SECTION 2

ADVANCED PRACTICAL RADIO ENGINEERING

THE PARALLEL LCR CIRCUIT

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THE PARALLEL LCR CIRCUIT

FOREWORD

In this technical assignment you will continue the study of the effects of resonance but with parallel LCR circuits. The mathematics and analytical processes will be no more difficult than in the study of series circuits. Many of the curves will appear similar to those obtained for series circuits—BUT THEY WILL BE OPPOSITE IN MEANING. For example, it was learned that at resonance a series LCR circuit offers minimum, often almost negligible, impedance which is purely resistive. Here you will learn that at resonance the impedance of a parallel LCR circuit also is purely resistive but has a *very high* value, often in the order of hundreds of thousands of ohms.

Again some of the results of a mathematical circuit analysis may be somewhat startling. For example, an r.f. voltage applied across a parallel resonant circuit consisting of L in one branch and C in the other, with very small resistive losses in each branch, may result in equal currents of *many amperes in each branch and negligible current through the device which produces the applied voltage.*

This phenomenon of parallel resonance makes possible a filter circuit which will block the passage of signals at the one frequency of parallel resonance while passing with negligible attenuation signals of all other frequencies—just the opposite to the effects at series resonance.

Parallel resonance is made use of in the design of transmitter plate tank circuits where high impedance together with large tank current and minimum circuit I^2R losses are desired. The applications of parallel resonant circuits are as numerous as those of series resonant circuits, and as important. Learn the essential differences between the two types and you will then know how to apply each.

E. H. Rietzke,
President.

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THE PARALLEL LCR CIRCUIT

SCOPE OF ASSIGNMENT

A preceding assignment on resistances in series and parallel brought out, in a general way, the differences between the operation of series and parallel circuits. Although the operation of an a-c circuit composed of inductance, capacity, and resistance differs considerably from that of a d-c circuit containing only resistance, many of the principal characteristics of the two are somewhat similar.

Series circuits containing capacity, inductance, and resistance have been discussed rather thoroughly in the preceding assignment. Parallel circuits containing capacity, inductance, and resistance will be analyzed in this assignment; and the solution of such circuits, explained. The various characteristics of parallel circuits (both resonant and non-resonant) will be described, and illustrated graphically.

Although parallel circuit calculations, in general, will be found to be quite simple to perform using the math which has been taught up to this point, circuit analyses of this type can be facilitated to a great extent by the use of so-called *j*-operators and a system known as complex algebra. In view of this fact, the application of complex algebra to alternating-current circuits (both series and parallel) will be explained in this assignment.

PARALLEL COMBINATIONS OF L, C, AND R

METHODS OF COMPUTING EQUIVALENT RESISTANCE.—The two general methods of computing resistances in parallel have been given previously. One way of determining the total resistance of two resistors in parallel is by the use of the formula

$$R_T = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}, \text{ or}$$

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$

The second method is to measure the current through each resistor (or calculate I from the applied voltage divided by the individual value of resistance); the total current will equal the sum of the two currents.

$$I_1 = E/R_1 \qquad (1)$$

$$I_2 = E/R_2 \qquad (2)$$

$$I_T = I_1 + I_2 \qquad (3)$$

$$R_T = E/I_T \qquad (4)$$

Since the student has previously worked problems containing resistances in parallel, no further discussion of that will be presented; however, parallel circuits containing components other than just resistances, will now be described.

PARALLEL RESONANCE.—Consider the condition existing in a parallel circuit consisting of a perfect capacity in one branch and a perfect inductance in the other, (assuming temporarily a condition of zero resistance), operated at the frequency at which $X_L = X_C$. The circuit is shown in Fig. 1. Assume that L and C are of such values that at this frequency both X_L and X_c equal 10 ohms. The applied voltage equals 100 volts. The current through any

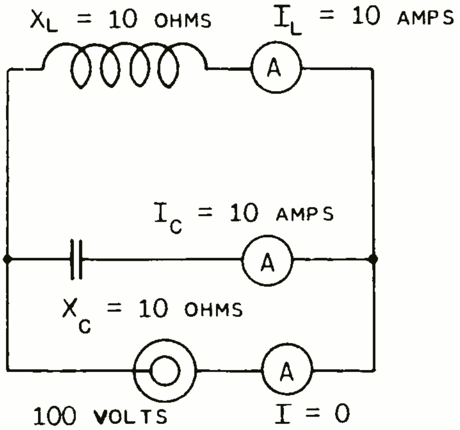


Fig. 1.—Parallel LC circuit.

one branch of a parallel circuit is determined only by the applied voltage and the impedance of that particular branch; thus, in the inductive branch, $I_L = E/X_L = 100/10 = 10$ amperes. In the capacitive branch, $I_c = E/X_c = 100/10 = 10$ amperes. Since the inductive branch contains, (theoretically), no resistance, the current through that branch lags the applied voltage by 90°. For a similar reason the current through the capacity branch leads the same applied voltage by 90°. This condition is shown in Fig. 2. With the two currents exactly equal and exactly opposite in direction of flow, their vector sum is exactly equal to zero. If the condition of *zero resistance* could be obtained it would be possible to construct a circuit in which the condition of ten amperes in each branch of the circuit and zero current in the external circuit could exist. In that case, the impedance of the parallel circuit operated at the frequency at which $X_L = X_C$, (resonant frequency), would equal the applied voltage divided by zero, or infinity. No energy is lost once the circuit is excited, since there is no resistance present.

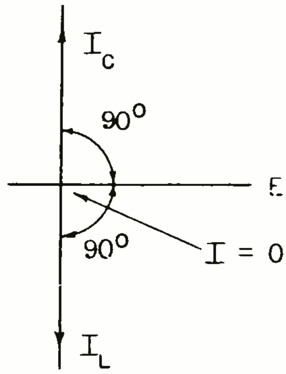


Fig. 2.—Vector diagram of I_L, I_C, and resultant I in circuit of Fig. 1.

In the study of inductance and capacity it was shown that it is impossible to construct a circuit having zero resistance losses. This means that the 90° Lead and 90° Lag are never quite realized. In either branch of the circuit the resistance R causes the lead or lag to be some angle less than 90° produced by X alone. The tangent of the angle equals X/R; if R is made as small as possible, the ratio of X to R may be very large and the angle will be correspondingly large. With a very low loss circuit the conditions existing are somewhat as shown by I_A, Fig. 3. This assumes that the losses in the L and C branches are equal. It will be seen that now, although the current in the external circuit is small compared to the current in each branch, it is NOT zero and the impedance therefore is not infinite. The introduction of a certain amount of resistance into

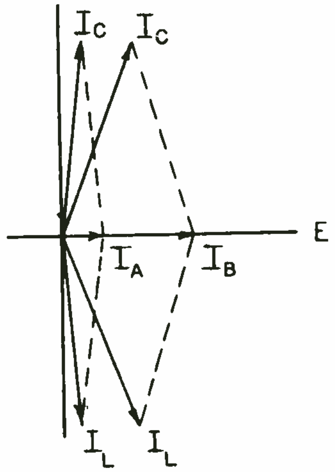


Fig. 3.—Vectors showing I_C, I_L, and resultant I in an LC circuit having some resistance.

the parallel circuit operating at resonance DECREASES the parallel impedance of the circuit.

If still more losses are introduced into each branch of the parallel circuit the condition becomes as shown by I_B, Fig. 3. The current angles of lead and lag are decreased, and the total current I becomes greater. If the current is greater, the impedance E/I becomes smaller. It will be seen that at the resonant frequency the resulting current is IN PHASE WITH THE VOLTAGE, the circuit therefore acts as a high value of resistance. The lower the losses in the circuit the HIGHER THIS EFFECTIVE RESISTANCE BECOMES. It should be noted that this effect is just the opposite to that of the series circuit operated at resonance.

In the circuits described here the total impedance of the circuit is equal to the applied voltage divided by the total current, this current being the *vector* sum of the currents through the individual branches.

Fig. 4(A) illustrates the con-

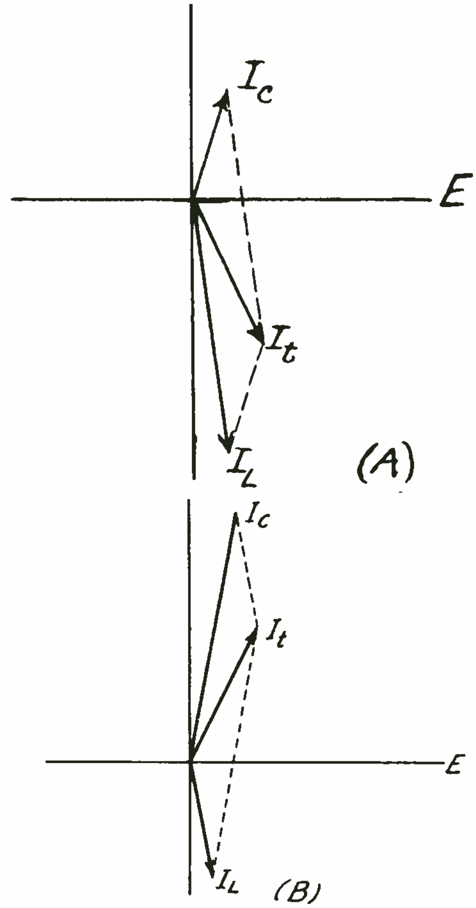


Fig. 4.—Vectors showing the current in a parallel circuit operated off resonance.

ditions existing in a parallel circuit when X_C is greater than X_L. Under these conditions I_L (the current through the inductive branch) is greater in value than is I_C (the current through the capacitive branch). The resultant current (I_T) lags the applied voltage; the *cir-*

cuit therefore acts as an inductance under the above circumstances.

Note that if there were no resistance in either branch, I_C would be vertically upward, I_L would be vertically downward, and their resultant I_T would also lie vertically downward. This means that the circuit would look like a purely inductive reactance, whereas that shown in Fig. 4 has some resistive component present.

In order for X_C to be greater than X_L, the operating frequency of a parallel circuit must be *lower* than the resonant frequency. As the frequency is lowered (L and C remaining constant), X_L decreases, whereas X_C increases. Thus, when a parallel circuit is operated *below its resonant frequency, the circuit appears inductive.*

In Fig. 4(B) are shown the conditions for a parallel circuit in which X_L is greater than X_C; I_L is therefore less than I_C. Under such conditions, the circuit acts as a capacitor.

When a parallel circuit is operated *above its resonant frequency, the circuit appears capacitive,* since as f increases, X_L increases and X_C decreases, thus permitting a greater current to flow through C than through L.

SUSCEPTANCE CURVES.—Probably the simplest method of illustrating the circuit characteristics is by means of a "Susceptance" curve. If the effect of resistance may be neglected, the susceptance curve demonstrates better than any other means the characteristics of the parallel circuit at resonance and at frequencies above and below resonance.

Such a set of curves is shown in Fig. 5. In this figure B_L and B_C are plotted in place of X_L and X_C. Susceptance B is the reciprocal of the reactance X and has the same relation to X that conductance G has to R. Thus the B curve indicates the ability of a reactance branch of the parallel circuit to pass alternating current.

If this curve is compared with the series circuit reactance curve shown in the previous assignment, it will be observed that the curves appear very similar.

Indeed, it is merely necessary to replace the symbols of the series circuit reactance curves shown in Fig. 5 with admittance symbols to

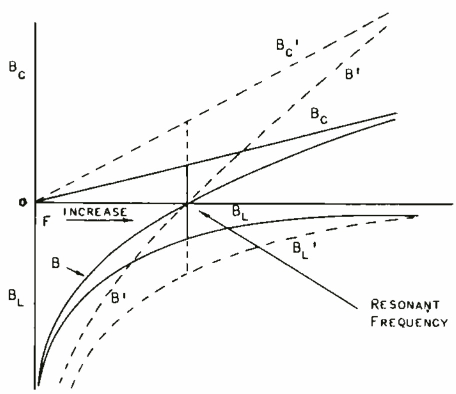


Fig. 5.—Susceptance curve.

obtain the curves. Thus, change the symbol X_C to B_L, X_L to B_C, and X to B' to obtain Fig. 5. This means that the *susceptance* of an inductance has the form $B_L = 1/2\pi fL$ which is mathematically similar to capacitive *reactance* $X_c = 1/2\pi fC$, and capacitive susceptance B_c has the form $1/(1/2\pi fC) = 2\pi fC$ which is mathematically similar to inductive reactance $X_L = 2\pi fL$. Hence B_c is a straight line when plotted against frequency similar to X_L, and B_L is a curved line like X_c.

Resonant frequency is indicated by the point where $B_L = B_c$. This point indicates zero susceptance or a zero ability to pass current through the parallel circuit. This would, if thought of in terms of impedance or resistance, indicate infinite impedance at that particular frequency. Since B_L and B_c are equal at resonance, the circuit would be neither inductive nor capacitive but would act rather as an infinite resistance.

Curve B passing through this point is the algebraic sum, or the arithmetic difference, of B_L and B_c. To the left of the resonance point, (lower frequency) B_L predominates and B is below the reference line. To the right, (higher frequency), B_c predominates and B is above the reference line. Curve B, extending into the region of B_L or B_c, clearly indicates at any frequency whether the parallel circuit is acting as an inductance or as a capacity.

As the frequency is decreased below resonance the circuit is plainly an inductive circuit due to the preponderance of B_L. At frequencies above resonance B_c is predominating and the circuit acts as a capacity.

This set of curves will also demonstrate the manner in which the selectivity of the parallel circuit can be increased. B_c varies directly as the value of capacity. Thus for a larger capacity the B_c curve will rise more steeply. In a similar manner the variation in B_L, for a given frequency variation, will be greater if the value of L is decreased. Since the selectivity of a parallel circuit is a function of the variation of B_L and B_c for a given frequency variation, it will be seen that by increasing the value of C and decreasing the value of L the selectivity of a parallel circuit will be increased. This is shown by the dotted curves B′, B_L', and B_c' in Fig. 5.

COMPARISON OF CHARACTERISTICS OF SERIES AND PARALLEL CIRCUITS.— A comparison with the characteristics of a series circuit will show that the parallel circuit characteristics are just the opposite to those of a series circuit:

At resonance the parallel circuit offers maximum impedance; the series circuit offers minimum impedance.

At frequencies lower than resonance the parallel circuit acts as an inductance; the series circuit acts as a capacity.

At frequencies higher than the resonant frequency the parallel circuit acts as a capacity; the series circuit acts as an inductance.

At resonance both parallel and series circuits act as resistances: the series circuit as a very low resistance, the parallel circuit as a very high resistance.

At resonance adding resistance to the series circuit increases the resulting resistance, while at resonance the effective resistance of a parallel circuit is decreased if a limited amount of resistance is added to each or either branch.

Fig. 6 shows the relative values of current and impedance at resonance and at frequencies above and below resonance. A comparison of these curves with the equivalent curves for the series circuit will show the difference between the characteristics of the two.

These curves may be sharpened by decreasing the losses in the circuit or by decreasing the L/C ratio, and broadened by increasing the losses or by increasing L and decreasing C. As shown in Fig. 5, with normal values of resistance the selectivity of a parallel circuit is mostly a function of the ratio of L/C. The *lower* this ratio the greater the degree of selectivity of the circuit. This is just the opposite to the condition in a series circuit.

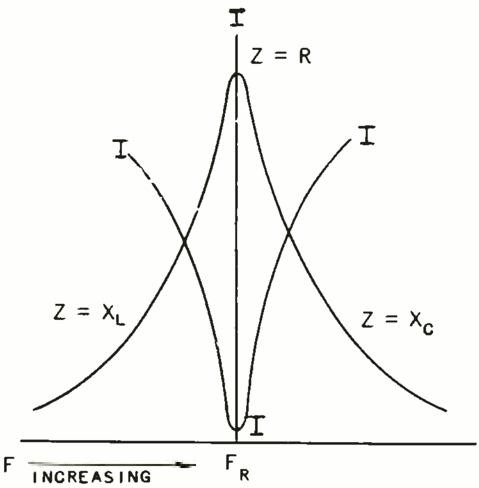


Fig. 6.—Resonance curves.

If the circuit contains NO resistance the external current will be zero and Z will be infinite at resonance. The addition of a limited amount of resistance will increase the value of the external current and decrease the value of Z at resonance.

APPLICATION OF COMPLEX ALGEBRA TO A.C. CIRCUITS

Up to this point, a.c. circuits have been analyzed on the basis of currents and voltages; however, as more complex a.c. circuits are encountered, it becomes cumbersome to utilize the methods of analysis on which the solutions of previous simpler problems were based. In view of that fact, a different (and less cumbersome) analytical viewpoint will now be explained here.

COMPLEX NUMBERS.—Although the following discussion may appear to be an apparent digression from the original trend of thought, it will be found to tie in with the a-c analysis. Recall the effect of multiplying a vector by -1; (See Fig. 7). In this example, assume that vector

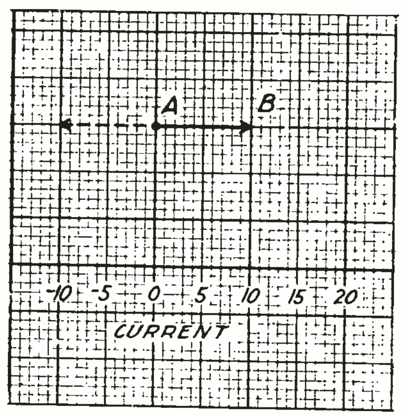


Fig. 7.—Vector showing effect of multiplying a number by -1.

AB represents a current of 10 amperes; it is drawn as shown by the solid line in the figure. Now, if 10 is multipled by -1, the product, of course, is -10; this value is shown by the dotted line in the diagram. It is evident that the effect of multiplying a vector by -1, is to reverse its direction; i.e., the vector is rotated 180° from its original position.

However, in a.c. work there is a need to rotate a vector through 90° instead of a full 180°. For example, if AB in Fig. 7 represents a current in phase with a voltage, then a factor or operator which would rotate AB counterclockwise through 90° would cause it to become a current 90° *leading* the voltage; and another operator that rotated the current 90° clockwise would produce a current that *lagged* the voltage by 90°.

The same convenience would be obtained in changing a resistive magnitude into an inductive or capacitive reactance; in short, it would be very desirable to have a mathematical symbol or tool that could produce 90° rotation in vectors instead of 180° rotations.

Fortunately, such a tool or operator is available. It has been found that the square root of -1, $\sqrt{-1}$, (sometimes known as an imaginary number), produces HALF the rotation of -1 itself; i.e., 90° instead of 180° rotation. By agreement or convention, $\sqrt{-1}$ is assumed to rotate a vector 90° *counterclockwise*; and $-\sqrt{-1}$ then rotates it 90° in the *opposite* direction; that is, *clockwise*.

In mathematics the cumbersome symbol $\sqrt{-1}$ is replaced by the letter *i*. However, in electrical calculations this symbol is used to denote electrical current, hence it has become customary to use the letter j to represent $\sqrt{-1}$, and the symbol for rotation is known as the j-operator.

Thus +j rotates a quantity 90° counterclockwise, and -j rotates it 90° clockwise. For example, in Fig. 8 is shown a vector of length 5. It is drawn horizontal and to the right. If it be multiplied by j; thus, j5, it becomes a vector rotated from its former position by 90° counterclockwise and now points vertically upward as shown in the figure. On the other hand, if it be multiplied by -j; thus -j5, it becomes a vector rotated 90° clockwise from its former position, and now points vertically downward.

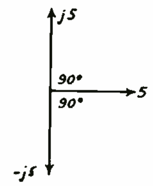


Fig. 8.—Rotation of a vector quantity through 90° by j and -j.

The operator can therefore be used to denote leading and lagging currents and voltages. It can also be used to represent inductive and capacitive REACTANCES. For example, suppose one has a resistance of 5 ohms, an inductive reactance of 5 ohms, and a capacitive reactance of 5 ohms. Then these can be written as 5 ohms, +j5 ohms, and -j5 ohms, respectively. The absence of the symbol j in front of a quantity means that the quantity is resistive; a +j in front of it means that the quantity represents an inductive reactance; and a -j in front of it means that the quantity represents a capacitive reactance. It will be seen later that a +j reactance produces a -j (lagging) current; and a -j reactance produces a +j (leading) current.

The question now arises, "How can we represent a resistance and reactance in series?" The answer is to write each quantity as if it were by itself, and then connect the two quantities with a plus sign. For example, suppose we have a resistance and inductance in series. The two can be written individually as R and jX_L; if they are in series, they are written as $R + jX_L$. On the other hand, if we have a capacitive reactance, it is written as $-jX_c$, and the two in series are written as $R + (-jX_c) = R - jX_c$. (Note the $-j$ when a capacitive reactance is to be denoted.)

The combination of a real and a j-number (also called imaginary number), is known as a *complex* number. A system of algebra has been developed to enable calculations to be made with complex numbers; the system is known as complex algebra. Its application and use in a.c. circuit calculations will now be given.

The *real* number is plotted on the horizontal axis, whereas the *j* number is plotted on the vertical axis. As an illustration, assume that in a series R-L circuit, there is a resistive drop of 20 volts, and an inductive drop of 10 volts. The vector expression for this voltage is $E = 20 + j10$ volts. (See Fig. 9).

Had the reactive drop been capacitive rather than inductive, only the sign of the j term would differ; i.e., $E = 20 - j10$ volts. The line voltage is the vector sum of these voltages.

$$\sqrt{(20)^2 + (10)^2} =$$

$$\sqrt{400 + 100} =$$

$$\sqrt{500} = 22.4 \text{ volts}$$

(length of vector in Fig. 9 = 22.4) In the first case, the line current lags the line voltage; in the second, I leads E.

COMPLEX ALGEBRA.—If it is desired to add two complex numbers, the procedure is as follows: Add the real parts of each vector with due regard to the sign of each term, and write the sum as the real part of the answer; then, add the j parts of each vector with due regard to the sign of each term, and write the sum as the j part of the answer. For example, when two voltages $30 + j12$ and $20 + j14$ are acting in series in a circuit and hence are additive, the result is $(30 + 20) + j\,(12 + 14) = 50 + j26$ volts.

A complex number can be subtracted from another by subtracting the real parts and the j parts separately; the algebraic rules for subtraction are followed.

The multiplication of complex

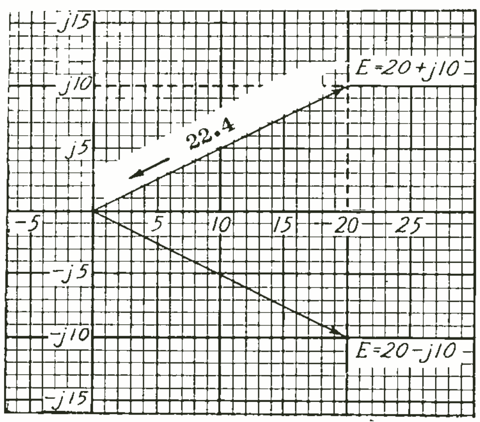


Fig. 9.—Method of locating complex numbers on graph paper.

numbers involves the same fundamental algebraic ideas as multiplying any polynomial expressions; i.e., the real and imaginary parts are multiplied as separate quantities, just like *x* and *y* or *a* and *b*.

For example, multiply $2 + j3$ by $4 - j2$.

$$\begin{array}{l} 2 + j3 \\ \underline{4 - j2} \\ 8 + j12 \\ \underline{\quad - j4 - j^2 6} \\ 8 + j8 - j^2 6 \end{array}$$

Note that 4 times $j3 = j12$, etc.

Since $j^2 = (\sqrt{-1})^2 = -1$, from the very meaning of square and square root, the final answer is :

$$8 + j8 - (-6) = 8 + j8 + 6 = 14 + j8$$

Thus, the product of two complex quantities is a complex quantity, as is also the quotient, square root, etc.

Now, consider a simple a.c. application of multiplication in complex algebra. Suppose a current of 5 amperes flows through a resistance of 1,000 ohms in series with an inductance of 150 μhenries, and that the frequency of the current is 1,000 kc (1,000,000 c.p.s. or 1 mc). This is illustrated in Fig. 10. It is desired to know the voltage drop E.

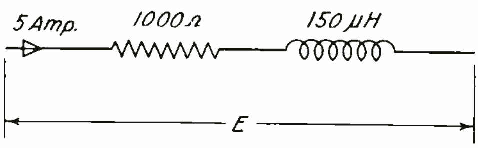


Fig. 10.—Calculation of the voltage drop across an R-L circuit through which a given current is passed.

By Ohm's law for a.c.,

$$E = IZ$$

Here $I = 5$ amperes, and $Z = R + jX_L$.

Further

$$X_L = 2\pi fL = 2\pi \times 10^6 \times 150 \times 10^{-6}$$

$$= 942 \text{ ohms.}$$

Hence

$$Z = 1{,}000 + j942$$

and

$$E = (5)\ (1{,}000 + j942)$$

$$= (5{,}000 + j4{,}710)$$

by merely multiplying through by 5.

In Fig. 11 is given the interpretation of the answer. The current is taken as the reference axis. The 5,000-volt component of E, being a real number, is in phase with the current, as shown (although the scales used for E and I may be entirely different). The j4,710-volt component is then 90° leading the 5,000-volt component (leading on account of the +j), and is accordingly drawn vertically upward. The

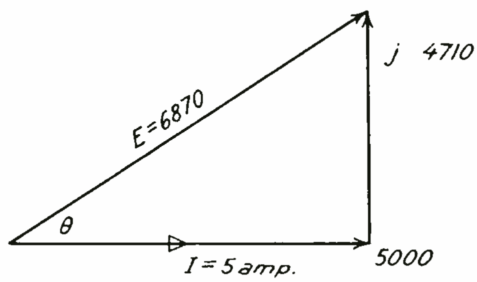


Fig. 11.—Vector relations for circuit shown in Fig. 10.

resultant or total value of E (that across R and L) is the hypotenuse of the right triangle formed, and therefore has the value

$$E = \sqrt{(5{,}000)^2 + (4{,}710)^2}$$
$$= 6{,}870 \text{ volts}$$

This voltage clearly leads the current; from the figure it is evident that

$$\tan\theta = 4{,}710/5{,}000 = 0.9420$$

or

$$\theta = 43°\ 17' \text{ leading}$$

Note how simply the result is obtained! However, of even greater utility is the use of j-operators in a.c. problems involving division. For example, suppose E and Z are given, then

$$I = E/Z$$

and division is involved in solving for I.

In order to solve such problems, it will be of interest to work out a sample problem. Suppose a potential of 10 volts is applied to an impedance of 15 + j22. How much current flows? First note that the impedance represents a 15 ohm resistance in series with an *inductive* reactance of 22 ohms. (It is inductive because +j is involved.)

The current is simply

$$I = 10/(15 + j22)$$

This, however, does not directly indicate the phase of the current nor, its magnitude, because the denominator is complex. If only the numerator were complex, and the denominator were real, the expression could readily be evaluated, as will be shown later.

Hence the first task is to attempt to "rationalize" the denominator, as it is called; i.e., to convert the denominator into a real number, even though at the same time the numerator may become complex.

This, oddly enough, is readily accomplished by multiplying the denominator (and also the numerator, to balance) by the conjugate of the demoninator. *The conjugate is an expression whose imaginary part has the opposite sign to the given expression.*

Hence, in the above example, multiply numerator and denominator by 15 - j22. Thus

$$I = \left(\frac{10}{15 + j22}\right)\left(\frac{15 - j22}{15 - j22}\right)$$

The numerator is readily evaluated by simply multiplying through by 10: (10 × 15) - (j22 × 10) = (150 - j220).

The denominator is multipled out as follows:

$$\begin{array}{r} 15 + j22 \\ \underline{15 - j22} \\ 225 - j330 \\ \underline{+\ j330 - j^2 484} \\ 225 + j0 \quad + \quad 484 \end{array}$$

or <u>709,</u> a REAL NUMBER. Note that $j^2 = \sqrt{(-1)}^2 = -1$, by definition, and $-j^2 = -(-1) = +1$, so that $-j^2 484$ is simply + 484.

Indeed, a careful inspection of the multiplication process shows that the product is essentially

$$(15)^2 + (22)^2 = 709$$

This is a general rule: the product of a complex number A + jB by its conjugate A -jB is simply the sum of the squares of the real and imaginary parts of the number, or $A^2 + B^2$, a REAL NUMBER.

Hence, whenever a complex expression is encountered in the denominator, it can be converted into a real number by simply multiplying numerator and denominator by the conjugate of the complex expression.

We have therefore finally obtained that

$$I = \frac{150 - j220}{709} = \frac{150}{709} - \frac{j220}{709}$$

$$= .212 - j.31$$

Note that once the denominator is a real number, the numerator may be split into a real and imaginary part, and that finally a single complex number is obtained: $(.212 - j.31)$.

This expression is readily interpreted in vector form. (See Fig. 12.) The current is shown to consist of two components: one of magnitude .212 amperes in phase with the voltage, and another of magnitude 0.31 ampere and lagging the voltage by 90° (owing to $-j$).

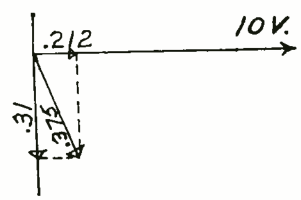


Fig. 12.—Vector diagram for current produced by a given voltage across a known impedance.

The actual value of the current is clearly

$$I = \sqrt{(.212)^2 + (.31)^2}$$

$$= .375 \text{ amperes}$$

and the angle is

$$\theta = \tan^{-1} \frac{-.31}{.212} = \tan^{-1} 1.461$$

$$= 55.6^\circ$$

where the minus sign, in conjunction with Fig. 12 indicates that the angle is one of lag.

PARALLEL IMPEDANCES.—So far it has been shown how to write the impedance of circuit elements in series; how to evaluate the voltage drop across a given impedance if the current through it is known; and finally how to evaluate the current flow through a given impedance when the voltage drop across it is known.

It will now be of interest to see how the equivalent or resultant impedance of two impedances in parallel can be evaluated by means of j-operators. The circuit is shown in Fig. 13; it involves a capacitor in one branch, and an inductance and resistance in series in the other branch.

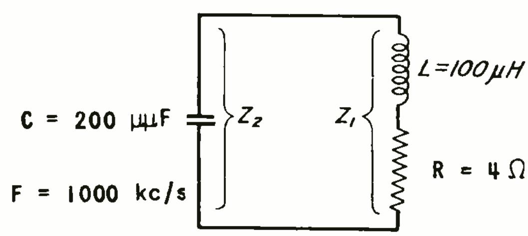


Fig. 13.—Parallel circuit having L and R in one branch, and C in the other.

The formula used to compute the total impedance is similar to that used for resistances in parallel; i.e.,

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2} \tag{5}$$

where Z_1 and Z_2 are the individual branch impedances. Using the system of complex notation as explained earlier in this assignment, the impedance of the inductive branch can be written as:

$$Z_1 = R + jX_L$$

Since the capacitive branch is considered to have no resistance, the impedance of this branch is simply $Z_2 = -jX_C$. The total impedance of the circuit will now be:

$$Z = \frac{Z_1\ Z_2}{Z_1 + Z_2} = \frac{(R + jX_L)\ (-jX_C)}{(R + jX_L) + (-jX_C)} =$$

$$\frac{(R + jX_L)\ (-jX_C)}{R + j\ (X_L - X_C)}$$

(factoring out the j-operator in the denominator). Multiplying Z_1 by Z_2 in the numerator, there is obtained:

$$\begin{array}{r} R + jX_L \\ -j\ X_C \\ \hline -jRX_C - j^2X_CX_L \end{array}$$

$$= X_CX_L - jRX_C$$

Therefore

$$Z = \frac{X_LX_C - jRX_C}{R + j(X_L - X_C)} \qquad (6)$$

The expression as it stands involves a complex denominator as well as a complex numerator. As has been indicated previously it is necessary to clear the denominator of j quantities before a physical interpretation can be given to the expression. The method is exactly the same as described previously: multiply numerator and denominator by the conjugate of the denominator.

In this way the expression in Eq. (6) can be reduced to an ordinary complex number. However, it will be well to proceed with numerical values for the various quantities. Thus from Fig. 13 it is seen that

$$X_L = 2\pi fL = 6.28 \times 1 \times 10^6 \times 100 \times 10^{-6} = 628 \text{ ohms}$$

and

$$X_C = 1/2\pi fC = 1/(6.28 \times 1 \times 10^6 \times 2.00 \times 10^{-10}) = 796 \text{ ohms}$$

Now, substitute these numerical values in Eq. (6):

$$Z = \frac{(628)\ (796) - j\,(4 \times 796)}{4 + j\,(628 - 796)} =$$

$$\frac{500{,}000 - j3{,}184}{4 - j168}$$

Now multiply numerator and denominator by the conjugate of the denominator;

$$\frac{(500{,}000 - j3{,}184)}{(4 - j168)}\ \frac{(4 + j\ 168)}{(4 + j\ 168)}$$

$$= \frac{(500{,}000 - j3{,}184)\ (4 + j\ 168)}{28{,}240}$$

To simplify the numbers, divide numerator (either factor) and the denominator by 10,000, and obtain

$$\frac{(50 - j.318)\ (4 + j168)}{2.82}$$

Now multiply out the numerator:

$$\begin{array}{r} 50 - j\ .318 \\ 4 + j\ 168 \\ \hline 200 - j\ 1.272 \qquad\qquad \\ + j\ 8,400 - j^2 53.4 \\ \hline 200 + j\ 8398.7 + 53.4 \end{array}$$

$$= 253.4 + j\ 8398.7$$

Hence

$$Z = \frac{253.4 + j\ 8398.7}{2.82} = 89.9 + j\ 2,978$$

This means that the two impedance arms correspond to a single impedance consisting of a resistance of 89.9 ohms IN SERIES WITH AN INDUCTANCE WHOSE REACTANCE IS 2,978 OHMS. This is illustrated in Fig. 14. The original circuit is shown

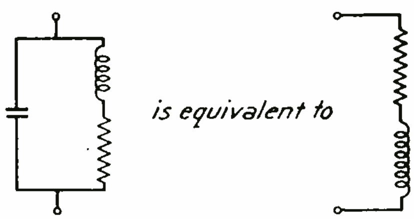


Fig. 14.—The left-hand circuit is equivalent to the right-hand circuit at 1,000 kc.

at the left; it is equivalent to the one at the right. However, the equivalence holds *only* at 1,000 kc; at some other frequency a different value of R and L, or even R and C in the right-hand circuit will be equivalent to the left-hand circuit.

This means that at each frequency under consideration, a separate calculation similar to the one given, must be made to obtain the equivalent impedance. Nevertheless, the method shown, using complex algebra, is a powerful tool for calculating the a-c impedance, current, etc., at any given frequency.

The next problem will be to calculate the same sort of parallel circuit as shown in Fig. 13, except the C value will be changed to have a capacitive reactance equal to the inductive reactance or 628 ohms. Let R remain a value of 4 ohms, and Q will be 628/4 = 157 which is a fairly high value. It has been indicated that parallel resonance *is obtained when the impedance looking into the circuit is purely resistive in nature.* Now it will be determined if our example problem is a condition of parallel resonance. Substitute values in Eq. (6):

$$Z = \frac{(628)^2 - j\ (4 \times 628)}{4 + j\ (628 - 628)}$$

$$= \frac{628^2 - j\ 2512}{4 + j0} = \frac{628^2}{4} - j\ \frac{2512}{4}$$

$$= 98,600 - j\ 628 \text{ ohms}$$

(4 + j0 is the same as 4, and j0 is dropped entirely)

Note carefully that although we have a reactive value of $-j628$ ohms the circuit is still for all practical purposes resonant. Tan $\theta = 628/98600 = 0.00637$ and $\theta = 22'$. A vector drawing would reveal the reactive component only if plotted on a very large scale, since the resistive component is 157 times longer than the reactive component. Presently it will be shown that a very slight change of X_L or X_c will remove even this small reactive value of Z.

It should be of interest to observe the effect of using a similar circuit with a lower value of Q. Let R = 62.8Ω so that Q = 10 for

$X_L = X_c = 628$ ohms.

$$Z = \frac{(628)^2 - j(62.8 \times 628)}{62.8 + j(628 - 628)}$$

$$= \frac{(628)^2}{62.8} - j\frac{(62.8 \times 628)}{62.8}$$

$$= 6280 - j\ 628 \text{ ohms}$$

In this case $\tan\theta = 628/6280 = .10$ and $\theta = 5°42'$. Here although R/X_c has a ratio of 10 to 1, or 1/15.7 of its previous value, the circuit from a practical viewpoint is still essentially at parallel resonance.

The series resistance 62.8Ω can be transformed to a value which can be considered as in parallel with L and C as follows:

$$R_{sh} = \frac{(\omega L)^2}{R_s} = \frac{(628)^2}{62.8} = 6280\Omega \qquad (7)$$

Similarly for the previous example:

$$R_{sh} = \frac{(628)^2}{4} = 98,600\Omega$$

For circuit applications such as transmitter tanks with a Q of 10 and also for high Q circuits of 150 this same approach will be found to give the resistive component of the impedance of the tank and is therefore a very useful relation in parallel resonant circuits. This relation means that at the resonant frequency a low resistance in series with L is stepped up to a high resistance across the terminals of L and C, so that the parallel resonant circuit acts as an impedance transforming device over a narrow range of frequencies centered on the resonant value. Its action is therefore very similar to that of a transformer, except that the latter acts over a greater (and usually lower) band of frequencies.

The impedance transforming property is illustrated in Fig. 15,

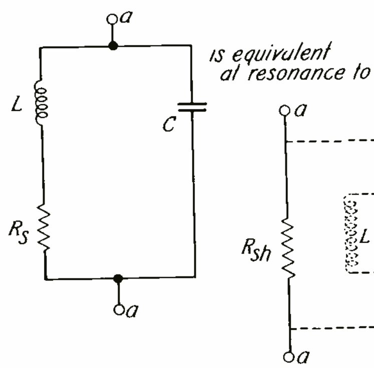


Fig. 15.—Step-up effect of a parallel-resonant circuit on a resistance.

where R_s is in series with L and the equivalent shunt resistance R_{sh} is in parallel with L and C across terminals a-a. Eq. (7) can be written in another form as

$$R_{sh} = \frac{L}{CR} \qquad (8)$$

This is derived from the general formula for Z as given by Eq. (5), and applies to parallel L-C-R circuits in the vicinity of resonance. It makes it possible to bypass the more involved Eqs. (5) or (6), which are used for any parallel circuit whether resonant or not.

Eqs. (7) and (8) can be easily transformed by the use of $Q = \omega L/R$ to arrive at the following expression:

$$R_{sh} = Q\omega L \qquad (9)$$

From this expression and by inspection of the results in the two examples of Z in parallel circuits just shown, it can be seen that the impedance of a parallel circuit for a given inductance and frequency is entirely a function of Q.

Equation (6) can be solved for a condition of no reactive component in Z by letting

$$X_L X_c (X_L - X_c) = -R(X_c R)$$

Solving for X_c,

$$X_c = \frac{X_L^2 + R^2}{X_L} \qquad (10)$$

or

$$C = \frac{L}{R^2 + X_L^2} \qquad (11)$$

By *careful computation to many significant figures* these equations (10) and (11) can be shown to give a purely resistive answer in (6) where X_c and X_L are not exactly equal. If $X_L = 628$ and $R = 4$ then

$$X_c = \frac{(628)^2 + 16}{628} = 628.02547\Omega$$

If R is made 62.8Ω then for $X_L = 628\Omega$ it is found that

$$X_c = 634.28\Omega$$

The frequency for resonance in a parallel circuit can be defined as that producing unity power factor (a purely resistive impedance). A formula can be developed from Eq. (5):

$$f_r = \frac{1}{2\pi\sqrt{LC}}\sqrt{1 - \frac{CR^2}{L}} \qquad (12)$$

This further demonstrates that R, as well as L and C, has an influence on the resonant frequency.

Another way of seeing the effect of R is to draw a vector diagram of the currents in a parallel circuit (Fig. 16). If I_c is shown leading E by 90° and I_L lagging by θ°, then I_c must equal $I_L \sin\theta$ in magnitude for the reactive components to cancel and I_t to be in phase with E, i.e., unity power factor. This will also give $Z = E/I = R$, a purely resistive impedance value.

PARALLEL CIRCUITS WITH R IN BOTH L AND C BRANCHES.—Consider the special case Fig. 17(A) where the resistance is in both branches of the parallel circuit. If $X_L = X_c$ and $R_1 = R_2$, then it is apparent that $Z_1 = Z_2$ and $I_L = I_c$ so that the input impedance will be purely resistive (I_t is in phase with E). I_t will be equal to $I_c \cos\theta$ plus $I_L \cos\theta$, since these components are both in phase with each other and with E.

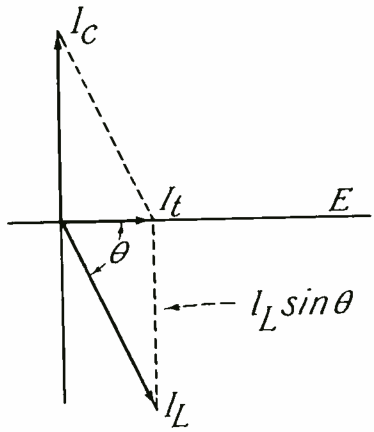


Fig. 16.—Vector diagram of the current in a parallel LCR circuit.

See Fig. 17(B).

The general expression for the input impedance of Fig. 17(A) was given by Eq. (5):

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Substituting for Z_1 and Z_2 there is obtained

$$Z = \frac{(R_1 + j\omega L)(R_2 + \frac{1}{j\omega C})}{R_1 + R_2 + j\omega L + \frac{1}{j\omega C})} \quad (13)$$

An example of a solution of this type of problem will now be shown. Let $R_1 = 5\Omega$, $R_2 = 3\Omega$, $j\omega L = 500\Omega$, $1/j\omega C = -j300$, find Z.

$$Z = \frac{(5 + j500)(3 - j300)}{5 + 3 + j500 - j300}$$

$$= \frac{15 + j1500 - j1500 - j^2 150,000}{8 + j200}$$

$$= \frac{150,015}{8 + j200}$$

Rationalizing

$$Z = \frac{150,015\ (8 - j200)}{(8 + j200)(8 - j200)}$$

$$= \frac{1,200,120 - j\ 30,003,000}{64 + 40,000}$$

$$= \frac{1,200,120 - j30,003,000}{40,064} = 30 - j750$$

Magnitude of the impedance is

$$|Z| = \sqrt{30^2 + 750^2} = \sqrt{900 + 562500}$$

$$= \sqrt{563400} = 751\Omega$$

It can be seen from this example that any parallel circuit can be solved by application of Eq. (5). As mentioned previously, for the special case of $R_1 = R_2$ and $X_L = X_c$, the value of Z will be a pure resistance.

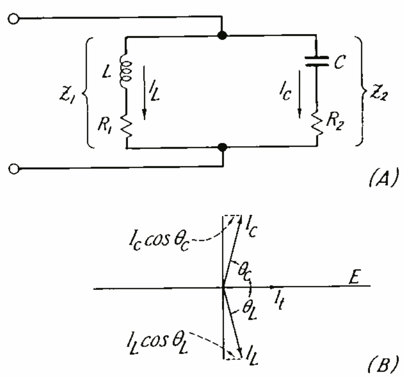


Fig. 17.—Parallel circuit with R in both L and C branches.

SOLUTION OF PARALLEL CIRCUITS BY ADMITTANCE METHOD.—Where three or more parallel branch circuits (see Fig. 18) are to be solved for the total parallel impedance, the solution can be obtained by using Eq. (5) in steps:

$$Z_A = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

then

$$Z_T = \frac{Z_A Z_3}{Z_A + Z_3}$$

or the circuit can be solved by the general formula:

$$Z_T = \frac{(Z_1)(Z_2)(Z_3)}{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3} \quad (14)$$

which is derived from the admittance form

$$\frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \qquad (15)$$

as has been shown for resistances in 'Ohm's and Kirchhoff's Laws'.

If Z_1, Z_2 and Z_3 are complex impedances containing both resistive and reactive elements, the individual elements in Eq. (15) can be solved

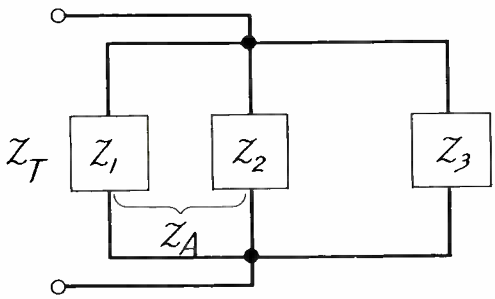


Fig. 18.—Three branches of impedances connected in parallel.

separately, and added together to get the total admittance Y_T, then $Z_T = 1/Y_T$. As an example see Fig. 19, which will now be solved as outlined above.

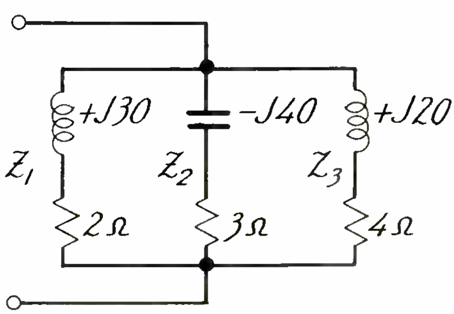


Fig. 19.—Three branches of complex impedances connected in parallel.

Since $Y_1 = \frac{1}{Z_1}$ or $\frac{1}{2 + j30}$

then $$Y_1 = \frac{1}{2 + j30}\left(\frac{2 - j30}{2 - j30}\right) = \frac{2 - j30}{4 + 900}$$

$$= \frac{2}{904} - j\frac{30}{904} = .00221 - j.0332$$

(notice the admittance Y has an opposite sign for the reactive component since it is a reciprocal of the impedance)

$$Y_2 = \frac{1}{3 - j40}\left(\frac{3 + j40}{3 + j40}\right) = \frac{3 + j40}{1609}$$

$$= .001864 + j.02484$$

$$Y_3 = \frac{1}{4 + j20}\left(\frac{4 - j20}{4 - j20}\right) = \frac{4 - j20}{416}$$

$$= .00962 - j.0481$$

$$\begin{aligned} Y_T = \; & .00221 - j.0332 \\ & .001864 + j.02484 \\ & .00962 - j.0481 \\ \hline & .013694 - j.05646 \end{aligned}$$

Z_T can then be found by taking the reciprocal of Y_T:

$$Z_T = \frac{1}{.013694 - j.05646} \times \left(\frac{.013694 + j.05646}{.013694 + j.05646}\right)$$

$$= \frac{.013694 + j.05646}{1.875 \times 10^{-4} + 31.877 \times 10^{-4}}$$

$$= \frac{136.94 \times 10^{-4}}{33.752 \times 10^{-4}} + j\frac{564.6 \times 10^{-4}}{33.752 \times 10^{-4}}$$

$$= 4.06 + j\,16.72$$

APPLICATION OF PARALLEL RESONANCE.—An interesting application of parallel resonance to receiver circuits is as a plate load resistance. This is illustrated in Fig. 20. Capacitor C is made up in part of a physical capacitor. The latter may

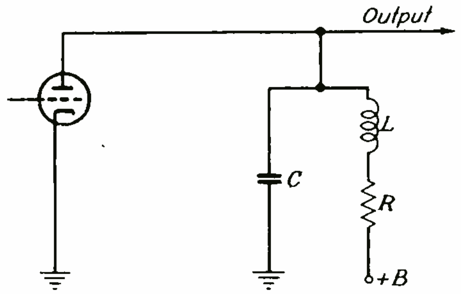


Fig. 20.—Application of parallel resonance to voltage amplifier stage.

be variable if tuning over a range is desired. The coil has an inductance L and a resistance R. This type of circuit will be found in some f.m. and television receivers, particularly those that are fixed tuned to certain channels, and employ a push-button or channel-selector switch.

The object of this circuit is to build up as high an impedance at resonance as possible, and yet not make the circuit tune too sharply, or it will not amplify uniformly over the desired band width.

The latter factor depends upon the Q of the circuit: the higher this is, the narrower the band width. The circuit Q or Q_c, can be expressed as

$$Q_c = \omega CR_{sh} \quad (16)$$

(in terms of C and the apparent shunting resistance R_{sh}. If the capacitor has no appreciable losses, then the apparent shunting resistance R_{sh} is, by Eq. (8),

$$R_{sh} = L/CR \quad (8)$$

Substitute this value of R_{sh} in Eq. (16) and obtain

$$Q_c = \omega CR_{sh} = \omega C(L/CR) = \omega L/R \quad (17)$$

But $\omega L/R$ is the Q of the coil; hence it is seen that the circuit Q of a parallel-resonant circuit is identical with its coil Q if there are no other losses in the circuit. If a high Q is desired, as in narrow-band a.m. broadcast practice, coils of low (a.c.) resistance are required to yield high coil and hence high circuit Q's.

At ultra-high frequencies there will be losses produced by the following tube owing to transit-time and cathode-lead inductance. These input losses of the following tube may be represented by a shunt resistance R_i across the parallel resonant circuit feeding the tube.

The circuit Q will now be lower than that produced by the coil itself. However, it can very readily be evaluated. Thus suppose, as in Fig. 21, tube I feeds tube II via the parallel-resonant circuit composed of capacitor C and the coil

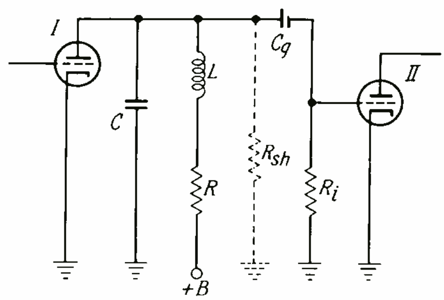


Fig. 21.—Parallel-resonant coupling circuit between tubes I and II. The input resistance of tube II is R_i; the equivalent shunt resistance of the coil resistance R is R_{sh}.

whose inductance is L and resistance is R. Note that C not only can include the output capacitance of tube I, but also the input capacitance of tube II.

The latter tube presents in addition a resistance R_i to the parallel-resonant circuit, even though the grid may be adequately biased negative. This is a property of u.h.f. operation, and will be discussed in a later assignment.

The effect of coil resistance R is to present an equivalent resistance R_{sh} across the tuned circuit, in accordance with Eq. (8). From Fig. 21 it is clear that R_{sh} is in parallel with R_i, so that the total net resistance across this circuit is

$$R_T = \frac{R_i R_{sh}}{R_i + R_{sh}} \tag{18}$$

where R_T is lower than R_i or R_{sh}.

It therefore follows from Eq. (8) that the circuit Q is lower than before:

$$Q_c = \omega C R_T \tag{19}$$

instead of $\omega C R_{sh}$, so that the band width will be broadened, and the circuit gain reduced. Indeed, at sufficiently high frequencies R_i becomes so low that the circuit gain is reduced to a value below unity. When this occurs, the tube no longer amplifies, but instead introduces a loss in the system.

It is at such frequencies that new and special tubes, such as the Klystron, enter the picture, and new and special circuits are required. With the releasing of the higher band of television frequencies, between 400 and 800 mc, we can expect even the miniature negative-grid tubes to show reduced gain, and if frequencies above 2,000 mc or so are employed, perhaps the more special klystron and similar tubes will have to be used.

CONCLUSIONS

This concludes the assignment on parallel LCR circuits. The methods of computing the total resistance of resistors in parallel were shown first. Then the condition of parallel resonance was explained. It was learned that a parallel circuit has characteristics which are exactly opposite to those of a series circuit. The characteristics of a parallel circuit may be summarized as follows:

(1) at resonance, the parallel circuit offers maximum impedance to the flow of current;

(2) at frequencies above resonance, the parallel circuit acts as a capacity;

(3) at frequencies below resonance, the parallel circuit acts as an inductance.

Parallel-resonant circuits present a very *high resistive* impedance at resonance if the components are of low loss, and are therefore well suited to provide a plate-coupling impedance in an r. f. amplifier stage, as well as to act as a series trap circuit to block the flow of a current at the resonant frequency.

The use of the j-operator, $\sqrt{-1}$, and of complex algebra to solve a-c circuit problems, has enormously facilitated the analysis of such problems. The basic ideas and methods have been presented here; further examples and applications will appear in subsequent assignments.

THE PARALLEL LCR CIRCUITS

EXERCISES

1. Given: $(2 + j5)$ and $(2 - j5)$; $(4 + j0)$ and $(3 - j0)$; $(3 + j1)$ and $(5 + j4)$.

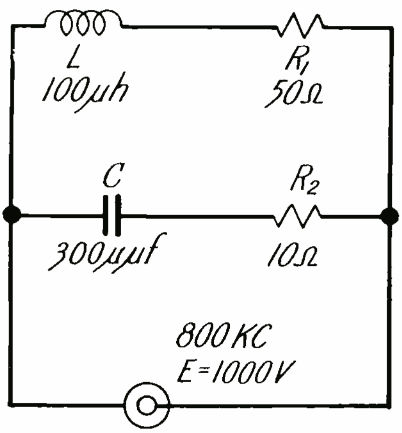
   (A) Add each pair of given complex numbers.

   (B) Subtract the second value from the first value of each pair of complex numbers.

2. Given: $(3 + j1)$ and $(3 - j1)$; $(2 + j1)$ and $(-2 + j1)$; $(4 + j10)$ and $(2 + j0)$.

   (A) Multiply each pair of complex numbers.

   (B) Divide the first complex number of each pair by the second complex number of the same pair.

3. 

   The plate tank circuit of a transmitter consists of a 100 μh inductance and a 300 μμf capacitance. The effective resistance of the inductive leg is 50 ohms and that of the capacitive leg is 10 ohms. Find the impedance of the tank at a frequency of 800 KC, if the voltage across the tank is 1000 V. Use the complex algebra method (Eq. 13).

4. In reference to Prob. 3, find the *approximate resonant impedance* by use of Equation 8. Note $R = 60\Omega$ (series value).

5. What is the impedance of the circuit given in Problem 3 as determined by the admittance method?

THE PARALLEL LCR CIRCUIT

ANSWERS EXERCISE PROBLEMS

1. (A) 4, 7, 8 + j5

   (B) j10; 1, -2 -j3

2. (A) 10, -5, 8 + j20

   (B) $\frac{4 + j3}{5}$, $\frac{-3 - j4}{5}$, 2 + j5

3. 1932 ohms $\theta = 64° 51'$ lag

4. 5556 ohms

5. 1939 ohms $\theta = 64° 51'$ lag

THE PARALLEL LCR CIRCUIT

EXAMINATION

1. A low-ohmage resistor of appreciable current-carrying capacity is required in the deflection circuit of a television receiver. It is made up of a 50-ohm, 100-ohm, and 300-ohm resistor, all in parallel. Find the total resistance of the combination by two methods.

2. In a shunt-peaking type of video amplifier the tube and stray-wiring capacity amounts to 30 $\mu\mu f$. This capacity is placed in parallel with a coil whose inductance is 13.17 μhenries.

(A) At what frequency will these two circuit elements resonate? (Use R of 3 ohms in Eq. (12), as a check on the effect of resistance).

(B) Suppose a generator developing 20 volts at 5 mc is connected across them. Neglecting the resistance of the circuit, find the current in each branch of the circuit, also the current in the external circuit to the generator, and the impedance of the circuit.

2. *(Continued)*

(C) Draw a vector diagram showing the total current and state whether the circuit is acting as an inductance or as a capacity.

3. What are the characteristics of a parallel circuit, inductive, capacitive or resistive, under the following conditions:

(A) At resonance? Why?

3. *(Continued)*

(B) At a frequency higher than resonance? Why?

(C) At a frequency lower than resonance? Why?

4. (A) How would you design a parallel circuit consisting of capacity and inductance to offer the highest impedance at resonance? Show this condition on a vector diagram.

4. *(Continued)*

(B) Also illustrate the conditions in a parallel circuit at resonance in which the impedance is not high. Explain the reasons for this decreased impedance.

5. How do the characteristics of a series circuit and a parallel circuit compare at: resonance, at frequencies above resonance, at frequencies below resonance? Show these differences by the use of appropriate curves.

6. You wish to trap out an interfering signal having a frequency of 50 mc from a television receiver. You have a 50 $\mu\mu f$ variable capacitor available.

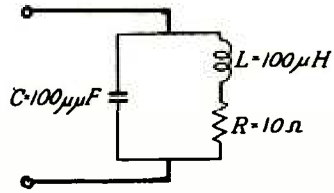
(A) What value of inductance would you use with this capacitor when set at 40 $\mu\mu f$?

(B) If the coil has a resistance of 1 ohm, what will be the approximate parallel impedance?

6. (continued)

(C) What value of inductance would be required if a 100 μμf capacitor were connected in parallel with the tuning capacitor, still set at 40 μμf?

7. Using the complex algebra method, calculate the equivalent impedance of the following circuit. f = 1.592 mc/s.
Use Eq. (6).



EXAMINATION, Page 10.

8. Calculate the equivalent impedance of the circuit of question 7, using the parallel resonant impedance formula. See Eq. (7), (8) or (9).

9. The plate tank circuit of a transmitter consists of L = 100 μh, C = 400 μμf. The effective resistance of the inductive leg is 10 ohms. Calculate by Z = L/CR the impedance of the circuit.

(A) Calculate further values of Z by adding to the 10 ohms in 5 ohm steps for values from 5 to 50 ohms or 15 to 60 ohms total R. Show sample calculation and tabulate your results.

(B) Draw a graph on linear paper using added value of resistance R as the abscissa and Z as ordinate.

10. Given the circuit and values shown. Find the total parallel impedance of the circuit. Use Eq. (13).