a frequency of 1000 kc

$$X_c = \frac{10^6}{6.28 \times 1 \times 266} = 598.8$$

Note: When frequency is expressed in megacycles and capacity is micromicrofarads the capacitive reactance formula may be written

$$X_c = \frac{10^6}{2\pi F_{mc} \; C_{\mu\mu F}} \text{ ohms}$$

The coil reactance at 1000 KC/s is

$$X_L = 6.28 \times 1 \times 91.8 = 576.5 \text{ ohms}$$

Note: When frequency is in megacycles and inductance in microhenries the inductive reactance formula may be written

$$X_L = 2\pi F_{mc} \; L_{\mu H}$$

The impedance of the capacitive branch is Z_c = -j598.8 ohms and of the inductive branch Z_L = 115 + j576.5 ohms. The total parallel circuit impedance at 1000 KC/s is

$$Z = \frac{(R + jX_L)\quad(-jX_c)}{(R + jX_L) + (-jX_c)} =$$

$$\frac{X_L X_c - jRX_c}{R + j(X_L - X_c)}$$

$$\frac{(576.5 \times 598.8) - j(115 \times 598.8)}{115 + j(576.5 - 598.8)} =$$

$$\frac{345{,}000 - j68900}{115 - j22.3}$$

The entire fraction is then multiplied by the conjugate of the denominator to complete the division.

$$Z = \frac{(345.000 - j68900)(115 + j22.3)}{(115 - j22.3)(115 + j22.3)} =$$

$$\frac{41{,}377{,}000 + j230{,}000}{115^2 + (22.3)^2} =$$

$$\frac{41{,}377{,}000 + j230{,}000}{13754} = 2995 + j16.74 \text{ ohms,}$$

for practical work

$$Z = 3000 \text{ ohms}$$

The result of the calculations show 1000 kc is very near to the frequency of resonance because the j term, that is the reactance, is very nearly zero.

For a frequency other than resonance it will be found that the resistive part of the impedance will be decreased while the reactive part of the complex expression will increase. If frequency is taken as 950 KC/s with all other factors remaining constant the reactances of C and L are found to be -630 and 547 ohms respectively. Z_c = -j630 ohms and Z_L = 115 + j547 ohms.

$$Z = \frac{(115 + j547)\quad(-j630)}{115 + j\ (547 - 630)} =$$

$$\frac{45720000 + j20250000}{20100} =$$

$$2270 + j1007 \text{ ohms}$$

The resistive part of the impedance has decreased to 2270 ohms but the reactive part has increased from almost zero to 1007 ohms. Note that the positive sign of the reactance agrees with the theory that a parallel circuit operated below resonance has an impedance that is inductive in character.

Referring to the discussion of Fig. 10, it was shown that the general expression for the impedance of a parallel circuit was

$$Z = \frac{X_L X_c - jRX_c}{R + j(X_L - X_c)}$$

It should now be evident that the next step in clarifying this expression is to eliminate the j number in the denominator of the fraction or in other words to carry out the division. This process is often referred to as rationalizing the denominator. The numerator and denominator will both be multiplied by $R - j(X_L - X_c)$, the conjugate of the denominator. This process will be carried out in detail.

$$Z = \frac{X_L X_c - jRX_c}{R + j(X_L - X_c)} \cdot \frac{R - j(X_L - X_c)}{R - j(X_L - X_c)}$$

The multiplication for the numerator is:

$$\begin{array}{rrrrl}
 & X_L X_c - jRX_c & & & \\
 & R - jX_L + jX_c & & & \text{(Remove parenthesis by multiplying term by term as in any algebraic multiplication.)} \\
\hline
 RX_L X_c & - jR^2X_c & & & \text{(Multiplying by R.)} \\
 -RX_L X_c & & -jX_L{}^2X_c & & \text{(Multiplying by } -jX_L\text{.)} \\
 & & & + jX_L X_c{}^2 + RX_c{}^2 & \text{(Multiplying by } jX_c\text{)} \\
\hline
 & - jR^2X_c & -jX_L{}^2X_c & + jX_L X_c{}^2 + RX_c{}^2 & \text{(Adding partial products.)}
\end{array}$$

The product of the numerator and the conjugate of the denominator is seen to be an expression of four terms one of which ($RX_c{}^2$) is the real number part and the other three are the j part. The j terms may be written as one term by factoring j from each term to get $j(X_L X_c{}^2 - X_L{}^2X_c - R^2X_c)$. The complex expression is then written

$$RX_c{}^2 + j(X_L X_c{}^2 - X_L{}^2X_c - R^2X_c).$$

The multiplication for the denominator is:

$$\begin{array}{ll}
R + j(X_L - X_c) & \\
R - j(X_L - X_c) & \\
\hline
R^2 + jR(X_L - X_c) & \\
\quad - jR(X_L - X_c) & - j^2(X_L - X_c)^2 \\
\hline
R^2 & - j^2(X_L - X_c)^2
\end{array}$$

This may be written $R^2 + (X_L - X_c)^2$ since $j^2 = -1$. The result is a RATIONILIZED denominator since it contains no j term.

The entire fraction now becomes

$$Z = \frac{RX_c{}^2 + j(X_L X_c{}^2 - X_L{}^2X_c - R^2X_c)}{R^2 + (X_L - X_c)^2}$$

The usual custom is to separate the real and j terms so

$$Z = \frac{RX_c{}^2}{R^2 + (X_L - X_c)^2} + j\,\frac{X_L X_c{}^2 - X_L{}^2X_c - R^2X_c}{R^2 + (X_L - X_c)^2}$$

This is the general expression for the impedance of a two branch parallel circuit and is of great importance to the radio engineer since nearly all tank circuits fall within this classification.

When a circuit is tuned to a condition of resonance it is understood that the process of tuning is to balance out any reactance which may be present before tuning. The first term

$$\frac{RX_c{}^2}{R^2 + (X_L - X_c)^2}$$

in the

above expression for impedance is the resistance or real part. The second term is the j or reactive part. For resonance (that is, unity power factor) the j term is reduced to zero. By reducing the j part of the expression to zero the necessary relationship of values R, C, and L for resonance is obtained. A fraction will equal zero if the numerator is equal to zero, provided the denominator has a value other than zero. Under these conditions the numerator of the j term can be made equal to zero.

$$X_L X_c^2 - X_L^2 X_c - R^2 X_c = 0$$

To solve for X_c first factor X_c from the left side of the equation.

$$X_c (X_L X_c - X_L^2 - R^2) = 0$$

It is evident that the product of two numbers can be zero only if one of the numbers is equal to zero. If

$$X_L X_c - X_L^2 - R^2 = 0$$

then by transposition

$$X_L X_c = R^2 + X_L^2$$

dividing by X_L

$$X_c = \frac{R^2 + X_L^2}{X_L}$$

Therefore at resonance the reactance of the capacitive branch must be equal to the complex expression

$$\frac{R^2 + X_L^2}{X_L}$$

and not simply the inductive reactance as was previously taught. But if R is small and the reactances are large then the effect of R in the equation

$$X_c = \frac{R^2 + X_L^2}{X_L}$$

may be neglected and

$$X_c = X_L^2/X_L = X_L.$$

Or at resonance $X_c = X_L$ if R is small enough in comparison with the individual reactances to be neglected. This condition is true in most radio circuits provided they are lightly loaded. It has previously been learned that resistance is reflected from one circuit to another by means of coupling. If a circuit is heavily loaded the reflected R may be quite large and, as a result, R is not small enough to be neglected. Transmitter tank and transmission line coupling circuits are usually well loaded so R must be taken into consideration when the resonant frequency of such circuits is to be calculated. Under these conditions resonant frequency *does not equal*

$$\frac{1}{2\pi\sqrt{LC}}$$

DERIVATION OF THE IMPEDANCE FORMULA Z = L/CR.—If the assumption is made that R in comparison to X is small enough to be neglected a very important series of expressions can be arrived at by considering the real part of the expression for impedance of a two branch parallel circuit. As previously explained at resonance the j term is equal to zero and

$$Z = \frac{RX_c^2}{R^2 + (X_L - X_c)^2}$$

Only the real term need be considered at resonance. If R is quite small then at resonance X_c and X_L are very nearly equal and their difference $X_L - X_c$ in the denominator of the term is so small it will have little effect when added to R^2. Hence $(X_L - X_c)$ may be neglected and

$$Z = \frac{RX_c^2}{R^2}$$

and by cancellation

$$Z = \frac{X_c^2}{R}.$$

If $X_c = X_L$ one may write $Z = X_L^2/R$. $X_L^2 = X_L X_L$ and since $X_c = X_L$

$$Z = \frac{X_L X_c}{R}$$

$X_L = 2\pi FL = \omega L$. Substituting equal values for X_L

$$Z = \frac{\omega L X_c}{R}$$

But $X_c = 1/2\pi FC = 1/\omega C$. Substituting for X_c,

$$Z = \frac{\omega L}{R} \cdot \frac{1}{\omega C} = \frac{\omega L}{\omega CR}$$

Cancelling ω, $Z = L/CR$. Therefore, *at resonance where R is small and is considered as being only in the inductive branch of a two branch parallel circuit* $Z = L/CR$.

The expressions

$$Z = \frac{X_L^2}{R} = \frac{X_c^2}{R} = \frac{X_L X_c}{R} = \frac{L}{CR}$$

are of great importance to the radio engineer and should be memorized. It is of still greater importance to remember under what conditions they can be used without introducing a major error. To the formula minded engineer the general expression for impedance is the more accurate and may be used to calculate the impedance of any *two branch* parallel circuit when the values of L, C, and R are known. The approximate expression finds its greatest use only when the circuit is at resonance.

The resistive part of the general expression for impedance

$$Z = \frac{RX_c^2}{R^2 + (X_L - X_c)^2}$$

may be used to *determine the impedance at resonance only.* (Because at resonance the reactance term is reduced to zero).

For the values of the circuit used in Fig. 11 when substituted in the above expression

$$Z = \frac{115 \times (598.8)^2}{115^2 + (576.5 - 598.8)^2} = 3008 \text{ ohms}$$

using the approximate formula $Z = L/CR$

$$Z = \frac{91.8}{266 \times 10^{-6} \times 115} = 3001 \text{ ohms}$$

Since X_L and X_c are also known

and

$$Z = \frac{X_L X_c}{R}$$

$$Z = \frac{576.5 \times 598.8}{115} = 3000 \text{ ohms}$$

The close correlation between the exact and approximate expressions for the impedance *at resonance* are clearly demonstrated in the above examples.

A very useful application of complex notation is in connection with the terminating network between a transmission line and an antenna. This illustration will deal with a relatively simple case where the antenna is operated at its fundamental frequency, that is, no loading is involved, and the antenna impedance is a pure resistance. Fig. 12(a) shows the circuit combination and Fig. 12(b) shows the equivalent circuit.

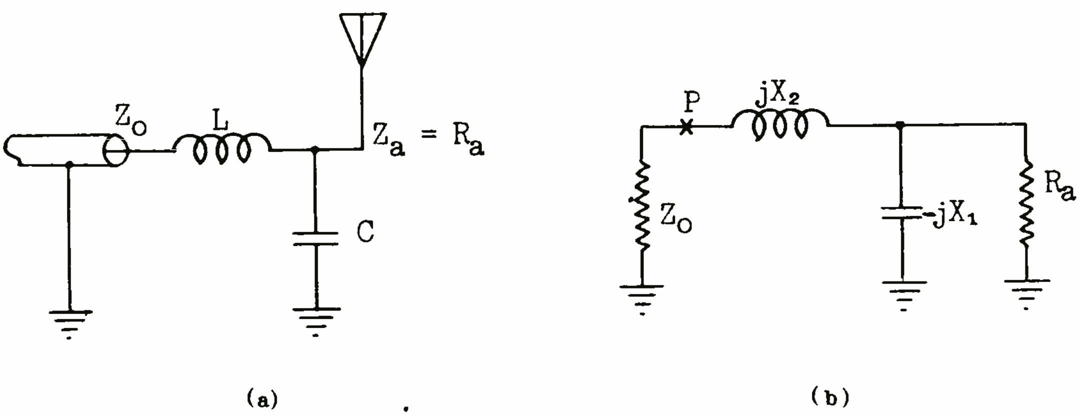


Fig. 12.—L circuit termination for line.

Fig. 12.—Equivalent circuit of Fig. 12(a).

The problem confronting the engineer is one of matching impedances, that is to make the impedance into which the transmission line "looks" of the same value as the surge impedance of the line. In other words Z_0 must equal the impedance of the combination to the right of point P Fig. 12(b). $-jX_1$ and R_a are in parallel hence the impedance of these two parameters is equal to their product divided by their sum or

$$\frac{-jX_1 R_a}{R_a + (-jX_1)} = \frac{-jX_1 R_a}{R_a - jX_1}.$$

Hence the impedance Z of the inductance in series with parallel combination CR must be

$$Z = jX_2 + \frac{-jX_1 R_a}{R_a - jX_1} \qquad (1)$$

By measurement or calculation the values of Z_0 (surge impedance of the transmission line) and $Z_a = R_a$ (the antenna impedance) are known. The purpose of the solution is to establish the values of X_1 and X_2 from which L and C can be calculated.

Rationalizing the fractional term in equation (1)

$$Z = jX_2 + \frac{-jX_1R_a}{R_a - jX_1} \cdot \frac{R_a + jX_1}{R_a + jX_1}$$

$$Z = jX_2 + \frac{X_1^2R_a - jX_1R_a^2}{R_a^2 + X_1^2}$$

$$Z = \frac{jX_2R_a^2 + jX_2X_1^2 + X_1^2R_a - jX_1R_a^2}{R_a^2 + X_1^2} \qquad (2)$$

Equation (2) may be written as follows:

$$Z = \frac{X_1^2R_a}{R_a^2 + X_1^2} + j\frac{X_2R_a^2 + X_2X_1^2 - X_1R_a^2}{R_a^2 + X_1^2} \qquad (3)$$

Equation 3 expresses the result of equation 2 but with the real and j parts separated.

For the transmission line to be terminated in a pure resistance, the j term must be zero. Then

$$Z_o = Z = \frac{X_1^2R_a}{R_a^2 + X_1^2}$$

This equation may be solved since Z_o and R_a are known by measurement or calculation.

The solution for X_1 follows:

$$Z_o = \frac{X_1^2R_a}{R_a^2 + X_1^2}$$

$$R_a^2Z_o + X_1^2Z_o = X_1^2R_a \quad \text{(clearing fractions)}$$

$$X_1^2R_a - X_1^2Z_o = R_a^2Z_o \quad \text{(collecting } X_1 \text{ terms)}$$

$$X_1^2(R_a - Z_o) = R_a^2Z_o \quad \text{(factoring)}$$

$$X_1^2 = \frac{R_a^2Z_o}{R_a - Z_o} \quad \text{(dividing by } R_a - Z_o\text{)}$$

$$X_1^2 = R_a^2 \frac{Z_o}{R_a - Z_o}$$

$$X_1 = \sqrt{R_a^2}\sqrt{\frac{Z_o}{R_a - Z_o}}$$

(taking square root of both sides)

$$X_1 = R_a\sqrt{\frac{Z_o}{R_a - Z_o}}$$

In equation 3 the j term must be equal to zero for proper termination. This means the j term may be equated to zero and the expression solved for X_2 since X_1 is known in terms of Z_o and R_a. As previously explained a fraction will be equal to zero if the numerator is zero.

$$X_2R_a^2 + X_2X_1^2 - X_1R_a^2 = 0$$

$$X_2R_a^2 + X_2X_1^2 = X_1R_a^2$$

(collecting X_2 terms)

$$X_2(R_a^2 + X_1^2) = X_1R_a^2$$

(factoring)

$$X_2 = \frac{X_1R_a^2}{R_a^2 + X_1^2}$$

(Dividing by $R_a^2 + X_1^2$)

The solution is complete since with R_a and Z_o known X_1 can be determined. With X_1 known X_2 may be calculated. For any given frequency L and C can be determined.

Suppose that measurements at the operating frequency of 1310 KC/s on the line and antenna at a broadcast station show that $Z_a = 120 + j0$ ohms for the antenna and that Z_o = 80 ohms for the transmission line. For a network like that in Fig. 12 determine the values of L and C.

$$X_1 = R_a\sqrt{\frac{Z_o}{R_a - Z_o}}$$

$$= 120\sqrt{\frac{80}{120 - 80}} = 170 \text{ ohms}$$

$$X_2 = \frac{X_1R_a^2}{R_a^2 + X_1^2} =$$

$$\frac{170 \times 120^2}{120^2 + 170^2} = 56.5 \text{ ohms}$$

at 1310 KC/s

$$C_1 = \frac{10^{-6}}{6.28 \times 1.31 \times 170} = 715 \ \mu\mu F$$

and

$$L_2 = \frac{56.5}{6.28 \times 1.31 \times 10^{-6}} = 6.88 \ \mu H$$

Fig. 12(c) shows the circuit and the calculated and known parameters.

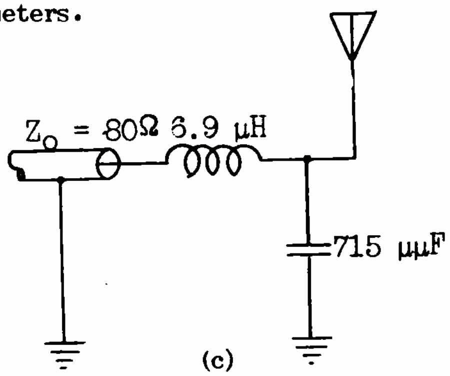


Fig. 12.—Values shown for circuit as in 12(a).

The above discussions afford examples demonstrating the ease with which a-c problems can be solved using the complex number. In the above condition, for proper matching between line and antenna, a more general condition will be established. Reactances will be used since they consume no power but nothing will be said about their character, that is, it will not be specified whether the reactances are inductive or capacitive but will be left to the choice of the engineer. The equivalent network is shown in Fig. 13.

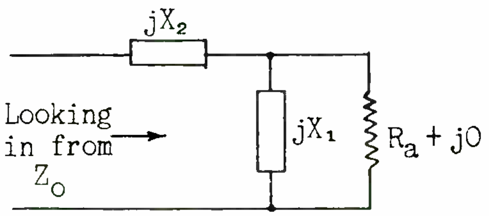


Fig. 13.—Circuit for finding Z_o looking into the network.

The impedance looking into the circuit from Z_o is

$$Z = jX_2 + \frac{jX_1R_a}{R_a + jX_1}$$

$$Z = jX_2 + \frac{X_1^2R_a + jX_1R_a^2}{R_a^2 + X_1^2}$$

(Rationalizing)

$$Z = \frac{jR_a^2X_2 + jX_1^2X_2 + X_1^2R_a + jX_1R_a^2}{R_a^2 + X_1^2}$$

(Multiplying jX_2 by $R_a^2 + X_1^2$)

$$Z = \frac{X_1^2R_a}{R_a^2 + X_1^2} + j\frac{X_1R_a^2 + X_1^2X_2 + R_a^2X_2}{R_a^2 + X_1^2}$$

(Separating real and j terms)

For Z to be a pure resistance equal to Z_o the j term must be zero.

So

$$Z_o = \frac{X_1^2R_a}{R_a^2 + X_1^2}$$

Solving X_1 in the same manner as previously shown

$$X_1^2 = R_a^2\frac{Z_o}{R_a - Z_o}$$

$$X_1 = R_a\sqrt{\frac{Z_o}{R_a - Z_o}}$$

(Where $R_a > Z_o$)

The character of X_1 is determined by the choice of sign, that is, if a positive sign is taken for X_1 then it becomes an inductance, if the sign is negative X_1 is a capacitive reactance.

Since X_2 is solved in terms of X_1 and R_a, it will then be determined by equating the j part of the expression for Z to zero and solving for X_2 in terms of X_1 and R_a.

$$X_1R_a^2 + X_1^2X_2 + R_a^2X_2 = 0$$

$$X_2(R_a^2 + X_1^2) = -X_1R_a^2$$

(Factoring and Transposing)

$$X_2 = -\frac{X_1R_a^2}{R_a^2 + X_1^2}$$

($R_a > Z_o$)

X_2 has an opposite sign to that of X_1 regardless of the choice of sign for X_1.

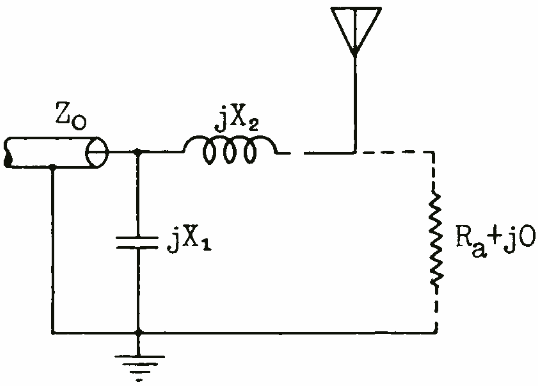
This solution is more powerful and general in scope because it removes the restriction as to the character of the impedance. It is evident that X_1 could be chosen either as an inductance or capacitance but X_1 is usually chosen as a capacitance and X_2 an inductance because of the better discrimination against harmonics. The series inductance opposes the higher harmonic frequencies while C tends to shunt them to ground thus keeping them out of the antenna.

It will be observed that the above solution holds true for the case where the antenna resistance is greater than the characteristic impedance of the line. When the antenna resistance is less than the line impedance, a similar arrangement is used but the positions of the source and "sink" (load) are reversed. This case will present the line looking into the side where the antenna was placed previously and with the antenna replacing the line position as shown in Fig. 14(a). This will enable the match to be made in the reverse direction. The impedance into which the line looks is

$$Z = \frac{-jX_1 (R_a + jX_2)}{R_a + j(X_2 - X_1)}$$

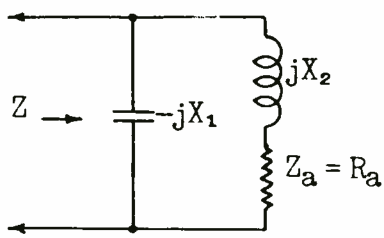
which will be recognized as the same solution as the two branch parallel circuit with the resistance all in the inductive branch. The solution for X_1 and X_2 in terms of Z_o and R_a is carried out in the same general manner as the preceding case.

Fig. 14(b) shows the equivalent circuit of 14(a). The general



(A)

Reversed connection of L Network.



(B)

Fig. 14.—Equivalent circuit of 14(A).

solution of this circuit has been developed earlier in this assignment and is

$$Z = \frac{RX_c^2}{R^2 + (X_L - X_c)^2} + j\frac{X_c^2 X_L - X_c X_L^2 - X_c R^2}{R^2 + (X_L - X_c)^2} \qquad (1)$$

Changing the subscript to conform to those used in Fig. 14(b)

$$Z = \frac{R_a X_1^2}{R_a^2 + (X_2 - X_1)^2} + j\frac{X_1^2X_2 - X_1X_2^2 - R_a^2X_1}{R_a^2 + (X_2 - X_1)^2} \qquad (2)$$

For the parallel circuit to represent a pure resistance equal to that of the transmission line the reactive term must be zero and

$$Z = Z_o = \frac{R_a X_1^2}{R_a^2 + (X_2 - X_1)^2} \qquad (3)$$

For the j term to be zero the numerator must be equal to zero or

$$X_1^2X_2 - X_1X_2^2 - R_a^2X_1 = 0 \qquad (4)$$

$$X_1 (X_1X_2 - X_2^2 - R_a^2) = 0 \qquad (5)$$

if

$$X_1X_2 - X_2^2 - R_a^2 = 0 \qquad (6)$$

then

$$X_1X_2 - X_2^2 = R_a^2 \qquad (7)$$

and

$$X_2 (X_1 - X_2) = R_a^2 \qquad (8)$$

dividing by X_2

$$X_1 - X_2 = R_a^2/X_2 \qquad (9)$$

multiplying by -1

$$X_2 - X_1 = -R_a^2/X_2 \qquad (10)$$

square both sides

$$(X_2 - X_1)^2 = R_a^4/X_2^2 \qquad (11)$$

substitute (11) in (3)

$$Z_o = \frac{R_a X_1^2}{R_a^2 + \frac{R_a^4}{X_2^2}} \qquad (12)$$

simplify compound fraction

$$Z_o = \frac{X_1^2X_2^2}{R_a (X_2^2 + R_a^2)} \qquad (13)$$

From (7)

$$X_1X_2 = R_a^2 + X_2^2 \qquad (14)$$

squaring both sides of (14)

$$X_1^2X_2^2 = (R_a^2 + X_2^2)^2 \qquad (15)$$

substituting for $X_1^2X_2^2$ in (13)

$$Z_o = \frac{(R_a^2 + X_2^2)^2}{R_a (X_2^2 + R_a^2)} \qquad (16)$$

cancelling

$$Z_o = \frac{R_a^2 + X_2^2}{R_a} \qquad (17)$$

simplifying

$$Z_oR_a = R_a^2 + X_2^2 \qquad (18)$$

solving for X_2^2

$$X_2^2 = Z_oR_a - R_a^2 \qquad (19)$$

$$X_2 = \sqrt{Z_oR_a - R_a^2} \qquad (20)$$

To solve for X_1 in terms of Z_o and R_a from equation (18)

$$Z_oR_a = R_a^2 + X_2^2 \qquad (21)$$

From (14)

$$X_1X_2 = R_a^2 + X_2^2 \qquad (22)$$

but

$$Z_oR_a = R_a^2 + X_2^2 \text{ from (21)}$$

so

$$X_1X_2 = Z_oR_a \qquad (23)$$

and

$$X_1 = \frac{Z_oR_a}{X_2}$$

but

$$X_2 = \sqrt{R_a(Z_o - R_a)} \text{ from (20)}$$

so

$$X_1 = \frac{Z_oR_a}{\sqrt{R_a(Z_o - R_a)}} \qquad (24)$$

squaring both sides

$$X_1^2 = \frac{Z_o^2R_a^2}{R_a(Z_o - R_a)} \qquad (25)$$

cancelling

$$X_1^2 = \frac{Z_o^2R_a}{Z_o - R_a} = Z_o^2\frac{R_a}{Z_o - R_a} \qquad (26)$$

extracting square root

$$X_1 = Z_o\sqrt{\frac{R_a}{Z_o - R_a}} \qquad (27)$$

therefore

$$X_2 = \sqrt{R_a(Z_o - R_a)}$$

and

$$X_1 = Z_o\sqrt{\frac{R_a}{Z_o - R_a}} \quad (R_a < Z_o)$$

which gives the value of X_1 and X_2 in terms of R_a and Z_o.

If the antenna impedance is $Z_a = 20 + j0$ ohms and the transmission line has an impedance of Z_o = 100 ohms, X_1 and X_2 may be found by substitution.

$$X_1 = 100\sqrt{\frac{20}{100 - 20}}$$

$$= 50 \text{ ohms}$$

$$X_2 = \sqrt{20(100 - 20)}$$

$$= 40 \text{ ohms}$$

L and C may be found for any frequency by the regular formulas. If the operating frequency is 1500 KC,

then

$$L = \frac{40}{6.28 \times 1.5} = 4.25 \ \mu H$$

and

$$C = \frac{10^{-6}}{6.28 \times 1.5 \times 50} = 5520 \ \mu\mu F$$

EXERCISES

Multiply:

51. $(15 + j8) \quad (10 - j6)$
52. $(.6 - j.2) \quad (.4 - j.03)$
53. $(A_1 + jB_1) \quad (A_2 - jB_2)$
54. $(6 + j8) \quad (4 - j5)$
55. $(10 - j5) \quad (5 - j10)$
56. $(12 - j5) \quad (.8 - j5)$
57. $(A + jB) \quad (C + jD)$
58. $(R + jX) \quad (R - jX)$
59. $(R_1 + jX_L) \quad (R_2 - jX_c)$

Divide:

60. $15 + j8$ by $10 - j6$
61. $.4 - j.03$ by $.6 - j.2$
62. 1 by $A_1 - jB_1$
63. 1 by $R - jX$
64. $-1 - j3$ by $-3 + j6$
65. $.50 + j80$ by $6 - j8$
66. $\frac{1}{6 + j8}$
67. $\frac{50 + j86.6}{6 - j8}$
68. $\frac{R + jX}{R - jX}$
69. $10/-j25$
70. $1/j$

ANSWERS TO EXERCISE PROBLEMS

1. 1.78, $\text{Log } 75$, $-\pi$, $-\sqrt{11}$, 2π, 7.2, -18, $10^{1.3}$

2. $10^{1.3}$, 7.2, 2π, $\text{Log } 75$, 1.78, $-\pi$, $-\sqrt{11}$, -18

3. Fractional: 10^{-2}, π, 1.36, $\sqrt{6}$, $3/2$
   Integral: -4

4. Rational: $\sqrt{64}$, $\sqrt[3]{64}$, $\sqrt[6]{64}$, $\sqrt[3]{27}$
   Irrational: $\sqrt{2\pi}$, $\sqrt[5]{125}$

5. (a) -16, $\sqrt{3^2 + 4^2}$

   (b) $3/2$, 19.26. $\sqrt{18/27}$

   (c) $\sqrt{A^2}$, $\sqrt{3^2 + 4^2}$, -16, 19.26

   (d) $-\sqrt{2}$, $\sqrt{18/27}$

   (e) $3/2$, 19.26, $\sqrt{3^2 + 4^2}$, $\sqrt{18/27}$

   (f) -16, $-\sqrt{2}$

6. Real: -16, $\sqrt{10}$, $-\sqrt{16}$, $21\sqrt{2}$
   Imaginary: $3\sqrt{-6}$

7. Real: 16, π, $-27^{-1/3}$
   Imaginary: $-27^{-1/2}$ $-27^{-1/2}$

8. Real.

9. Imaginary.

10. Real.

11. Real: $j^2 15$, $5 + j^2 3$
    Imaginary: $j8$, $j^3 20$, $\sqrt{j^2}$

12. $3 - j6$ 4th Quadrant
    $5 + j5$ 1st Quadrant
    $16 + j0$ Positive X axis
    $-\sqrt{16}$ Negative X axis

13. $j8$ on + j axis
    $-2 - j13$ 3rd Quadrant
    $-12 - j12$ 3rd Quadrant
    $-3 + j10$ 2nd Quadrant

14. $-1 - j10$ 3rd Quadrant
-6 Negative X axis
$10 + j^2$ 9 units on positive X axis
$15 + j$ 1st Quadrant
$17 - j^3$ 4th Quadrant

15. $2.22 + j0$
$-2.553 + j0$

16. $8 + j0$
$1 + j0$

17. $-.143 \pm j.515$

18. $1.81 + j0$
$-3.31 + j0$

19. $1.618 + j0$
$-.618 + j0$

20. $\pm j4$

21. $j, 1, 1, -j, -j$

22. $5 - j7$ 4th Quadrant
$7 + j5$ 1st Quadrant
$-5 + j7$ 2nd Quadrant
$-7 - j5$ 3rd Quadrant

23. $55 + j88.7$ volts 1st Quadrant

24. $E = 19 - j5V$
$I = 8 - j8A$ Inductive

25. $\theta > 90^\circ$ an impossible condition.

26. I leads E by 30°, $E = 200$ V at 30°, $I = 10A$ at 60°.

27. $11.5 + j9.64A$

28. $9.85 - j1.74$ (-10° in 4th Quad. Sin is − and Cos + in 4th Quad.)

29. $10 + j25.12V$ if $I = 1A$

30. $60 - j32$ volts 4th Quadrant

31. $120 + j68$ ohms

32. Z = 120 + j0 ohms

33. Z = 120 - j123 ohms

34. 120 +j47 ohms

35. 15 - j15

36. 14 - j5

37. 58.9 - j65.4

38. 53 - j9.8

39. 357 - j558

40. 134.8 + j74.85

41. 143 + j130

42. 8 - j24.7

43. 96 - j113

44. 367 - j76

45. -25.9 - j55

46. 183 - j38

47. 653

48. 672 - j976

49. -20.3 + j187

50. -462 + j2730

51. 198 - j10

52. .234 - j.098

53. $A_1A_2 + B_1B_2 + j(A_2B_1 - A_1B_2)$

54. 64 + j2

55. -j125

56. -15.4 - j64

57. $AC - BD + j(BC + AD)$

58. $R^2 + X^2$

59. $R_1R_2 + X_LX_c + j(R_2X_L - R_1X_c)$

60. $.75 + j1.25$

61. $.615 + j.155$

62. $\frac{A_1}{A_1^2 + B_1^2} + j\frac{B_1}{A_1^2 + B_1^2}$

63. $\frac{R}{R^2 + X^2} + j\frac{X}{R^2 + X^2}$

64. $-.333 + j.333$

65. $-6.37 + j4.84$

66. $.06 - j.08$

67. $-3.93 + j9.2$

68. $\frac{R^2 - X^2}{R^2 + X^2} + j\frac{2RX}{R^2 + X^2}$

69. $j.4$

70. $-j$

A T NETWORK FOR ANTENNA COUPLING

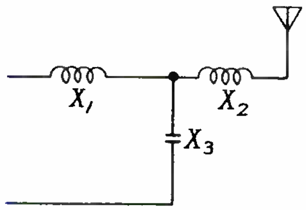


Fig. 1.

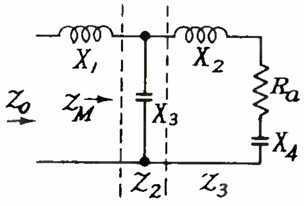


Fig. 2.

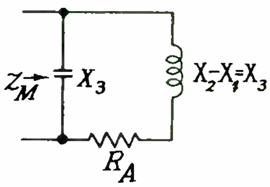


Fig. 3.

A simple coupling method is sometimes used as shown in Fig. 1.

The broadcast vertical tower antenna is usually worked at slightly greater than a half wavelength and therefore has capacitive reactance. This is shown as X_4 in Fig. 2. In this particular coupling method X_3 is chosen to transform R_a to Z_o and X_2 is chosen to obtain series resonance in the circuit X_3, X_2, X_4.

Therefore

$$X_2 - X_3 - X_4 = 0 \qquad (1)$$

or

$$X_3 = X_2 - X_4 \qquad (2)$$

$$Z_m = \frac{Z_2 Z_3}{Z_2 + Z_3} = \frac{-JX_3\,(JX_2 - JX_4 + R_a)}{-JX_3 + JX_2 - JX_4 + R_a} \qquad (3)$$

Substituting (2) in (3)

$$Z_m = \frac{X_3^{\,2} - JX_3 R_a}{R_a} = \frac{X_3^{\,2}}{R_a} - jX_3 \qquad (4)$$

Then, to make Z_o resistive, X_1 must be inductive and equal to X_3 in magnitude.

Then

$$Z_o = Z_m + jX_1 = \qquad (5)$$

$$\frac{X_3^{\,2}}{R_a} - jX_3 + jX_1 = \frac{X_3^{\,2}}{R_a}$$

Solving for X_3,

$$Z_o = \frac{X_3^{\,2}}{R_a} \text{ or } X_3 = \sqrt{Z_o R_a} \qquad (6)$$

Since Z_o and R_a are fixed, X_3 would be calculated first. X_2 would be chosen to tune the antenna to series resonance according to Equation (1). Then X_1 is inserted so that the transmission line will be terminated in a pure resistive load. Equation (4) holds true for any parallel circuit such as Fig. 3 where $X_L = X_c$, and all the resistance is considered to be in the L branch.

SUPPLEMENTARY NOTES

Supplementary notes on the impedance of a resistor shunted by a *capacitive* or an *inductive* reactance:

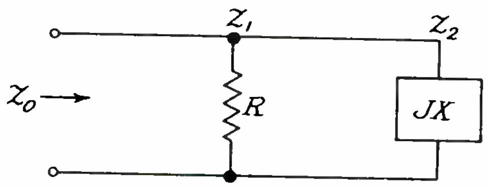


Fig. 1.

$$Z_o = \frac{Z_1 \; Z_2}{Z_1 + Z_2}, \qquad Z_1 = R, \qquad Z_2 = JX,$$

Fig. 1.

Substituting:

$$Z_o = \frac{R(jX)}{R + jX} \qquad (1)$$

If the denominator consisted of either a real term alone or a j term alone, we could divide without further manipulation. But we cannot divide directly by a sum of a real term and a j term. Therefore we must eliminate the j term in the denominator by rationalizing as demonstrated in problem 7 and 8.

Therefore:

$$Z_o = \frac{j\;RX}{R + jX} \times \frac{R - jX}{R - jX} =$$

$$\frac{jR^2X - j^2RX^2}{R^2 + X^2} = \frac{jR^2X + RX^2}{R^2 + X^2} =$$

$$= \frac{RX^2}{R^2 + X^2} + \frac{j\;XR^2}{R^2 + X^2} \qquad (2)$$

We have now mathematically transformed Fig. 1 to an equivalent circuit as shown in Fig. 2.

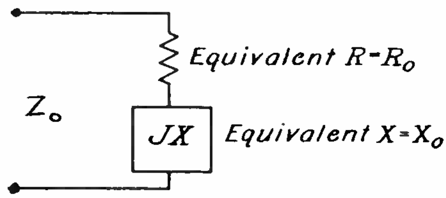


Fig. 2.—

R_o (Equivalent) of Fig. 2 will be the apparent resistance of Fig. 1 and will be equal to the first term of Eq. (2).

$$R_o = R \frac{X^2}{R^2 + X^2} \qquad (3)$$

X_o (Equivalent) of Fig. 2 will

be the apparent reactance of Fig. 1 and will be equal to the j term of (2).

$$X_o = X \frac{R^2}{R^2 + X^2} \qquad (4)$$

The equivalent inductance or capacitance, as the case may be, can be calculated from Eq. (4). Note that in Eq. (4) the j notation can be dropped since this will have no significance concerning the magnitude of X_o.

From Fig. 2, it will be observed that the impedance of either Fig. 1 or 2 is equal to the square root of the sum of the squares of the equivalent resistance and reactance:

$$Z_o = \sqrt{R_o^2 + X_o^2} \qquad (5)$$

$$Z_o = \sqrt{\left(\frac{RX^2}{R^2 + X^2}\right)^2 + \left(\frac{XR^2}{R^2 + X^2}\right)^2} =$$

$$\sqrt{\frac{(RX^2)^2 + (XR^2)^2}{(R^2 + X^2)^2}} =$$

$$\sqrt{\frac{X^2R^2\ (X^2 + R^2)}{(X^2 + R^2)^2}} =$$

$$\sqrt{\frac{X^2\ R^2}{X^2 + R^2}} = \frac{XR}{\sqrt{X^2 + R^2}}$$

Note that we can drop the j notation from Eq. (2) when combining with Eq. (5) because in the latter we are computing the magnitude only.

This is a good location to pause and review the mathematical development to obtain a mental picture of what has been accomplished. Let us consider Eq. (3). This will be encountered many times in various communication networks, and illustrates how a resistance may be effectively reduced in value by shunting it with either a capacity or an inductance. This is often done in reducing the resistance of an antenna to match the characteristic impedance of a transmission line. In case of the antenna, an additional reactance of opposite sign would be inserted in the transmission line to cancel the equivalent reactance X_o as illustrated in Fig. 12(b), Page 22.

Since the formulas given by Eq. (3) and (4) will be often encountered in radio work, let us see how they can be easily committed to memory. Note that Eq. (3), the equivalent *resistance* is *R* times a fraction, while in Eq. (4) the equivalent reactance is X times a fraction. The denominator in both cases is identical. This now suggests that the positions of R and X are interchanged in the numerator of the two formulas and may be easily remembered.

Other more extended formulas will sometimes contain Eq. (3) and (4) and thus are more readily analyzed when the student is able to recognize the various parts at a glance.

For example note formula (3) page 23. The real part of this formula is the same as Eq. (3) of this paper.

The power factor of the circuit in Fig. 1 will be the cosine of the angle whose tangent is the ratio of the reactance to the resistance (X_o/R_o) of the equivalent circuit 2.

Now suppose that it is desired to calculate the value of X to reduce R to a given impedance Z_o, understanding that in this case Z_o will contain a reactive component. We can do this easily by solving Eq. (5) for X.

From Eq. (5)

$$Z_o = \frac{XR}{\sqrt{X^2 + R^2}}$$

Squaring both sides;

$$Z_o^2 = \frac{X^2R^2}{X^2 + R^2}$$

Clearing of fractions:

$$Z_o^2X^2 + Z_o^2R^2 = X^2R^2$$

Transposing $Z_o^2X^2$;

$$Z_o^2R^2 = X^2 \; (R^2 - Z_o^2)$$

Dividing both members by ($R^2 - Z_o^2$);

$$X^2 = \frac{Z_o^2R^2}{R^2 - Z_o^2}$$

Extracting square root:

$$X = \frac{Z_oR}{\sqrt{R^2 - Z_o^2}} \qquad (6)$$

Although the reactance X in Eq. (6) may be either capacitive or inductive, a capacitor is invariably used when the only object is to reduce the resistive component. This is because there will be less losses in a good capacitor than in a coil of the same reactance. However, Eq. (6) is useful in calculating the reactance of a coil when the circuit requires such an arrangement.

It should be emphasized again that Eq. (6) solves for X only under the conditions of the circuit in Fig. 1. If a capacitor is shunted by a coil and a resistance in series, the equations shown in pages 16 and 18 of this assignment must be used. Note also that Z of Fig. 1 will always have a reactive component and this is included in the result of Equation (5).

However, in practice it is more often desired to calculate the required X to reduce R to an equivalent *resistive* component. This can be done by solving Eq. (3) for X.

$$R_o = R\frac{X^2}{R^2 + X^2}$$

$$R_o(R^2 + X^2) = RX^2$$

$$R_oR^2 = RX^2 - R_oX^2;$$

$$X^2 = \frac{R_oR^2}{R - R_o}$$

$$X = R\sqrt{\frac{R_o}{R - R_o}} \qquad (7)$$

The required capacity may then be calculated from X as given in Eq. (7). The equivalent reactance of the combination would then be given by Eq. (4) and, if desired, this could be cancelled by a reactance of opposite sign as mentioned on Page 2.

Page 1 shows the development of formulas to convert a parallel circuit to a series circuit, but it is sometimes desired to do the opposite. Suppose Fig. 2 represents an antenna with capacitive reactance, and it is desired to couple it to a transmission line of lower impedance by the use of an L network as shown in Fig. 7.

The antenna resistance may be reduced to the transmission line impedance, Z_o, by shunting with a capacity. But first the series antenna circuit must be converted to the equivalent parallel circuit (Fig. 5) so that it may be seen what equivalent *capacity* is already in parallel with the equivalent parallel *resistance* of the antenna and what this resistance will be.

The formulas for this purpose can be simply developed by reversing the process used on page one. Bear in mind that in this case R_o and X_o are known while X and R are unknown.

Squaring Eqs. (3) and (4) and adding:

$$R_o{}^2 + X_o{}^2 = \frac{(X^2R)^2 + (XR^2)^2}{(R^2 + X^2)^2} = \qquad (8)$$

$$\frac{X^2R^2(R^2 + X^2)}{(R^2 + X^2)^2} = \frac{X^2R^2}{R^2 + X^2}$$

To solve for R, divide Eq. (8) by Eq. (3). (Invert right hand side of Eq. (3) and multiply.)

$$\frac{R_o{}^2 + X_o{}^2}{R_o} = \frac{X^2R^2}{R^2 + X^2} \cdot \frac{R^2 + X^2}{RX^2} = R_p \qquad (9)$$

To solve for X, divide Eq. (8) by Eq. (4). (Invert right hand side of Eq. (4) and multiply.)

$$\frac{R_o{}^2 + X_o{}^2}{X_o} = \frac{X^2R^2}{R^2 + X^2} \cdot \frac{R^2 + X^2}{XR^2} = X_p$$

Thus if the given antenna has an impedance $Z_a = R_a - jX_a$, as shown in Fig. 4 the first step in making calculations for an *L coupling network* would be to convert to an equivalent parallel circuit as shown in Fig. 5.

Note the change in subscripts indicated in the diagram.

The following calculations must be based on Fig. 5. In calculating the capacitive reactance X_c, necessary to reduce R′ (Fig. 5) to Z_o, Eq. (7) may be used, substituting R′ for R and Z_o for R_o. But we already have C′ (Fig. 5) which can be calculated from X′. C" will then be equal to C − C′. Thus the resistance of the antenna of Fig. 6. X_f of the combination X", X′ and R′ may be calculated from Eq. (4) by making the proper substitutions.

This is a particularly convenient method of coupling to use when the transmitter power is low so

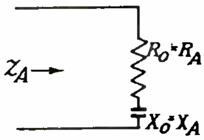


Fig. 4.

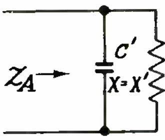


Fig. 5.

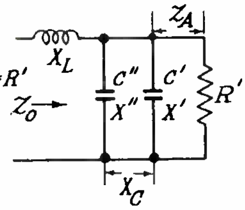


Fig. 6.

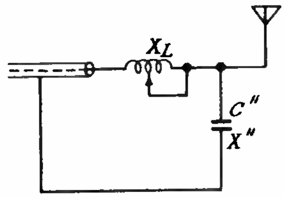


Fig. 7.--Actual Circuit.

has been reduced to the impedance of the transmission line. But the line must be terminated in a pure resistive impedance so X_L must be inserted to cancel X_c that X" can be a variable condenser. In high power installations where X" must be a fixed condenser the method shown on the sheet "A T NETWORK FOR ANTENNA COUPLING" is more often used.

OPERATOR J

EXAMINATION

1. Evaluate the following powers of j: j^5, j^9, j^{12}, j^{15}, j^{38}.

2. Plot the vector 5 + j12. Operate on it by -j and plot the new vector. Operate on the new vector by j^2. Show the values of the 3 vectors and plot on the complex plane.

3. Add:

   (a) 3 + j5 and 7 + j2.

   (b) 14 + j11 and -6 - j4.

4. In Problem 3 subtract the second complex number from the first in both (a) and (b) parts.

4. *(Continued)*

5. Show the following subtractions performed geometrically on the Argand diagram on graph paper.

   (a) Subtract 7 + j2 from 3 + j5.

   (b) Subtract -6 -j4 from 14 + j11.

6. Multiply:

   (a) 3 + j5 by 7 + j2.

   (b) A - jB by -C + jD.

EXAMINATION, Page 3.

7. Divide and express result in the same form.

(a) (3 + j5) by (7 + j2)

(b) 173 + j100 by 5 + j18.

8. Find the reciprocal of 5 + j3 and express in the form R + jX.

9. Show the position of the following vectors on the complex plane.

   (a) -5

   (b) -j5

   (c) 6 + j3 — Plot to scale on graph paper.

   (d) 7 - j5

   (e) -7 + j5

10. A resistance of 100 ohms terminates a certain transmission line. It is desired to reduce the apparent value of this resistance to a lower value. It is accordingly shunted by a 100 μμf condenser. The frequency of the supply is 15 mc/s.

    (a) What is the apparent resistance of the combination?

    (b) What is the apparent capacitance of the condenser?

    (These values are indicated by the equivalent series impedance determined by rationalizing the parallel impedance)

EXAMINATION, Page 5.

10. *(Continued)*