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A Mathematical Theory of Communication

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Introduction

THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist[1] and Hartley[2] on this subject. In the present paper we will extend the theory to include a number of new factors, in particular the effect of noise in the channel, and the savings possible due to the statistical structure of the original message and due to the nature of the final destination of the information.

The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point. Frequently the messages have *meaning*; that is they refer to or are correlated according to some system with certain physical or conceptual entities. These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one *selected from a set* of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design.

If the number of messages in the set is finite then this number or any monotonic function of this number can be regarded as a measure of the information produced when one message is chosen from the set, all choices being equally likely. As was pointed out by Hartley the most natural choice is the logarithmic function. Although this definition must be generalized considerably when we consider the influence of the statistics of the message and when we have a continuous range of messages, we will in all cases use an essentially logarithmic measure.

The logarithmic measure is more convenient for various reasons:

1. It is practically more useful. Parameters of engineering importance

[1] Nyquist, H., "Certain Factors Affecting Telegraph Speed," *Bell System Technical Journal*, April 1924, p. 324; "Certain Topics in Telegraph Transmission Theory," *A. I. E. E. Trans.*, v. 47, April 1928, p. 617.

[2] Hartley, R. V. L., "Transmission of Information," *Bell System Technical Journal*, July 1928, p. 535.

such as time, bandwidth, number of relays, etc., tend to vary linearly with the logarithm of the number of possibilities. For example, adding one relay to a group doubles the number of possible states of the relays. It adds 1 to the base 2 logarithm of this number. Doubling the time roughly squares the number of possible messages, or doubles the logarithm, etc.

2. It is nearer to our intuitive feeling as to the proper measure. This is closely related to (1) since we intuitively measure entities by linear comparison with common standards. One feels, for example, that two punched cards should have twice the capacity of one for information storage, and two identical channels twice the capacity of one for transmitting information.

3. It is mathematically more suitable. Many of the limiting operations are simple in terms of the logarithm but would require clumsy restatement in terms of the number of possibilities.

The choice of a logarithmic base corresponds to the choice of a unit for measuring information. If the base 2 is used the resulting units may be called binary digits, or more briefly *bits*, a word suggested by J. W. Tukey. A device with two stable positions, such as a relay or a flip-flop circuit, can store one bit of information. N such devices can store N bits, since the total number of possible states is 2^N and $\log_2 2^N = N$. If the base 10 is used the units may be called decimal digits. Since

$$\begin{aligned} \log_2 M &= \log_{10} M/\log_{10} 2 \\ &= 3.32 \log_{10} M, \end{aligned}$$

a decimal digit is about $3\frac{1}{3}$ bits. A digit wheel on a desk computing machine has ten stable positions and therefore has a storage capacity of one decimal digit. In analytical work where integration and differentiation are involved the base e is sometimes useful. The resulting units of information will be called natural units. Change from the base a to base b merely requires multiplication by $\log_b a$.

By a communication system we will mean a system of the type indicated schematically in Fig. 1. It consists of essentially five parts:

1. An *information source* which produces a message or sequence of messages to be communicated to the receiving terminal. The message may be of various types: e.g. (a) A sequence of letters as in a telegraph or teletype system; (b) A single function of time $f(t)$ as in radio or telephony; (c) A function of time and other variables as in black and white television—here the message may be thought of as a function $f(x, y, t)$ of two space coordinates and time, the light intensity at point (x, y) and time t on a pickup tube plate; (d) Two or more functions of time, say $f(t)$, $g(t)$, $h(t)$—this is the case in "three dimensional" sound transmission or if the system is intended to service several individual channels in multiplex; (e) Several functions of

several variables—in color television the message consists of three functions $f(x, y, t)$, $g(x, y, t)$, $h(x, y, t)$ defined in a three-dimensional continuum—we may also think of these three functions as components of a vector field defined in the region—similarly, several black and white television sources would produce "messages" consisting of a number of functions of three variables; (f) Various combinations also occur, for example in television with an associated audio channel.

2. A *transmitter* which operates on the message in some way to produce a signal suitable for transmission over the channel. In telephony this operation consists merely of changing sound pressure into a proportional electrical current. In telegraphy we have an encoding operation which produces a sequence of dots, dashes and spaces on the channel corresponding to the message. In a multiplex PCM system the different speech functions must be sampled, compressed, quantized and encoded, and finally interleaved properly to construct the signal. Vocoder systems, television, and frequency modulation are other examples of complex operations applied to the message to obtain the signal.

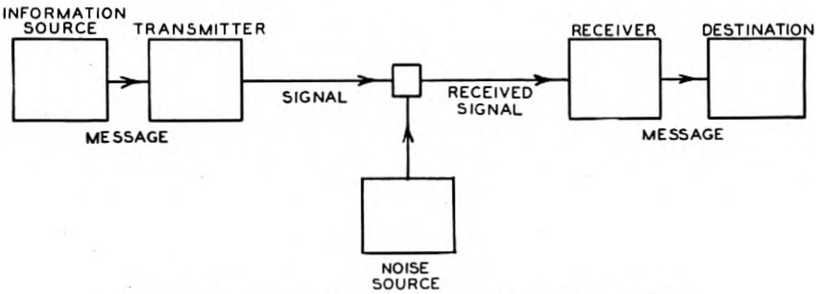

Fig. 1—Schematic diagram of a general communication system.

3. The *channel* is merely the medium used to transmit the signal from transmitter to receiver. It may be a pair of wires, a coaxial cable, a band of radio frequencies, a beam of light, etc.

4. The *receiver* ordinarily performs the inverse operation of that done by the transmitter, reconstructing the message from the signal.

5. The *destination* is the person (or thing) for whom the message is intended.

We wish to consider certain general problems involving communication systems. To do this it is first necessary to represent the various elements involved as mathematical entities, suitably idealized from their physical counterparts. We may roughly classify communication systems into three main categories: discrete, continuous and mixed. By a discrete system we will mean one in which both the message and the signal are a sequence of discrete symbols. A typical case is telegraphy where the message is a sequence of letters and the signal a sequence of dots, dashes and spaces. A continuous system is one in which the message and signal are both treated as continuous functions, e.g. radio or television. A mixed system is one in which both discrete and continuous variables appear, e.g., PCM transmission of speech.

We first consider the discrete case. This case has applications not only in communication theory, but also in the theory of computing machines, the design of telephone exchanges and other fields. In addition the discrete case forms a foundation for the continuous and mixed cases which will be treated in the second half of the paper.

PART I: DISCRETE NOISELESS SYSTEMS

1. The Discrete Noiseless Channel

Teletype and telegraphy are two simple examples of a discrete channel for transmitting information. Generally, a discrete channel will mean a system whereby a sequence of choices from a finite set of elementary symbols $S_1 \cdots S_n$ can be transmitted from one point to another. Each of the symbols S_i is assumed to have a certain duration in time t_i seconds (not necessarily the same for different S_i, for example the dots and dashes in telegraphy). It is not required that all possible sequences of the S_i be capable of transmission on the system; certain sequences only may be allowed. These will be possible signals for the channel. Thus in telegraphy suppose the symbols are: (1) A dot, consisting of line closure for a unit of time and then line open for a unit of time; (2) A dash, consisting of three time units of closure and one unit open; (3) A letter space consisting of, say, three units of line open; (4) A word space of six units of line open. We might place the restriction on allowable sequences that no spaces follow each other (for if two letter spaces are adjacent, it is identical with a word space). The question we now consider is how one can measure the capacity of such a channel to transmit information.

In the teletype case where all symbols are of the same duration, and any sequence of the 32 symbols is allowed the answer is easy. Each symbol represents five bits of information. If the system transmits n symbols per second it is natural to say that the channel has a capacity of $5n$ bits per second. This does not mean that the teletype channel will always be transmitting information at this rate—this is the maximum possible rate and whether or not the actual rate reaches this maximum depends on the source of information which feeds the channel, as will appear later.

In the more general case with different lengths of symbols and constraints on the allowed sequences, we make the following definition:
Definition: The capacity C of a discrete channel is given by

$$C = \operatorname*{Lim}_{T \to \infty} \frac{\log N(T)}{T}$$

where $N(T)$ is the number of allowed signals of duration T.

It is easily seen that in the teletype case this reduces to the previous result. It can be shown that the limit in question will exist as a finite number in most cases of interest. Suppose all sequences of the symbols $S_1, \cdots, S_n$ are allowed and these symbols have durations $t_1, \cdots, t_n$. What is the channel capacity? If $N(t)$ represents the number of sequences of duration t we have

$$N(t) = N(t - t_1) + N(t - t_2) + \cdots + N(t - t_n)$$

The total number is equal to the sum of the numbers of sequences ending in $S_1, S_2, \cdots, S_n$ and these are $N(t - t_1), N(t - t_2), \cdots, N(t - t_n)$, respectively. According to a well known result in finite differences, $N(t)$ is then asymptotic for large t to X_0^t where X_0 is the largest real solution of the characteristic equation:

$$X^{-t_1} + X^{-t_2} + \cdots + X^{-t_n} = 1$$

and therefore

$$C = \log X_0$$

In case there are restrictions on allowed sequences we may still often obtain a difference equation of this type and find C from the characteristic equation. In the telegraphy case mentioned above

$$N(t) = N(t - 2) + N(t - 4) + N(t - 5) + N(t - 7) + N(t - 8) + N(t - 10)$$

as we see by counting sequences of symbols according to the last or next to the last symbol occurring. Hence C is $-\log \mu_0$ where μ_0 is the positive root of $1 = \mu^2 + \mu^4 + \mu^5 + \mu^7 + \mu^8 + \mu^{10}$. Solving this we find $C = 0.539$.

A very general type of restriction which may be placed on allowed sequences is the following: We imagine a number of possible states $a_1, a_2, \cdots, a_m$. For each state only certain symbols from the set $S_1, \cdots, S_n$ can be transmitted (different subsets for the different states). When one of these has been transmitted the state changes to a new state depending both on the old state and the particular symbol transmitted. The telegraph case is a simple example of this. There are two states depending on whether or not a space was the last symbol transmitted. If so then only a dot or a dash can be sent next and the state always changes. If not, any symbol can be transmitted and the state changes if a space is sent, otherwise it remains the same. The conditions can be indicated in a linear graph as shown in Fig. 2. The junction points correspond to the states and the lines indicate the symbols possible in a state and the resulting state. In Appendix I it is shown that if the conditions on allowed sequences can be described in this form C will exist and can be calculated in accordance with the following result:

Theorem 1: Let $b_{ij}^{(s)}$ be the duration of the s^{th} symbol which is allowable in state i and leads to state j. Then the channel capacity C is equal to log W where W is the largest real root of the determinant equation:

$$\left| \sum_s W^{-b_{ij}^{(s)}} - \delta_{ij} \right| = 0.$$

where $\delta_{ij} = 1$ if $i = j$ and is zero otherwise.

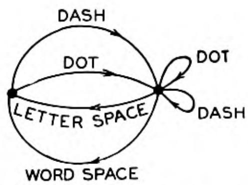

Fig. 2—Graphical representation of the constraints on telegraph symbols.

For example, in the telegraph case (Fig. 2) the determinant is:

$$\begin{vmatrix} -1 & (W^{-2} + W^{-4}) \\ (W^{-3} + W^{-6}) & (W^{-2} + W^{-4} - 1) \end{vmatrix} = 0$$

On expansion this leads to the equation given above for this case.

2. The Discrete Source of Information

We have seen that under very general conditions the logarithm of the number of possible signals in a discrete channel increases linearly with time. The capacity to transmit information can be specified by giving this rate of increase, the number of bits per second required to specify the particular signal used.

We now consider the information source. How is an information source to be described mathematically, and how much information in bits per second is produced in a given source? The main point at issue is the effect of statistical knowledge about the source in reducing the required capacity of the channel, by the use of proper encoding of the information. In telegraphy, for example, the messages to be transmitted consist of sequences of letters. These sequences, however, are not completely random. In general, they form sentences and have the statistical structure of, say, English. The letter E occurs more frequently than Q, the sequence TH more frequently than XP, etc. The existence of this structure allows one to make a saving in time (or channel capacity) by properly encoding the message sequences into signal sequences. This is already done to a limited extent in telegraphy by using the shortest channel symbol, a dot, for the most common English letter E; while the infrequent letters, Q, X, Z are represented by longer sequences of dots and dashes. This idea is carried still further in certain commercial codes where common words and phrases are represented by four- or five-letter code groups with a considerable saving in average time. The standardized greeting and anniversary telegrams now in use extend this to the point of encoding a sentence or two into a relatively short sequence of numbers.

We can think of a discrete source as generating the message, symbol by symbol. It will choose successive symbols according to certain probabilities depending, in general, on preceding choices as well as the particular symbols in question. A physical system, or a mathematical model of a system which produces such a sequence of symbols governed by a set of probabilities is known as a stochastic process.[3] We may consider a discrete source, therefore, to be represented by a stochastic process. Conversely, any stochastic process which produces a discrete sequence of symbols chosen from a finite set may be considered a discrete source. This will include such cases as:

1. Natural written languages such as English, German, Chinese.
2. Continuous information sources that have been rendered discrete by some quantizing process. For example, the quantized speech from a PCM transmitter, or a quantized television signal.
3. Mathematical cases where we merely define abstractly a stochastic process which generates a sequence of symbols. The following are examples of this last type of source.
   (A) Suppose we have five letters A, B, C, D, E which are chosen each with probability .2, successive choices being independent. This would lead to a sequence of which the following is a typical example.
   B D C B C E C C C A D C B D D A A E C E E A
   A B B D A E E C A C E E B A E E C B C E A D
   This was constructed with the use of a table of random numbers.[4]

[3] See, for example, S. Chandrasekhar, "Stochastic Problems in Physics and Astronomy," *Reviews of Modern Physics*, v. 15, No. 1, January 1943, p. 1.
[4] Kendall and Smith, "Tables of Random Sampling Numbers," Cambridge, 1939.

(B) Using the same five letters let the probabilities be .4, .1, .2, .2, .1 respectively, with successive choices independent. A typical message from this source is then:
A A A C D C B D C E A A D A D A C E D A
E A D C A B E D A D D C E C A A A A A D

(C) A more complicated structure is obtained if successive symbols are not chosen independently but their probabilities depend on preceding letters. In the simplest case of this type a choice depends only on the preceding letter and not on ones before that. The statistical structure can then be described by a set of transition probabilities $p_i(j)$, the probability that letter i is followed by letter j. The indices i and j range over all the possible symbols. A second equivalent way of specifying the structure is to give the "digram" probabilities $p(i, j)$, i.e., the relative frequency of the digram $i\ j$. The letter frequencies $p(i)$, (the probability of letter i), the transition probabilities $p_i(j)$ and the digram probabilities $p(i, j)$ are related by the following formulas.

$$p(i) = \sum_j p(i, j) = \sum_j p(j, i) = \sum_j p(j) p_j(i)$$

$$p(i, j) = p(i) p_i(j)$$

$$\sum_j p_i(j) = \sum_i p(i) = \sum_{i,j} p(i, j) = 1.$$

As a specific example suppose there are three letters A, B, C with the probability tables:

$p_i(j)$		j: A	B	C
	A	0	$\frac{4}{5}$	$\frac{1}{5}$
i	B	$\frac{1}{2}$	$\frac{1}{2}$	0
	C	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{10}$

i	$p(i)$
A	$\frac{9}{27}$
B	$\frac{16}{27}$
C	$\frac{2}{27}$

$p(i, j)$		j: A	B	C
	A	0	$\frac{4}{15}$	$\frac{1}{15}$
i	B	$\frac{8}{27}$	$\frac{8}{27}$	0
	C	$\frac{1}{27}$	$\frac{4}{135}$	$\frac{1}{135}$

A typical message from this source is the following:
A B B A B A B A B A B A B A B B B A B B B B B A B
A B A B A B A B B B A C A C A B B A B B B B A B B
A B A C B B B A B A

The next increase in complexity would involve trigram frequencies but no more. The choice of a letter would depend on the preceding two letters but not on the message before that point. A set of trigram frequencies $p(i, j, k)$ or equivalently a set of transition probabilities $p_{ij}(k)$ would be required. Continuing in this way one obtains successively more complicated stochastic processes. In the general n-gram case a set of n-gram probabilities $p(i_1, i_2, \cdots, i_n)$ or of transition probabilities $p_{i_1, i_2, \cdots, i_{n-1}}(i_n)$ is required to specify the statistical structure.

(D) Stochastic processes can also be defined which produce a text consisting of a sequence of "words." Suppose there are five letters A, B, C, D, E and 16 "words" in the language with associated probabilities:

.10 A	.16 BEBE	.11 CABED	.04 DEB
.04 ADEB	.04 BED	.05 CEED	.15 DEED
.05 ADEE	.02 BEED	.08 DAB	.01 EAB
.01 BADD	.05 CA	.04 DAD	.05 EE

Suppose successive "words" are chosen independently and are separated by a space. A typical message might be:
DAB EE A BEBE DEED DEB ADEE ADEE EE DEB BEBE BEBE BEBE ADEE BED DEED DEED CEED ADEE A DEED DEED BEBE CABED BEBE BED DAB DEED ADEB
If all the words are of finite length this process is equivalent to one of the preceding type, but the description may be simpler in terms of the word structure and probabilities. We may also generalize here and introduce transition probabilities between words, etc.

These artificial languages are useful in constructing simple problems and examples to illustrate various possibilities. We can also approximate to a natural language by means of a series of simple artificial languages. The zero-order approximation is obtained by choosing all letters with the same probability and independently. The first-order approximation is obtained by choosing successive letters independently but each letter having the same probability that it does in the natural language.[5] Thus, in the first-order approximation to English, E is chosen with probability .12 (its frequency in normal English) and W with probability .02, but there is no influence between adjacent letters and no tendency to form the preferred digrams such as *TH*, *ED*, etc. In the second-order approximation, digram structure is introduced. After a letter is chosen, the next one is chosen in accordance with the frequencies with which the various letters follow the first one. This requires a table of digram frequencies $p_i(j)$. In the third-order approximation, trigram structure is introduced. Each letter is chosen with probabilities which depend on the preceding two letters.

[5] Letter, digram and trigram frequencies are given in "Secret and Urgent" by Fletcher Pratt, Blue Ribbon Books 1939. Word frequencies are tabulated in "Relative Frequency of English Speech Sounds," G. Dewey, Harvard University Press, 1923.

3. The Series of Approximations to English

To give a visual idea of how this series of processes approaches a language, typical sequences in the approximations to English have been constructed and are given below. In all cases we have assumed a 27-symbol "alphabet," the 26 letters and a space.

1. Zero-order approximation (symbols independent and equi-probable).

   XFOML RXKHRJFFJUJ ZLPWCFWKCYJ FFJEYVKCQSGXYD QPAAMKBZAACIBZLHJQD

2. First-order approximation (symbols independent but with frequencies of English text).

   OCRO HLI RGWR NMIELWIS EU LL NBNESEBYA TH EEI ALHENHTTPA OOBTTVA NAH BRL

3. Second-order approximation (digram structure as in English).

   ON IE ANTSOUTINYS ARE T INCTORE ST BE S DEAMY ACHIN D ILONASIVE TUCOOWE AT TEASONARE FUSO TIZIN ANDY TOBE SEACE CTISBE

4. Third-order approximation (trigram structure as in English).

   IN NO IST LAT WHEY CRATICT FROURE BIRS GROCID PONDENOME OF DEMONSTURES OF THE REPTAGIN IS REGOACTIONA OF CRE

5. First-Order Word Approximation. Rather than continue with tetragram, $\cdots$, n-gram structure it is easier and better to jump at this point to word units. Here words are chosen independently but with their appropriate frequencies.

   REPRESENTING AND SPEEDILY IS AN GOOD APT OR COME CAN DIFFERENT NATURAL HERE HE THE A IN CAME THE TO OF TO EXPERT GRAY COME TO FURNISHES THE LINE MESSAGE HAD BE THESE.

6. Second-Order Word Approximation. The word transition probabilities are correct but no further structure is included.

   THE HEAD AND IN FRONTAL ATTACK ON AN ENGLISH WRITER THAT THE CHARACTER OF THIS POINT IS THEREFORE ANOTHER METHOD FOR THE LETTERS THAT THE TIME OF WHO EVER TOLD THE PROBLEM FOR AN UNEXPECTED

The resemblance to ordinary English text increases quite noticeably at each of the above steps. Note that these samples have reasonably good structure out to about twice the range that is taken into account in their construction. Thus in (3) the statistical process insures reasonable text for two-letter sequence, but four-letter sequences from the sample can usually be fitted into good sentences. In (6) sequences of four or more words can easily be placed in sentences without unusual or strained constructions. The particular sequence of ten words "attack on an English writer that the character of this" is not at all unreasonable. It appears then that a sufficiently complex stochastic process will give a satisfactory representation of a discrete source.

The first two samples were constructed by the use of a book of random numbers in conjunction with (for example 2) a table of letter frequencies. This method might have been continued for (3), (4), and (5), since digram, trigram, and word frequency tables are available, but a simpler equivalent method was used. To construct (3) for example, one opens a book at random and selects a letter at random on the page. This letter is recorded. The book is then opened to another page and one reads until this letter is encountered. The succeeding letter is then recorded. Turning to another page this second letter is searched for and the succeeding letter recorded, etc. A similar process was used for (4), (5), and (6). It would be interesting if further approximations could be constructed, but the labor involved becomes enormous at the next stage.

4. Graphical Representation of a Markoff Process

Stochastic processes of the type described above are known mathematically as discrete Markoff processes and have been extensively studied in the literature.[6] The general case can be described as follows: There exist a finite number of possible "states" of a system; $S_1, S_2, \cdots, S_n$. In addition there is a set of transition probabilities; $p_i(j)$ the probability that if the system is in state S_i it will next go to state S_j. To make this Markoff process into an information source we need only assume that a letter is produced for each transition from one state to another. The states will correspond to the "residue of influence" from preceding letters.

The situation can be represented graphically as shown in Figs. 3, 4 and 5. The "states" are the junction points in the graph and the probabilities and letters produced for a transition are given beside the corresponding line. Figure 3 is for the example B in Section 2, while Fig. 4 corresponds to the example C. In Fig. 3 there is only one state since successive letters are independent. In Fig. 4 there are as many states as letters. If a trigram example were constructed there would be at most n^2 states corresponding to the possible pairs of letters preceding the one being chosen. Figure 5 is a graph for the case of word structure in example D. Here S corresponds to the "space" symbol.

[6] For a detailed treatment see M. Frechet, "Methods des fonctions arbitraires. Theorie des énénements en chaine dans le cas d'un nombre fini d'états possibles." Paris, Gauthier-Villars, 1938.

5. Ergodic and Mixed Sources

As we have indicated above a discrete source for our purposes can be considered to be represented by a Markoff process. Among the possible discrete Markoff processes there is a group with special properties of significance in

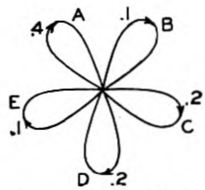

Fig. 3—A graph corresponding to the source in example B.

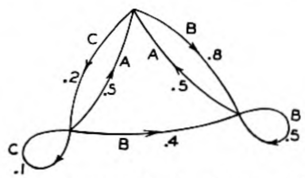

Fig. 4—A graph corresponding to the source in example C.

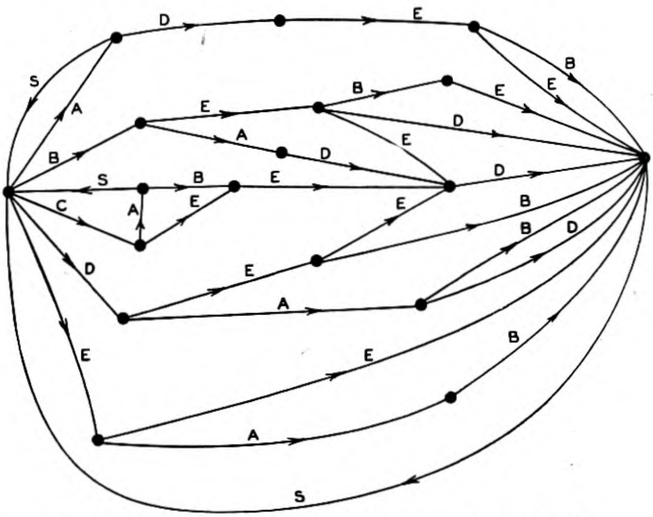

Fig. 5—A graph corresponding to the source in example D.

communication theory. This special class consists of the "ergodic" processes and we shall call the corresponding sources ergodic sources. Although a rigorous definition of an ergodic process is somewhat involved, the general idea is simple. In an ergodic process every sequence produced by the process is the same in statistical properties. Thus the letter frequencies, digram frequencies, etc., obtained from particular sequences will, as the lengths of the sequences increase, approach definite limits independent of the particular sequence. Actually this is not true of every sequence but the set for which it is false has probability zero. Roughly the ergodic property means statistical homogeneity.

All the examples of artificial languages given above are ergodic. This property is related to the structure of the corresponding graph. If the graph has the following two properties[7] the corresponding process will be ergodic:

1. The graph does not consist of two isolated parts A and B such that it is impossible to go from junction points in part A to junction points in part B along lines of the graph in the direction of arrows and also impossible to go from junctions in part B to junctions in part A.
2. A closed series of lines in the graph with all arrows on the lines pointing in the same orientation will be called a "circuit." The "length" of a circuit is the number of lines in it. Thus in Fig. 5 the series BEBES is a circuit of length 5. The second property required is that the greatest common divisor of the lengths of all circuits in the graph be one.

If the first condition is satisfied but the second one violated by having the greatest common divisor equal to $d > 1$, the sequences have a certain type of periodic structure. The various sequences fall into d different classes which are statistically the same apart from a shift of the origin (i.e., which letter in the sequence is called letter 1). By a shift of from 0 up to $d - 1$ any sequence can be made statistically equivalent to any other. A simple example with $d = 2$ is the following: There are three possible letters a, b, c. Letter a is followed with either b or c with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively. Either b or c is always followed by letter a. Thus a typical sequence is

a b a c a c a c a b a c a b a b a c a c

This type of situation is not of much importance for our work.

If the first condition is violated the graph may be separated into a set of subgraphs each of which satisfies the first condition. We will assume that the second condition is also satisfied for each subgraph. We have in this case what may be called a "mixed" source made up of a number of pure components. The components correspond to the various subgraphs. If L_1, L_2, L_3, $\cdots$ are the component sources we may write

$$L = p_1 L_1 + p_2 L_2 + p_3 L_3 + \cdots$$

where p_i is the probability of the component source L_i.

[7] These are restatements in terms of the graph of conditions given in Frechet.

Physically the situation represented is this: There are several different sources $L_1, L_2, L_3, \cdots$ which are each of homogeneous statistical structure (i.e., they are ergodic). We do not know *a priori* which is to be used, but once the sequence starts in a given pure component L_i it continues indefinitely according to the statistical structure of that component.

As an example one may take two of the processes defined above and assume $p_1 = .2$ and $p_2 = .8$. A sequence from the mixed source

$$L = .2\,L_1 + .8\,L_2$$

would be obtained by choosing first L_1 or L_2 with probabilities .2 and .8 and after this choice generating a sequence from whichever was chosen.

Except when the contrary is stated we shall assume a source to be ergodic. This assumption enables one to identify averages along a sequence with averages over the ensemble of possible sequences (the probability of a discrepancy being zero). For example the relative frequency of the letter A in a particular infinite sequence will be, with probability one, equal to its relative frequency in the ensemble of sequences.

If P_i is the probability of state i and $p_i(j)$ the transition probability to state j, then for the process to be stationary it is clear that the P_i must satisfy equilibrium conditions:

$$P_j = \sum_i P_i\, p_i(j).$$

In the ergodic case it can be shown that with any starting conditions the probabilities $P_j(N)$ of being in state j after N symbols, approach the equilibrium values as $N \rightarrow \infty$.

6. Choice, Uncertainty and Entropy

We have represented a discrete information source as a Markoff process. Can we define a quantity which will measure, in some sense, how much information is "produced" by such a process, or better, at what rate information is produced?

Suppose we have a set of possible events whose probabilities of occurrence are $p_1, p_2, \cdots, p_n$. These probabilities are known but that is all we know concerning which event will occur. Can we find a measure of how much "choice" is involved in the selection of the event or of how uncertain we are of the outcome?

If there is such a measure, say $H(p_1, p_2, \cdots, p_n)$, it is reasonable to require of it the following properties:

1. H should be continuous in the p_i.
2. If all the p_i are equal, $p_i = \frac{1}{n}$, then H should be a monotonic increasing

function of n. With equally likely events there is more choice, or uncertainty, when there are more possible events.

3. If a choice be broken down into two successive choices, the original H should be the weighted sum of the individual values of H. The meaning of this is illustrated in Fig. 6. At the left we have three possibilities $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$, $p_3 = \frac{1}{6}$. On the right we first choose between two possibilities each with probability $\frac{1}{2}$, and if the second occurs make another choice with probabilities $\frac{2}{3}$, $\frac{1}{3}$. The final results have the same probabilities as before. We require, in this special case, that

$$H(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}) = H(\tfrac{1}{2}, \tfrac{1}{2}) + \tfrac{1}{2}H(\tfrac{2}{3}, \tfrac{1}{3})$$

The coefficient $\frac{1}{2}$ is because this second choice only occurs half the time.

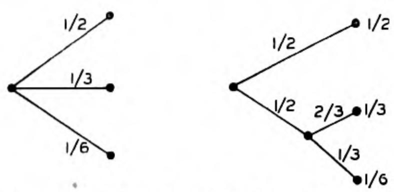

Fig. 6—Decomposition of a choice from three possibilities.

In Appendix II, the following result is established:

Theorem 2: The only H satisfying the three above assumptions is of the form:

$$H = -K \sum_{i=1}^{n} p_i \log p_i$$

where K is a positive constant.

This theorem, and the assumptions required for its proof, are in no way necessary for the present theory. It is given chiefly to lend a certain plausibility to some of our later definitions. The real justification of these definitions, however, will reside in their implications.

Quantities of the form $H = -\Sigma\, p_i \log p_i$ (the constant K merely amounts to a choice of a unit of measure) play a central role in information theory as measures of information, choice and uncertainty. The form of H will be recognized as that of entropy as defined in certain formulations of statistical mechanics[8] where p_i is the probability of a system being in cell i of its phase space. H is then, for example, the H in Boltzmann's famous H theorem. We shall call $H = -\Sigma\, p_i \log p_i$ the entropy of the set of probabilities

[8] See, for example, R. C. Tolman, "Principles of Statistical Mechanics," Oxford, Clarendon, 1938.

$p_1, \cdots, p_n$. If x is a chance variable we will write $H(x)$ for its entropy; thus x is not an argument of a function but a label for a number, to differentiate it from $H(y)$ say, the entropy of the chance variable y.

The entropy in the case of two possibilities with probabilities p and $q = 1 - p$, namely

$$H = -(p \log p + q \log q)$$

is plotted in Fig. 7 as a function of p.

The quantity H has a number of interesting properties which further substantiate it as a reasonable measure of choice or information.

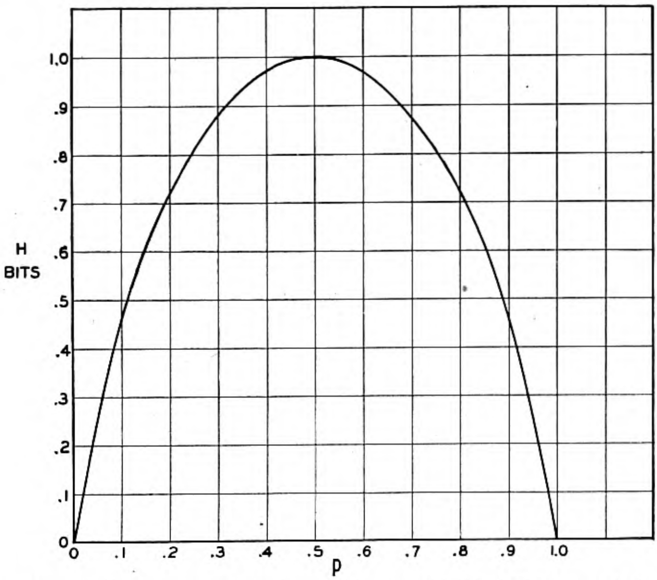

Fig. 7—Entropy in the case of two possibilities with probabilities p and $(1 - p)$.

1. $H = 0$ if and only if all the p_i but one are zero, this one having the value unity. Thus only when we are certain of the outcome does H vanish. Otherwise H is positive.

2. For a given n, H is a maximum and equal to $\log n$ when all the p_i are equal $\left(\text{i.e., } \frac{1}{n}\right)$. This is also intuitively the most uncertain situation.

3. Suppose there are two events, x and y, in question with m possibilities for the first and n for the second. Let $p(i, j)$ be the probability of the joint occurrence of i for the first and j for the second. The entropy of the joint event is

$$H(x, y) = -\sum_{i,j} p(i, j) \log p(i, j)$$

while

$$H(x) = -\sum_{i,j} p(i, j) \log \sum_{j} p(i, j)$$

$$H(y) = -\sum_{i,j} p(i, j) \log \sum_{i} p(i, j).$$

It is easily shown that

$$H(x, y) \leq H(x) + H(y)$$

with equality only if the events are independent (i.e., $p(i, j) = p(i)\, p(j)$). The uncertainty of a joint event is less than or equal to the sum of the individual uncertainties.

4. Any change toward equalization of the probabilities $p_1, p_2, \cdots, p_n$ increases H. Thus if $p_1 < p_2$ and we increase p_1, decreasing p_2 an equal amount so that p_1 and p_2 are more nearly equal, then H increases. More generally, if we perform any "averaging" operation on the p_i of the form

$$p_i' = \sum_j a_{ij}\, p_j$$

where $\sum_i a_{ij} = \sum_j a_{ij} = 1$, and all $a_{ij} \geq 0$, then H increases (except in the special case where this transformation amounts to no more than a permutation of the p_j with H of course remaining the same).

5. Suppose there are two chance events x and y as in 3, not necessarily independent. For any particular value i that x can assume there is a conditional probability $p_i(j)$ that y has the value j. This is given by

$$p_i(j) = \frac{p(i, j)}{\sum_j p(i, j)}.$$

We define the *conditional entropy* of y, $H_x(y)$ as the average of the entropy of y for each value of x, weighted according to the probability of getting that particular x. That is

$$H_x(y) = -\sum_{i,j} p(i, j) \log p_i(j).$$

This quantity measures how uncertain we are of y on the average when we know x. Substituting the value of $p_i(j)$ we obtain

$$\begin{aligned} H_x(y) &= -\sum_{ij} p(i, j) \log p(i, j) + \sum_{ij} p(i, j) \log \sum_j p(i, j) \\ &= H(x, y) - H(x) \end{aligned}$$

or

$$H(x, y) = H(x) + H_x(y)$$

The uncertainty (or entropy) of the joint event x, y is the uncertainty of x plus the uncertainty of y when x is known.

6. From 3 and 5 we have

$$H(x) + H(y) \geq H(x, y) = H(x) + H_x(y)$$

Hence

$$H(y) \geq H_x(y)$$

The uncertainty of y is never increased by knowledge of x. It will be decreased unless x and y are independent events, in which case it is not changed.

7. The Entropy of an Information Source

Consider a discrete source of the finite state type considered above. For each possible state i there will be a set of probabilities $p_i(j)$ of producing the various possible symbols j. Thus there is an entropy H_i for each state. The entropy of the source will be defined as the average of these H_i weighted in accordance with the probability of occurrence of the states in question:

$$\begin{aligned} H &= \sum_i P_i H_i \\ &= -\sum_{i,j} P_i p_i(j) \log p_i(j) \end{aligned}$$

This is the entropy of the source per symbol of text. If the Markoff process is proceeding at a definite time rate there is also an entropy per second

$$H' = \sum_i f_i H_i$$

where f_i is the average frequency (occurrences per second) of state i. Clearly

$$H' = mH$$

where m is the average number of symbols produced per second. H or H' measures the amount of information generated by the source per symbol or per second. If the logarithmic base is 2, they will represent bits per symbol or per second.

If successive symbols are independent then H is simply $-\Sigma p_i \log p_i$ where p_i is the probability of symbol i. Suppose in this case we consider a long message of N symbols. It will contain with high probability about $p_1 N$ occurrences of the first symbol, $p_2 N$ occurrences of the second, etc. Hence the probability of this particular message will be roughly

$$p = p_1^{p_1 N} p_2^{p_2 N} \cdots p_n^{p_n N}$$

or

$$\log p \doteq N \sum_i p_i \log p_i$$

$$\log p \doteq -NH$$

$$H \doteq \frac{\log 1/p}{N}.$$

H is thus approximately the logarithm of the reciprocal probability of a typical long sequence divided by the number of symbols in the sequence. The same result holds for any source. Stated more precisely we have (see Appendix III):

Theorem 3: Given any $\epsilon > 0$ and $\delta > 0$, we can find an N_0 such that the sequences of any length $N \geq N_0$ fall into two classes:

1. A set whose total probability is less than ϵ.
2. The remainder, all of whose members have probabilities satisfying the inequality

$$\left| \frac{\log p^{-1}}{N} - H \right| < \delta$$

In other words we are almost certain to have $\frac{\log p^{-1}}{N}$ very close to H when N is large.

A closely related result deals with the number of sequences of various probabilities. Consider again the sequences of length N and let them be arranged in order of decreasing probability. We define $n(q)$ to be the number we must take from this set starting with the most probable one in order to accumulate a total probability q for those taken.

Theorem 4:

$$\lim_{N \to \infty} \frac{\log n(q)}{N} = H$$

when q does not equal 0 or 1.

We may interpret $\log n(q)$ as the number of bits required to specify the sequence when we consider only the most probable sequences with a total probability q. Then $\frac{\log n(q)}{N}$ is the number of bits per symbol for the specification. The theorem says that for large N this will be independent of q and equal to H. The rate of growth of the logarithm of the number of reasonably probable sequences is given by H, regardless of our interpretation of "reasonably probable." Due to these results, which are proved in appendix III, it is possible for most purposes to treat the long sequences as though there were just 2^{HN} of them, each with a probability 2^{-HN}.

The next two theorems show that H and H' can be determined by limiting operations directly from the statistics of the message sequences, without reference to the states and transition probabilities between states.

Theorem 5: Let $p(B_i)$ be the probability of a sequence B_i of symbols from the source. Let

$$G_N = -\frac{1}{N} \sum_i p(B_i) \log p(B_i)$$

where the sum is over all sequences B_i containing N symbols. Then G_N is a monotonic decreasing function of N and

$$\lim_{N \to \infty} G_N = H.$$

Theorem 6: Let $p(B_i, S_j)$ be the probability of sequence B_i followed by symbol S_j and $p_{B_i}(S_j) = p(B_i, S_j)/p(B_i)$ be the conditional probability of S_j after B_i. Let

$$F_N = -\sum_{i,j} p(B_i, S_j) \log p_{B_i}(S_j)$$

where the sum is over all blocks B_i of $N - 1$ symbols and over all symbols S_j. Then F_N is a monotonic decreasing function of N,

$$F_N = NG_N - (N - 1)\, G_{N-1} ,$$

$$G_N = \frac{1}{N} \sum_1^n F_N ,$$

$$F_N \leq G_N ,$$

and $\lim_{N \to \infty} F_N = H$.

These results are derived in appendix III. They show that a series of approximations to H can be obtained by considering only the statistical structure of the sequences extending over 1, 2, $\cdots$ N symbols. F_N is the better approximation. In fact F_N is the entropy of the N^{th} order approximation to the source of the type discussed above. If there are no statistical influences extending over more than N symbols, that is if the conditional probability of the next symbol knowing the preceding $(N - 1)$ is not changed by a knowledge of any before that, then $F_N = H$. F_N of course is the conditional entropy of the next symbol when the $(N - 1)$ preceding ones are known, while G_N is the entropy per symbol of blocks of N symbols.

The ratio of the entropy of a source to the maximum value it could have while still restricted to the same symbols will be called its *relative entropy*. This is the maximum compression possible when we encode into the same alphabet. One minus the relative entropy is the *redundancy*. The redundancy of ordinary English, not considering statistical structure over greater distances than about eight letters is roughly 50%. This means that when we write English half of what we write is determined by the structure of the language and half is chosen freely. The figure 50% was found by several independent methods which all gave results in this neighborhood. One is by calculation of the entropy of the approximations to English. A second method is to delete a certain fraction of the letters from a sample of English text and then let someone attempt to restore them. If they can be restored when 50% are deleted the redundancy must be greater than 50%. A third method depends on certain known results in cryptography.

Two extremes of redundancy in English prose are represented by Basic English and by James Joyces' book "Finigans Wake." The Basic English vocabulary is limited to 850 words and the redundancy is very high. This is reflected in the expansion that occurs when a passage is translated into Basic English. Joyce on the other hand enlarges the vocabulary and is alleged to achieve a compression of semantic content.

The redundancy of a language is related to the existence of crossword puzzles. If the redundancy is zero any sequence of letters is a reasonable text in the language and any two dimensional array of letters forms a crossword puzzle. If the redundancy is too high the language imposes too many constraints for large crossword puzzles to be possible. A more detailed analysis shows that if we assume the constraints imposed by the language are of a rather chaotic and random nature, large crossword puzzles are just possible when the redundancy is 50%. If the redundancy is 33%, three dimensional crossword puzzles should be possible, etc.

8. Representation of the Encoding and Decoding Operations

We have yet to represent mathematically the operations performed by the transmitter and receiver in encoding and decoding the information. Either of these will be called a discrete transducer. The input to the transducer is a sequence of input symbols and its output a sequence of output symbols. The transducer may have an internal memory so that its output depends not only on the present input symbol but also on the past history. We assume that the internal memory is finite, i.e. there exists a finite number m of possible states of the transducer and that its output is a function of the present state and the present input symbol. The next state will be a second function of these two quantities. Thus a transducer can be described by two functions:

$$y_n = f(x_n, \alpha_n)$$

$$\alpha_{n+1} = g(x_n, \alpha_n)$$

where: x_n is the n^{th} input symbol,

α_n is the state of the transducer when the n^{th} input symbol is introduced,

y_n is the output symbol (or sequence of output symbols) produced when x_n is introduced if the state is α_n.

If the output symbols of one transducer can be identified with the input symbols of a second, they can be connected in tandem and the result is also a transducer. If there exists a second transducer which operates on the output of the first and recovers the original input, the first transducer will be called non-singular and the second will be called its inverse.

Theorem 7: The output of a finite state transducer driven by a finite state statistical source is a finite state statistical source, with entropy (per unit time) less than or equal to that of the input. If the transducer is non-singular they are equal.

Let α represent the state of the source, which produces a sequence of symbols x_i ; and let β be the state of the transducer, which produces, in its output, blocks of symbols y_j. The combined system can be represented by the "product state space" of pairs (α, β). Two points in the space, (α_1, β_1) and $(\alpha_2 \beta_2)$, are connected by a line if α_1 can produce an x which changes β_1 to β_2, and this line is given the probability of that x in this case. The line is labeled with the block of y_j symbols produced by the transducer. The entropy of the output can be calculated as the weighted sum over the states. If we sum first on β each resulting term is less than or equal to the corresponding term for α, hence the entropy is not increased. If the transducer is non-singular let its output be connected to the inverse transducer. If H'_1, H'_2 and H'_3 are the output entropies of the source, the first and second transducers respectively, then $H'_1 \geq H'_2 \geq H'_3 = H'_1$ and therefore $H'_1 = H'_2$.

Suppose we have a system of constraints on possible sequences of the type which can be represented by a linear graph as in Fig. 2. If probabilities $p_{ij}^{(s)}$ were assigned to the various lines connecting state i to state j this would become a source. There is one particular assignment which maximizes the resulting entropy (see Appendix IV).

Theorem 8: Let the system of constraints considered as a channel have a capacity C. If we assign

$$p_{ij}^{(s)} = \frac{B_j}{B_i} C^{-\ell_{ij}^{(s)}}$$

where $\ell_{ij}^{(s)}$ is the duration of the s^{th} symbol leading from state i to state j and the B_i satisfy

$$B_i = \sum_{s,j} B_j C^{-\ell_{ij}^{(s)}}$$

then H is maximized and equal to C.

By proper assignment of the transition probabilities the entropy of symbols on a channel can be maximized at the channel capacity.

9. The Fundamental Theorem for a Noiseless Channel

We will now justify our interpretation of H as the rate of generating information by proving that H determines the channel capacity required with most efficient coding.

Theorem 9: Let a source have entropy H (bits per symbol) and a channel have a capacity C (bits per second). Then it is possible to encode the output of the source in such a way as to transmit at the average rate $\frac{C}{H} - \epsilon$ symbols per second over the channel where ϵ is arbitrarily small. It is not possible to transmit at an average rate greater than $\frac{C}{H}$.

The converse part of the theorem, that $\frac{C}{H}$ cannot be exceeded, may be proved by noting that the entropy of the channel input per second is equal to that of the source, since the transmitter must be non-singular, and also this entropy cannot exceed the channel capacity. Hence $H' \leq C$ and the number of symbols per second $= H'/H \leq C/H$.

The first part of the theorem will be proved in two different ways. The first method is to consider the set of all sequences of N symbols produced by the source. For N large we can divide these into two groups, one containing less than $2^{(H+\eta)N}$ members and the second containing less than 2^{RN} members (where R is the logarithm of the number of different symbols) and having a total probability less than μ. As N increases η and μ approach zero. The number of signals of duration T in the channel is greater than $2^{(C-\theta)T}$ with θ small when T is large. If we choose

$$T = \left(\frac{H}{C} + \lambda\right) N$$

then there will be a sufficient number of sequences of channel symbols for the high probability group when N and T are sufficiently large (however small λ) and also some additional ones. The high probability group is coded in an arbitrary one to one way into this set. The remaining sequences are represented by larger sequences, starting and ending with one of the sequences not used for the high probability group. This special sequence acts as a start and stop signal for a different code. In between a sufficient time is allowed to give enough different sequences for all the low probability messages. This will require

$$T_1 = \left(\frac{R}{C} + \varphi\right) N$$

where φ is small. The mean rate of transmission in message symbols per second will then be greater than

$$\left[(1-\delta)\frac{T}{N}+\delta\frac{T_1}{N}\right]^{-1}=\left[(1-\delta)\left(\frac{H}{C}+\lambda\right)+\delta\left(\frac{R}{C}+\varphi\right)\right]^{-1}$$

As N increases δ, λ and φ approach zero and the rate approaches $\frac{C}{H}$.

Another method of performing this coding and proving the theorem can be described as follows: Arrange the messages of length N in order of decreasing probability and suppose their probabilities are $p_1 \geq p_2 \geq p_3 \ldots \geq p_n$. Let $P_s = \sum_1^{s-1} p_i$; that is P_s is the cumulative probability up to, but not including, p_s. We first encode into a binary system. The binary code for message s is obtained by expanding P_s as a binary number. The expansion is carried out to m_s places, where m_s is the integer satisfying:

$$\log_2 \frac{1}{p_s} \leq m_s < 1 + \log_2 \frac{1}{p_s}$$

Thus the messages of high probability are represented by short codes and those of low probability by long codes. From these inequalities we have

$$\frac{1}{2^{m_s}} \leq p_s < \frac{1}{2^{m_s-1}}.$$

The code for P_s will differ from all succeeding ones in one or more of its m_s places, since all the remaining P_i are at least $\frac{1}{2^{m_s}}$ larger and their binary expansions therefore differ in the first m_s places. Consequently all the codes are different and it is possible to recover the message from its code. If the channel sequences are not already sequences of binary digits, they can be ascribed binary numbers in an arbitrary fashion and the binary code thus translated into signals suitable for the channel.

The average number H' of binary digits used per symbol of original message is easily estimated. We have

$$H' = \frac{1}{N}\Sigma m_s p_s$$

But,

$$\frac{1}{N}\Sigma\left(\log_2\frac{1}{p_s}\right)p_s \leq \frac{1}{N}\Sigma m_s p_s < \frac{1}{N}\Sigma\left(1+\log_2\frac{1}{p_s}\right)p_s$$

and therefore,

$$-\Sigma p_s \log p_s \leq H' < \frac{1}{N} - \Sigma p_s \log p_s$$

As N increases $-\Sigma p_s \log p_s$ approaches H, the entropy of the source and H' approaches H.

We see from this that the inefficiency in coding, when only a finite delay of N symbols is used, need not be greater than $\frac{1}{N}$ plus the difference between the true entropy H and the entropy G_N calculated for sequences of length N. The per cent excess time needed over the ideal is therefore less than

$$\frac{G_N}{H} + \frac{1}{HN} - 1.$$

This method of encoding is substantially the same as one found independently by R. M. Fano.[9] His method is to arrange the messages of length N in order of decreasing probability. Divide this series into two groups of as nearly equal probability as possible. If the message is in the first group its first binary digit will be 0, otherwise 1. The groups are similarly divided into subsets of nearly equal probability and the particular subset determines the second binary digit. This process is continued until each subset contains only one message. It is easily seen that apart from minor differences (generally in the last digit) this amounts to the same thing as the arithmetic process described above.

10. Discussion

In order to obtain the maximum power transfer from a generator to a load a transformer must in general be introduced so that the generator as seen from the load has the load resistance. The situation here is roughly analogous. The transducer which does the encoding should match the source to the channel in a statistical sense. The source as seen from the channel through the transducer should have the same statistical structure as the source which maximizes the entropy in the channel. The content of Theorem 9 is that, although an exact match is not in general possible, we can approximate it as closely as desired. The ratio of the actual rate of transmission to the capacity C may be called the efficiency of the coding system. This is of course equal to the ratio of the actual entropy of the channel symbols to the maximum possible entropy.

In general, ideal or nearly ideal encoding requires a long delay in the transmitter and receiver. In the noiseless case which we have been considering, the main function of this delay is to allow reasonably good

[9] Technical Report No. 65, The Research Laboratory of Electronics, M. I. T.

matching of probabilities to corresponding lengths of sequences. With a good code the logarithm of the reciprocal probability of a long message must be proportional to the duration of the corresponding signal, in fact

$$\left| \frac{\log p^{-1}}{T} - C \right|$$

must be small for all but a small fraction of the long messages.

If a source can produce only one particular message its entropy is zero, and no channel is required. For example, a computing machine set up to calculate the successive digits of π produces a definite sequence with no chance element. No channel is required to "transmit" this to another point. One could construct a second machine to compute the same sequence at the point. However, this may be impractical. In such a case we can choose to ignore some or all of the statistical knowledge we have of the source. We might consider the digits of π to be a random sequence in that we construct a system capable of sending any sequence of digits. In a similar way we may choose to use some of our statistical knowledge of English in constructing a code, but not all of it. In such a case we consider the source with the maximum entropy subject to the statistical conditions we wish to retain. The entropy of this source determines the channel capacity which is necessary and sufficient. In the π example the only information retained is that all the digits are chosen from the set 0, 1, . . . , 9. In the case of English one might wish to use the statistical saving possible due to letter frequencies, but nothing else. The maximum entropy source is then the first approximation to English and its entropy determines the required channel capacity.

11. Examples

As a simple example of some of these results consider a source which produces a sequence of letters chosen from among A, B, C, D with probabilities $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{8}$, successive symbols being chosen independently. We have

$$\begin{aligned} H &= -\left(\tfrac{1}{2} \log \tfrac{1}{2} + \tfrac{1}{4} \log \tfrac{1}{4} + \tfrac{2}{8} \log \tfrac{1}{8}\right) \\ &= \tfrac{7}{4} \text{ bits per symbol.} \end{aligned}$$

Thus we can approximate a coding system to encode messages from this source into binary digits with an average of $\frac{7}{4}$ binary digit per symbol. In this case we can actually achieve the limiting value by the following code (obtained by the method of the second proof of Theorem 9):

A	0
B	10
C	110
D	111

The average number of binary digits used in encoding a sequence of N symbols will be

$$N(\tfrac{1}{2} \times 1 + \tfrac{1}{4} \times 2 + \tfrac{2}{8} \times 3) = \tfrac{7}{4}N$$

It is easily seen that the binary digits 0, 1 have probabilities $\frac{1}{2}$, $\frac{1}{2}$ so the H for the coded sequences is one bit per symbol. Since, on the average, we have $\frac{7}{4}$ binary symbols per original letter, the entropies on a time basis are the same. The maximum possible entropy for the original set is $\log 4 = 2$, occurring when A, B, C, D have probabilities $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$. Hence the relative entropy is $\frac{7}{8}$. We can translate the binary sequences into the original set of symbols on a two-to-one basis by the following table:

00	A'
01	B'
10	C'
11	D'

This double process then encodes the original message into the same symbols but with an average compression ratio $\frac{7}{8}$.

As a second example consider a source which produces a sequence of A's and B's with probability p for A and q for B. If $p << q$ we have

$$\begin{aligned} H &= -\log p^p(1-p)^{1-p} \\ &= -p \log p\,(1-p)^{(1-p)/p} \\ &\doteq p \log \frac{e}{p} \end{aligned}$$

In such a case one can construct a fairly good coding of the message on a 0, 1 channel by sending a special sequence, say 0000, for the infrequent symbol A and then a sequence indicating the *number* of B's following it. This could be indicated by the binary representation with all numbers containing the special sequence deleted. All numbers up to 16 are represented as usual; 16 is represented by the next binary number after 16 which does not contain four zeros, namely $17 = 10001$, etc.

It can be shown that as $p \rightarrow 0$ the coding approaches ideal provided the length of the special sequence is properly adjusted.

PART II: THE DISCRETE CHANNEL WITH NOISE

11. Representation of a Noisy Discrete Channel

We now consider the case where the signal is perturbed by noise during transmission or at one or the other of the terminals. This means that the received signal is not necessarily the same as that sent out by the transmitter. Two cases may be distinguished. If a particular transmitted signal always produces the same received signal, i.e. the received signal is a definite function of the transmitted signal, then the effect may be called distortion. If this function has an inverse—no two transmitted signals producing the same received signal—distortion may be corrected, at least in principle, by merely performing the inverse functional operation on the received signal.

The case of interest here is that in which the signal does not always undergo the same change in transmission. In this case we may assume the received signal E to be a function of the transmitted signal S and a second variable, the noise N.

$$E = f(S, N)$$

The noise is considered to be a chance variable just as the message was above. In general it may be represented by a suitable stochastic process. The most general type of noisy discrete channel we shall consider is a generalization of the finite state noise free channel described previously. We assume a finite number of states and a set of probabilities

$$p_{\alpha, i}(\beta, j).$$

This is the probability, if the channel is in state α and symbol i is transmitted, that symbol j will be received and the channel left in state β. Thus α and β range over the possible states, i over the possible transmitted signals and j over the possible received signals. In the case where successive symbols are independently perturbed by the noise there is only one state, and the channel is described by the set of transition probabilities $p_i(j)$, the probability of transmitted symbol i being received as j.

If a noisy channel is fed by a source there are two statistical processes at work: the source and the noise. Thus there are a number of entropies that can be calculated. First there is the entropy $H(x)$ of the source or of the input to the channel (these will be equal if the transmitter is non-singular). The entropy of the output of the channel, i.e. the received signal, will be denoted by $H(y)$. In the noiseless case $H(y) = H(x)$. The joint entropy of input and output will be $H(xy)$. Finally there are two conditional entropies $H_x(y)$ and $H_y(x)$, the entropy of the output when the input is known and conversely. Among these quantities we have the relations

$$H(x, y) = H(x) + H_x(y) = H(y) + H_y(x)$$

All of these entropies can be measured on a per-second or a per-symbol basis.

12. Equivocation and Channel Capacity

If the channel is noisy it is not in general possible to reconstruct the original message or the transmitted signal with *certainty* by any operation on the received signal E. There are, however, ways of transmitting the information which are optimal in combating noise. This is the problem which we now consider.

Suppose there are two possible symbols 0 and 1, and we are transmitting at a rate of 1000 symbols per second with probabilities $p_0 = p_1 = \frac{1}{2}$. Thus our source is producing information at the rate of 1000 bits per second. During transmission the noise introduces errors so that, on the average, 1 in 100 is received incorrectly (a 0 as 1, or 1 as 0). What is the rate of transmission of information? Certainly less than 1000 bits per second since about 1% of the received symbols are incorrect. Our first impulse might be to say the rate is 990 bits per second, merely subtracting the expected number of errors. This is not satisfactory since it fails to take into account the recipient's lack of knowledge of where the errors occur. We may carry it to an extreme case and suppose the noise so great that the received symbols are entirely independent of the transmitted symbols. The probability of receiving 1 is $\frac{1}{2}$ whatever was transmitted and similarly for 0. Then about half of the received symbols are correct due to chance alone, and we would be giving the system credit for transmitting 500 bits per second while actually no information is being transmitted at all. Equally "good" transmission would be obtained by dispensing with the channel entirely and flipping a coin at the receiving point.

Evidently the proper correction to apply to the amount of information transmitted is the amount of this information which is missing in the received signal, or alternatively the uncertainty when we have received a signal of what was actually sent. From our previous discussion of entropy as a measure of uncertainty it seems reasonable to use the conditional entropy of the message, knowing the received signal, as a measure of this missing information. This is indeed the proper definition, as we shall see later. Following this idea the rate of actual transmission, R, would be obtained by subtracting from the rate of production (i.e., the entropy of the source) the average rate of conditional entropy.

$$R = H(x) - H_y(x)$$

The conditional entropy $H_y(x)$ will, for convenience, be called the equivocation. It measures the average ambiguity of the received signal.

In the example considered above, if a 0 is received the *a posteriori* probability that a 0 was transmitted is .99, and that a 1 was transmitted is .01. These figures are reversed if a 1 is received. Hence

$$H_y(x) = -[.99 \log .99 + 0.01 \log 0.01]$$
$$= .081 \text{ bits/symbol}$$

or 81 bits per second. We may say that the system is transmitting at a rate $1000 - 81 = 919$ bits per second. In the extreme case where a 0 is equally likely to be received as a 0 or 1 and similarly for 1, the a posteriori probabilities are $\frac{1}{2}$, $\frac{1}{2}$ and

$$H_y(x) = -[\tfrac{1}{2} \log \tfrac{1}{2} + \tfrac{1}{2} \log \tfrac{1}{2}]$$
$$= 1 \text{ bit per symbol}$$

or 1000 bits per second. The rate of transmission is then 0 as it should be.

The following theorem gives a direct intuitive interpretation of the equivocation and also serves to justify it as the unique appropriate measure. We consider a communication system and an observer (or auxiliary device) who can see both what is sent and what is recovered (with errors due to noise). This observer notes the errors in the recovered message and transmits data to the receiving point over a "correction channel" to enable the receiver to correct the errors. The situation is indicated schematically in Fig. 8.

Theorem 10: If the correction channel has a capacity equal to $H_y(x)$ it is possible to so encode the correction data as to send it over this channel and correct all but an arbitrarily small fraction ϵ of the errors. This is not possible if the channel capacity is less than $H_y(x)$.

Roughly then, $H_y(x)$ is the amount of additional information that must be supplied per second at the receiving point to correct the received message.

To prove the first part, consider long sequences of received message M' and corresponding original message M. There will be logarithmically $TH_y(x)$ of the M's which could reasonably have produced each M'. Thus we have $TH_y(x)$ binary digits to send each T seconds. This can be done with ϵ frequency of errors on a channel of capacity $H_y(x)$.

The second part can be proved by noting, first, that for any discrete chance variables x, y, z

$$H_y(x, z) \geq H_y(x)$$

The left-hand side can be expanded to give

$$H_y(z) + H_{yz}(x) \geq H_y(x)$$
$$H_{yz}(x) \geq H_y(x) - H_y(z) \geq H_y(x) - H(z)$$

If we identify x as the output of the source, y as the received signal and z as the signal sent over the correction channel, then the right-hand side is the equivocation less the rate of transmission over the correction channel. If the capacity of this channel is less than the equivocation the right-hand side will be greater than zero and $H_{yz}(x) \geq 0$. But this is the uncertainty of what was sent, knowing both the received signal and the correction signal. If this is greater than zero the frequency of errors cannot be arbitrarily small.

Example:

Suppose the errors occur at random in a sequence of binary digits: probability p that a digit is wrong and $q = 1 - p$ that it is right. These errors can be corrected if their position is known. Thus the correction channel need only send information as to these positions. This amounts to trans-

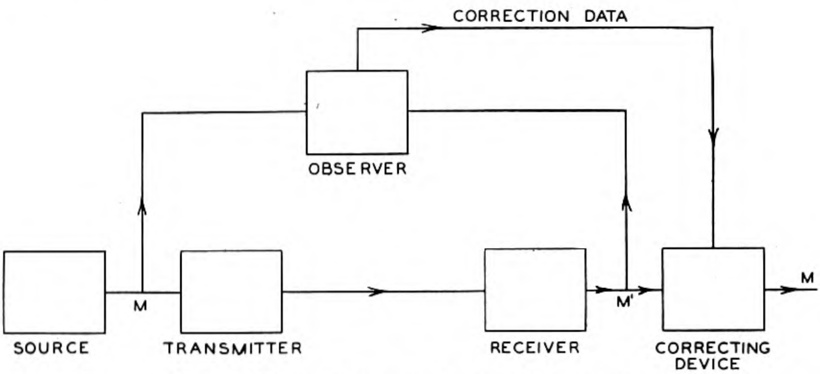

Fig. 8—Schematic diagram of a correction system.

mitting from a source which produces binary digits with probability p for 1 (correct) and q for 0 (incorrect). This requires a channel of capacity

$$-[p \log p + q \log q]$$

which is the equivocation of the original system.

The rate of transmission R can be written in two other forms due to the identities noted above. We have

$$\begin{aligned} R &= H(x) - H_y(x) \\ &= H(y) - H_x(y) \\ &= H(x) + H(y) - H(x, y). \end{aligned}$$

The first defining expression has already been interpreted as the amount of information sent less the uncertainty of what was sent. The second measures the amount received less the part of this which is due to noise. The third is the sum of the two amounts less the joint entropy and therefore in a sense is the number of bits per second common to the two. Thus all three expressions have a certain intuitive significance.

The capacity C of a noisy channel should be the maximum possible rate of transmission, i.e., the rate when the source is properly matched to the channel. We therefore define the channel capacity by

$$C = \text{Max}\,(H(x) - H_y(x))$$

where the maximum is with respect to all possible information sources used as input to the channel. If the channel is noiseless, $H_y(x) = 0$. The definition is then equivalent to that already given for a noiseless channel since the maximum entropy for the channel is its capacity.

13. The Fundamental Theorem for a Discrete Channel with Noise

It may seem surprising that we should define a definite capacity C for a noisy channel since we can never send certain information in such a case. It is clear, however, that by sending the information in a redundant form the probability of errors can be reduced. For example, by repeating the message many times and by a statistical study of the different received versions of the message the probability of errors could be made very small. One would expect, however, that to make this probability of errors approach zero, the redundancy of the encoding must increase indefinitely, and the rate of transmission therefore approach zero. This is by no means true. If it were, there would not be a very well defined capacity, but only a capacity for a given frequency of errors, or a given equivocation; the capacity going down as the error requirements are made more stringent. Actually the capacity C defined above has a very definite significance. It is possible to send information at the rate C through the channel *with as small a frequency of errors or equivocation as desired* by proper encoding. This statement is not true for any rate greater than C. If an attempt is made to transmit at a higher rate than C, say $C + R_1$, then there will necessarily be an equivocation equal to a greater than the excess R_1. Nature takes payment by requiring just that much uncertainty, so that we are not actually getting any more than C through correctly.

The situation is indicated in Fig. 9. The rate of information into the channel is plotted horizontally and the equivocation vertically. Any point above the heavy line in the shaded region can be attained and those below cannot. The points on the line cannot in general be attained, but there will usually be two points on the line that can.

These results are the main justification for the definition of C and will now be proved.

Theorem 11. Let a discrete channel have the capacity C and a discrete source the entropy per second H. If $H \leq C$ there exists a coding system such that the output of the source can be transmitted over the channel with an arbitrarily small frequency of errors (or an arbitrarily small equivocation). If $H > C$ it is possible to encode the source so that the equivocation is less than $H - C + \epsilon$ where ϵ is arbitrarily small. There is no method of encoding which gives an equivocation less than $H - C$.

The method of proving the first part of this theorem is not by exhibiting a coding method having the desired properties, but by showing that such a code must exist in a certain group of codes. In fact we will average the frequency of errors over this group and show that this average can be made less than ϵ. If the average of a set of numbers is less than ϵ there must exist at least one in the set which is less than ϵ. This will establish the desired result.

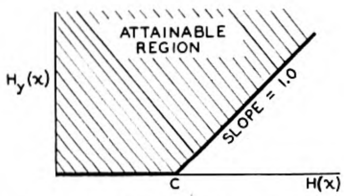

Fig. 9—The equivocation possible for a given input entropy to a channel.

The capacity C of a noisy channel has been defined as

$$C = \text{Max}\ (H(x) - H_y(x))$$

where x is the input and y the output. The maximization is over all sources which might be used as input to the channel.

Let S_0 be a source which achieves the maximum capacity C. If this maximum is not actually achieved by any source let S_0 be a source which approximates to giving the maximum rate. Suppose S_0 is used as input to the channel. We consider the possible transmitted and received sequences of a long duration T. The following will be true:

1. The transmitted sequences fall into two classes, a high probability group with about $2^{TH(x)}$ members and the remaining sequences of small total probability.
2. Similarly the received sequences have a high probability set of about $2^{TH(y)}$ members and a low probability set of remaining sequences.
3. Each high probability output could be produced by about $2^{TH_y(x)}$ inputs. The probability of all other cases has a small total probability.

All the ϵ's and δ's implied by the words "small" and "about" in these statements approach zero as we allow T to increase and S_0 to approach the maximizing source.

The situation is summarized in Fig. 10 where the input sequences are points on the left and output sequences points on the right. The fan of cross lines represents the range of possible causes for a typical output.

Now suppose we have another source producing information at rate R with $R < C$. In the period T this source will have 2^{TR} high probability outputs. We wish to associate these with a selection of the possible channe

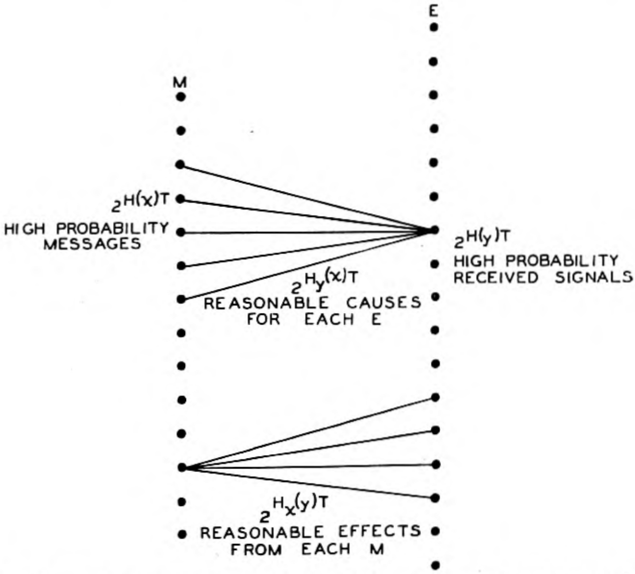

Fig. 10—Schematic representation of the relations between inputs and outputs in a channel.

inputs in such a way as to get a small frequency of errors. We will set up this association in all possible ways (using, however, only the high probability group of inputs as determined by the source S_0) and average the frequency of errors for this large class of possible coding systems. This is the same as calculating the frequency of errors for a random association of the messages and channel inputs of duration T. Suppose a particular output y_1 is observed. What is the probability of more than one message in the set of possible causes of y_1? There are 2^{TR} messages distributed at random in $2^{TH(x)}$ points. The probability of a particular point being a message is thus

$$2^{T(R-H(x))}$$

The probability that none of the points in the fan is a message (apart from the actual originating message) is

$$P = [1 - 2^{T(R-H(x))}]^{2^{TH_y(x)}}$$

Now $R < H(x) - H_y(x)$ so $R - H(x) = -H_y(x) - \eta$ with η positive. Consequently

$$P = [1 - 2^{-TH_y(x)-T\eta}]^{2^{TH_y(x)}}$$

approaches (as $T \rightarrow \infty$)

$$1 - 2^{-T\eta}.$$

Hence the probability of an error approaches zero and the first part of the theorem is proved.

The second part of the theorem is easily shown by noting that we could merely send C bits per second from the source, completely neglecting the remainder of the information generated. At the receiver the neglected part gives an equivocation $H(x) - C$ and the part transmitted need only add ϵ. This limit can also be attained in many other ways, as will be shown when we consider the continuous case.

The last statement of the theorem is a simple consequence of our definition of C. Suppose we can encode a source with $R = C + a$ in such a way as to obtain an equivocation $H_y(x) = a - \epsilon$ with ϵ positive. Then $R = H(x) = C + a$ and

$$H(x) - H_y(x) = C + \epsilon$$

with ϵ positive. This contradicts the definition of C as the maximum of $H(x) - H_y(x)$.

Actually more has been proved than was stated in the theorem. If the average of a set of numbers is within ϵ of their maximum, a fraction of at most $\sqrt{\epsilon}$ can be more than $\sqrt{\epsilon}$ below the maximum. Since ϵ is arbitrarily small we can say that almost all the systems are arbitrarily close to the ideal.

14. Discussion

The demonstration of theorem 11, while not a pure existence proof, has some of the deficiencies of such proofs. An attempt to obtain a good approximation to ideal coding by following the method of the proof is generally impractical. In fact, apart from some rather trivial cases and certain limiting situations, no explicit description of a series of approximation to the ideal has been found. Probably this is no accident but is related to the difficulty of giving an explicit construction for a good approximation to a random sequence.

An approximation to the ideal would have the property that if the signal is altered in a reasonable way by the noise, the original can still be recovered. In other words the alteration will not in general bring it closer to another reasonable signal than the original. This is accomplished at the cost of a certain amount of redundancy in the coding. The redundancy must be introduced in the proper way to combat the particular noise structure involved. However, any redundancy in the source will usually help if it is utilized at the receiving point. In particular, if the source already has a certain redundancy and no attempt is made to eliminate it in matching to the channel, this redundancy will help combat noise. For example, in a noiseless telegraph channel one could save about 50% in time by proper encoding of the messages. This is not done and most of the redundnacy of English remains in the channel symbols. This has the advantage, however, of allowing considerable noise in the channel. A sizable fraction of the letters can be received incorrectly and still reconstructed by the context. In fact this is probably not a bad approximation to the ideal in many cases, since the statistical structure of English is rather involved and the reasonable English sequences are not too far (in the sense required for theorem) from a random selection.

As in the noiseless case a delay is generally required to approach the ideal encoding. It now has the additional function of allowing a large sample of noise to affect the signal before any judgment is made at the receiving point as to the original message. Increasing the sample size always sharpens the possible statistical assertions.

The content of theorem 11 and its proof can be formulated in a somewhat different way which exhibits the connection with the noiseless case more clearly. Consider the possible signals of duration T and suppose a subset of them is selected to be used. Let those in the subset all be used with equal probability, and suppose the receiver is constructed to select, as the original signal, the most probable cause from the subset, when a perturbed signal is received. We define $N(T, q)$ to be the maximum number of signals we can choose for the subset such that the probability of an incorrect interpretation is less than or equal to q.

Theorem 12: $\operatorname{Lim}_{T \to \infty} \dfrac{\log N(T, q)}{T} = C$, where C is the channel capacity, provided that q does not equal 0 or 1.

In other words, no matter how we set our limits of reliability, we can distinguish reliably in time T enough messages to correspond to about CT bits, when T is sufficiently large. Theorem 12 can be compared with the definition of the capacity of a noiseless channel given in section 1.

15. Example of a Discrete Channel and Its Capacity

A simple example of a discrete channel is indicated in Fig. 11. There are three possible symbols. The first is never affected by noise. The second and third each have probability p of coming through undisturbed, and q of being changed into the other of the pair. We have (letting $\alpha = -[p \log p + q \log q]$ and P and Q be the probabilities of using the first or second symbols)

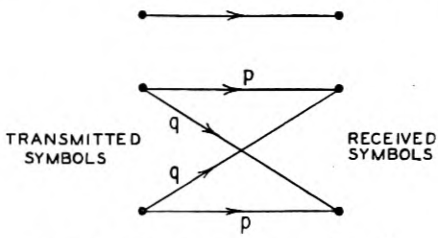

Fig. 11—Example of a discrete channel.

$$H(x) = -P \log P - 2Q \log Q$$

$$H_y(x) = 2Q\alpha$$

We wish to choose P and Q in such a way as to maximize $H(x) - H_y(x)$, subject to the constraint $P + 2Q = 1$. Hence we consider

$$U = -P \log P - 2Q \log Q - 2Q\alpha + \lambda(P + 2Q)$$

$$\frac{\partial U}{\partial P} = -1 - \log P + \lambda = 0$$

$$\frac{\partial U}{\partial Q} = -2 - 2 \log Q - 2\alpha + 2\lambda = 0 .$$

Eliminating λ

$$\log P = \log Q + \alpha$$

$$P = Qe^{\alpha} = Q\beta$$

$$P = \frac{\beta}{\beta + 2} \qquad Q = \frac{1}{\beta + 2} .$$

The channel capacity is then

$$C = \log \frac{\beta + 2}{\beta} .$$

Note how this checks the obvious values in the cases $p = 1$ and $p = \frac{1}{2}$. In the first, $\beta = 1$ and $C = \log 3$, which is correct since the channel is then noiseless with three possible symbols. If $p = \frac{1}{2}$, $\beta = 2$ and $C = \log 2$. Here the second and third symbols cannot be distinguished at all and act together like one symbol. The first symbol is used with probability $P = \frac{1}{2}$ and the second and third together with probability $\frac{1}{2}$. This may be distributed in any desired way and still achieve the maximum capacity.

For intermediate values of p the channel capacity will lie between log 2 and log 3. The distinction between the second and third symbols conveys some information but not as much as in the noiseless case. The first symbol is used somewhat more frequently than the other two because of its freedom from noise.

16. The Channel Capacity in Certain Special Cases

If the noise affects successive channel symbols independently it can be described by a set of transition probabilities p_{ij}. This is the probability, if symbol i is sent, that j will be received. The maximum channel rate is then given by the maximum of

$$\sum_{i,j} P_i p_{ij} \log \sum_i P_i p_{ij} - \sum_{i,j} P_i p_{ij} \log p_{ij}$$

where we vary the P_i subject to $\Sigma P_i = 1$. This leads by the method of Lagrange to the equations,

$$\sum_j p_{sj} \log \frac{p_{sj}}{\sum_i P_i p_{ij}} = \mu \qquad s = 1, 2, \cdots .$$

Multiplying by P_s and summing on s shows that $\mu = -C$. Let the inverse of p_{sj} (if it exists) be h_{st} so that $\sum_s h_{st} p_{sj} = \delta_{tj}$. Then:

$$\sum_{s,j} h_{st} p_{sj} \log p_{sj} - \log \sum_i P_i p_{it} = -C \sum_s h_{st} .$$

Hence:

$$\sum_i P_i p_{it} = \exp \left[C \sum_s h_{st} + \sum_{s,j} h_{st} p_{sj} \log p_{sj}\right]$$

or,

$$P_i = \sum_t h_{it} \exp \left[C \sum_s h_{st} + \sum_{s,j} h_{st} p_{sj} \log p_{sj}\right].$$

This is the system of equations for determining the maximizing values of P_i, with C to be determined so that $\Sigma P_i = 1$. When this is done C will be the channel capacity, and the P_i the proper probabilities for the channel symbols to achieve this capacity.

If each input symbol has the same set of probabilities on the lines emerging from it, and the same is true of each output symbol, the capacity can be easily calculated. Examples are shown in Fig. 12. In such a case $H_x(y)$ is independent of the distribution of probabilities on the input symbols, and is given by $-\Sigma\, p_i \log p_i$ where the p_i are the values of the transition probabilities from any input symbol. The channel capacity is

$$\begin{aligned}&\text{Max}\quad [H(y) \quad - \quad H_x(y)]\\ &= \text{Max}\, H(y) + \Sigma\, p_i \log p_i\,.\end{aligned}$$

The maximum of $H(y)$ is clearly $\log m$ where m is the number of output

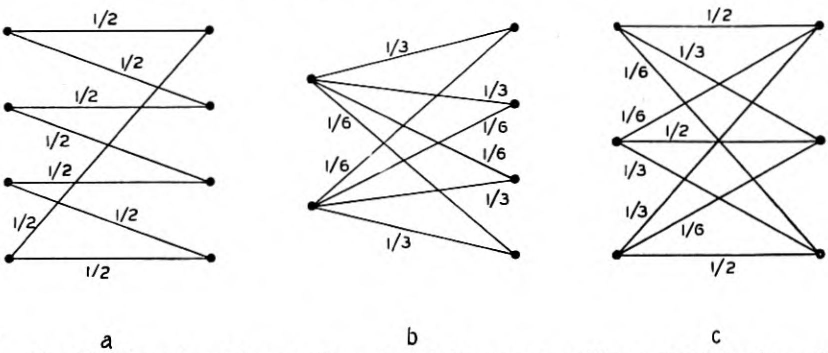

Fig. 12—Examples of discrete channels with the same transition probabilities for each input and for each output.

symbols, since it is possible to make them all equally probable by making the input symbols equally probable. The channel capacity is therefore

$$C = \log m + \Sigma\, p_i \log p_i\,.$$

In Fig. 12a it would be

$$C = \log 4 - \log 2 = \log 2\,.$$

This could be achieved by using only the 1st and 3d symbols. In Fig. 12b

$$\begin{aligned}C &= \log 4 - \tfrac{2}{3} \log 3 - \tfrac{1}{3} \log 6\\ &= \log 4 - \log 3 - \tfrac{1}{3} \log 2\\ &= \log \tfrac{1}{3}\, 2^{\frac{5}{3}}\,.\end{aligned}$$

In Fig. 12c we have

$$\begin{aligned}C &= \log 3 - \tfrac{1}{2} \log 2 - \tfrac{1}{3} \log 3 - \tfrac{1}{6} \log 6\\ &= \log \frac{3}{2^{\frac{1}{2}}\, 3^{\frac{1}{3}}\, 6^{\frac{1}{6}}}\,.\end{aligned}$$

Suppose the symbols fall into several groups such that the noise never causes a symbol in one group to be mistaken for a symbol in another group. Let the capacity for the nth group be C_n when we use only the symbols in this group. Then it is easily shown that, for best use of the entire set, the total probability P_n of all symbols in the nth group should be

$$P_n = \frac{2^{C_n}}{\Sigma 2^{C_n}}.$$

Within a group the probability is distributed just as it would be if these were the only symbols being used. The channel capacity is

$$C = \log \Sigma 2^{C_n}.$$

17. An Example of Efficient Coding

The following example, although somewhat unrealistic, is a case in which exact matching to a noisy channel is possible. There are two channel symbols, 0 and 1, and the noise affects them in blocks of seven symbols. A block of seven is either transmitted without error, or exactly one symbol of the seven is incorrect. These eight possibilities are equally likely. We have

$$\begin{aligned} C &= \text{Max}\,[H(y) - H_x(y)] \\ &= \tfrac{1}{7}\,[7 + \tfrac{8}{8}\log \tfrac{1}{8}] \\ &= \tfrac{4}{7}\ \text{bits/symbol}. \end{aligned}$$

An efficient code, allowing complete correction of errors and transmitting at the rate C, is the following (found by a method due to R. Hamming):

Let a block of seven symbols be $X_1, X_2, \ldots X_7$. Of these X_3, X_5, X_6 and X_7 are message symbols and chosen arbitrarily by the source. The other three are redundant and calculated as follows:

$$\begin{array}{llll} X_4 & \text{is chosen to make} & \alpha = X_4 + X_5 + X_6 + X_7 & \text{even} \\ X_2 & \text{“ \quad “ \quad “ \quad “} & \beta = X_2 + X_3 + X_6 + X_7 & \text{“} \\ X_1 & \text{“ \quad “ \quad “ \quad “} & \gamma = X_1 + X_3 + X_5 + X_7 & \text{“} \end{array}$$

When a block of seven is received α, β and γ are calculated and if even called zero, if odd called one. The binary number $\alpha\,\beta\,\gamma$ then gives the subscript of the X_i that is incorrect (if 0 there was no error).

APPENDIX 1

The Growth of the Number of Blocks of Symbols With A Finite State Condition

Let $N_i(L)$ be the number of blocks of symbols of length L ending in state i. Then we have

$$N_j(L) = \sum_{is} N_i(L - b_{ij}^{(s)})$$

where $b_{ij}^1, b_{ij}^2, \ldots b_{ij}^m$ are the length of the symbols which may be chosen in state i and lead to state j. These are linear difference equations and the behavior as $L \to \infty$ must be of the type

$$N_j = A_j W^L$$

Substituting in the difference equation

$$A_j W^L = \sum_{i,S} A_i W^{L - b_{ji}^{(s)}}$$

or

$$A_j = \sum_{iS} A_i W^{-b_{ij}^{(s)}}$$

$$\sum_i \left(\sum_S W^{-b_{ij}^{(s)}} - \delta_{ij}\right) A_i = 0.$$

For this to be possible the determinant

$$D(W) = |a_{ij}| = \left|\sum_S W^{-b_{ij}^{(s)}} - \delta_{ij}\right|$$

must vanish and this determines W, which is, of course, the largest real root of $D = 0$.

The quantity C is then given by

$$C = \operatorname*{Lim}_{L \to \infty} \frac{\log \Sigma A_j W^L}{L} = \log W$$

and we also note that the same growth properties result if we require that all blocks start in the same (arbitrarily chosen) state.

APPENDIX 2

Derivation of $H = -\Sigma\, p_i \log p_i$

Let $H\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right) = A(n)$. From condition (3) we can decompose a choice from s^m equally likely possibilities into a series of m choices each from s equally likely possibilities and obtain

$$A(s^m) = m\, A(s)$$

Similarly

$$A(t^n) = n\, A(t)$$

We can choose n arbitrarily large and find an m to satisfy

$$s^m \le t^n < s^{(m+1)}$$

Thus, taking logarithms and dividing by $n \log s$,

$$\frac{m}{n} \leq \frac{\log t}{\log s} \leq \frac{m}{n} + \frac{1}{n} \quad \text{or} \quad \left| \frac{m}{n} - \frac{\log t}{\log s} \right| < \epsilon$$

where ϵ is arbitrarily small.

Now from the monotonic property of $A(n)$

$$A(s^m) \leq A(t^n) \leq A(s^{m+1})$$

$$m\,A(s) \leq nA(t) \leq (m+1)\,A(s)$$

Hence, dividing by $nA(s)$,

$$\frac{m}{n} \leq \frac{A(t)}{A(s)} \leq \frac{m}{n} + \frac{1}{n} \quad \text{or} \quad \left| \frac{m}{n} - \frac{A(t)}{A(s)} \right| < \epsilon$$

$$\left| \frac{A(t)}{A(s)} - \frac{\log t}{\log s} \right| \leq 2\epsilon \qquad A(t) = -K \log t$$

where K must be positive to satisfy (2).

Now suppose we have a choice from n possibilities with commeasurable probabilities $p_i = \dfrac{n_i}{\Sigma n_i}$ where the n_i are integers. We can break down a choice from Σn_i possibilities into a choice from n possibilities with probabilities $p_i \ldots p_n$ and then, if the ith was chosen, a choice from n_i with equal probabilities. Using condition 3 again, we equate the total choice from Σn_i as computed by two methods

$$K \log \Sigma n_i = H(p_1, \ldots, p_n) + K\Sigma\, p_i \log n_i$$

Hence

$$H = K\,[\Sigma\, p_i \log \Sigma\, n_i - \Sigma\, p_i \log n_i]$$

$$= -K\,\Sigma\, p_i \log \frac{n_i}{\Sigma n_i} = -K\,\Sigma\, p_i \log p_i\,.$$

If the p_i are incommeasurable, they may be approximated by rationals and the same expression must hold by our continuity assumption. Thus the expression holds in general. The choice of coefficient K is a matter of convenience and amounts to the choice of a unit of measure.

APPENDIX 3

Theorems on Ergodic Sources

If it is possible to go from any state with $P > 0$ to any other along a path of probability $p > 0$, the system is ergodic and the strong law of large numbers can be applied. Thus the number of times a given path p_{ij} in the network is traversed in a long sequence of length N is about proportional to the probability of being at i and then choosing this path, $P_i p_{ij} N$. If N is large enough the probability of percentage error $\pm\,\delta$ in this is less than ϵ so that for all but a set of small probability the actual numbers lie within the limits

$$(P_i p_{ij} \pm \delta)N$$

Hence nearly all sequences have a probability p given by

$$p = \Pi p_{ij}^{(P_i p_{ij} \pm \delta)N}$$

and $\frac{\log p}{N}$ is limited by

$$\frac{\log p}{N} = \Sigma(P_i\, p_{ij} \pm \delta) \log p_{ij}$$

or

$$\left| \frac{\log p}{N} - \Sigma P_i\, p_{ij} \log p_{ij} \right| < \eta.$$

This proves theorem 3.

Theorem 4 follows immediately from this on calculating upper and lower bounds for $n(q)$ based on the possible range of values of p in Theorem 3.

In the mixed (not ergodic) case if

$$L = \Sigma\, p_i\, L_i$$

and the entropies of the components are $H_1 \geq H_2 \geq \ldots \geq H_n$ we have the

Theorem: $\lim_{N\to\infty} \frac{\log n(q)}{N} = \varphi(q)$ is a decreasing step function,

$$\varphi(q) = H_s \quad \text{in the interval} \quad \sum_1^{s-1} \alpha_i < q < \sum_1^{s} \alpha_i\,.$$

To prove theorems 5 and 6 first note that F_N is monotonic decreasing because increasing N adds a subscript to a conditional entropy. A simple substitution for $p_{B_i}(S_j)$ in the definition of F_N shows that

$$F_N = N\,G_N - (N-1)\,G_{N-1}$$

and summing this for all N gives $G_N = \frac{1}{N}\Sigma\, F_N$. Hence $G_N \geq F_N$ and G_N monotonic decreasing. Also they must approach the same limit. By using theorem 3 we see that $\lim_{N\to\infty} G_N = H$.

APPENDIX 4

Maximizing the Rate for a System of Constraints

Suppose we have a set of constraints on sequences of symbols that is of the finite state type and can be represented therefore by a linear graph.

Let $\ell_{ij}^{(s)}$ be the lengths of the various symbols that can occur in passing from state i to state j. What distribution of probabilities P_i for the different states and $p_{ij}^{(s)}$ for choosing symbol s in state i and going to state j maximizes the rate of generating information under these constraints? The constraints define a discrete channel and the maximum rate must be less than or equal to the capacity C of this channel, since if all blocks of large length were equally likely, this rate would result, and if possible this would be best. We will show that this rate can be achieved by proper choice of the P_i and $p_{ij}^{(s)}$.

The rate in question is

$$\frac{-\Sigma P_i p_{ij}^{(s)} \log p_{ij}^{(s)}}{\Sigma P_{(i)} p_{ij}^{(s)} \ell_{ij}^{(s)}} = \frac{N}{M}.$$

Let $\ell_{ij} = \sum_s \ell_{ij}^{(s)}$. Evidently for a maximum $p_{ij}^{(s)} = k \exp \ell_{ij}^{(s)}$. The constraints on maximization are $\Sigma P_i = 1$, $\sum_j p_{ij} = 1$, $\Sigma P_i(p_{ij} - \delta_{ij}) = 0$. Hence we maximize

$$U = \frac{-\Sigma P_i p_{ij} \log p_{ij}}{\Sigma P_i p_{ij} \ell_{ij}} + \lambda \sum_i P_i + \Sigma \mu_i p_{ij} + \Sigma \eta_j P_i(p_{ij} - \delta_{ij})$$

$$\frac{\partial U}{\partial p_{ij}} = -\frac{MP_i(1 + \log p_{ij}) + NP_i \ell_{ij}}{M^2} + \lambda + \mu_i + \eta_i P_i = 0.$$

Solving for p_{ij}

$$p_{ij} = A_i B_j D^{-\ell_{ij}}.$$

Since

$$\sum_j p_{ij} = 1, \qquad A_i^{-1} = \sum_j B_j D^{-\ell_{ij}}$$

$$p_{ij} = \frac{B_j D^{-\ell_{ij}}}{\sum_s B_s D^{-\ell_{is}}}.$$

The correct value of D is the capacity C and the B_j are solutions of

$$B_i = \Sigma B_j C^{-\ell_{ij}}$$

for then

$$p_{ij} = \frac{B_j}{B_i} C^{-\ell_{ij}}$$

$$\Sigma P_i \frac{B_j}{B_i} C^{-\ell_{ij}} = P_j$$

or

$$\Sigma \frac{P_i}{B_i} C^{-\ell_{ij}} = \frac{P_j}{B_i}$$

So that if λ_i satisfy

$$\Sigma \gamma_i C^{-\ell_{ij}} = \gamma_j$$

$$P_i = B_i \gamma_i$$

Both of the sets of equations for B_i and γ_i can be satisfied since C is such that

$$| C^{-\ell_{ij}} - \delta_{ij} | = 0$$

In this case the rate is

$$-\frac{\Sigma P_i p_{ij} \log \dfrac{B_j}{B_i} C^{-\ell_{ij}}}{\Sigma P_i p_{ij} \ell_{ij}}$$

$$= C - \frac{\Sigma P_i p_{ij} \log \dfrac{B_j}{B_i}}{\Sigma P_i p_{ij} \ell_{ij}}$$

but

$$\Sigma P_i p_{ij} (\log B_j - \log B_i) = \sum_j P_j \log B_j - \Sigma P_i \log B_i = 0$$

Hence the rate is C and as this could never be exceeded this is the maximum, justifying the assumed solution.

(To be continued)

An Aspect of the Dialing Behavior of Subscribers and Its Effect on the Trunk Plant

By CHARLES CLOS

Introduction

DURING the war it became necessary for the Bell System Companies to lower many service standards. Among these was the standard for the provision of trunks for handling subscriber-dialed calls. In the interest of economy the number of trunks for a given volume of traffic was lowered. It is evident that for any given case there is a lower limit to the number of trunks that should be provided for handling subscriber-dialed calls. Below this limit congestion of calls gets beyond control. The control of congestion is important. In the case of operator-handled calls it is possible to control congestion by filing tickets and placing calls in an orderly fashion. In the case of subscriber-dialed calls the subscriber may with impunity make many, indeed very many, successive dialing attempts to complete a call that is blocked due to a shortage of trunks. If, in a particular office enough subscribers do this simultaneously, a sender shortage may develop with its resulting reaction on the whole office.

From the foregoing it is evident that the standard of service for providing trunks in trunk groups handling subscriber-dialed calls is of importance. During the war years, the New York Telephone Company undertook a study to determine the limits below which it would be undesirable to degrade the service. This study was designed to test the reasonableness of the reduction in the inter-office trunk standard from the pre-war basis of providing enough trunks to delay only one out of a hundred calls in the busy hour to a wartime basis of providing enough trunks to delay two calls in every hundred during the busy hour. The conclusion from this study was that it was safe to use wartime standards.

The study reported herein is an analysis of the effect of repeated attempts when subscriber-dialed calls are blocked due to trunk shortages. The data upon which the results are based indicate that dial subscribers after encountering a *busy* condition make new attempts sooner and much more often than has been generally believed. The results indicate that one can reconstruct what happens when trunk groups carrying subscriber-dialed calls encounter serious overloads and that trunk capacity tables for such situations can be developed.

The study is based on extensive service observations taken at the New

York City Service Observing Bureaus during the winter of 1943–44. These observations dealt with the behavior of subscribers who encounter a *busy* on a dialed call. This behavior is assumed to apply to the situation when subscribers encounter an *all-trunks-busy* condition.

Inadequacy of the Poisson and Erlang B Formulae To Express the Situation when Shortages Occur in Trunk Groups Handling Subscriber Dialed Calls

In connection with the provision of trunks in the exchange plant, two sets of trunk-call-carrying-capacity tables are currently in use. One set of these tables is computed from the Poisson Formula and the other from the Erlang B Formula. The Poisson tables are used for trunk groups carrying non-alternate route traffic, whereas the Erlang B tables are used for trunk groups carrying traffic subject to alternate routing.

The assumption underlying the Poisson Formula, when a shortage of trunks occurs, is that of a partial delay. A call which encounters *all trunks busy* waits but not longer than a holding time interval for a trunk to become available.

The corresponding assumption underlying the Erlang B Formula is that of no delay. A call which encounters *all trunks busy* is cleared out. The call may be abandoned by the subscriber or advanced to an alternate route.

With respect to non-alternate route trunk groups handling subscriber dialed calls neither of the above two assumptions is realized in practice. When all trunks are busy, the dial equipment is arranged to return an *all-trunks-busy* signal to the subscriber rather than hold the call pending the outcome of a subsequent test for an idle trunk. The subscriber upon encountering an *all-trunks-busy* signal does not necessarily abandon the call. In most cases he redials the call.

The degree by which the assumptions are not realized depends upon the relative number of trunks that are provided for a given volume of traffic. For instance if, during an hour, 150 calls having an average holding time of 100 seconds are submitted to ten trunks and an equivalent volume of traffic is submitted to five trunks, the following theoretical results follow from the Poisson and Erlang B Formulae:—

Table I
Theoretical Results from Poisson and Erlang B Formulae

150 Calls of 100 Seconds Average Holding Time Submitted during an hour to	Number of Calls that Are Delayed on the Basis of the Poisson Formula	Number of Calls that Are Cleared Out on the Basis of the Erlang B Formula
10 trunks	1.6	1.0
5 trunks	60.6	32.0

The values in Table I indicate that, when a liberal number of trunks, i.e., ten trunks, is provided, the numerical difference between the results of the two formulae is small and the results of either formula can be used as an approximation of the number of calls affected by an *all-trunks-busy* condition. There are undoubtedly repetitious attempts, but because the number is small their effect can be neglected.

When, however, there is a serious shortage of trunks, as when only five trunks are provided, the numerical difference between the theoretical results of the two formulae is large. In addition, the repetitious attempts will be too numerous to ignore. Some of the repetitious attempts will encounter *all trunks busy* again and again. Other repetitious attempts will seize idle trunks thereby causing new calls to encounter *all trunks busy*. The effect is cumulative. Neither the Poisson nor the Erlang B Formula indicates to what extent the repetitious attempts take place nor their effect. A preliminary glimpse at the results of this study indicates that 150 calls of 100 seconds average holding time when submitted during an hour to five trunks become inflated by 99 repetitious attempts and appear as 249 calls being submitted to the trunks. Of these 249 calls, 108 encounter *all trunks busy*. Of the 108 calls, 99 become the aforementioned repetitious attempts and nine are abandoned. It is evident that neither formula presents this picture. For studies considering the effect of overloads due to trunk shortages, this is the type of information needed. A new approach is required to obtain such data. To do this, it is desirable to examine the habits of dial subscribers who have encountered *busies*.

The Dialing Behavior of Subscribers upon Encountering a *Busy*

In order to investigate the grade of service given to dial subscribers when trunk shortages occur it is desirable to know something about their behavior when they encounter *all-trunks-busy* signals. Specifically there are four items that need investigation; these are:—

1. How soon after encountering an *all-trunks-busy* signal does the subscriber redial his call?
2. What percentage of the subscribers make subsequent attempts?
3. How do the time intervals between successive subsequent attempts compare with each other; that is, are they about the same or do they differ widely?
4. What differences, if any, exist between classes of subscribers?

The first three items are answered from the results of service observations. The fourth item is answered indirectly.

The service observations consisted of 1,107 cases where line *busies* were observed (except for 35 cases of *all-trunks-busy* signals). Observations on line *busies* were used instead of *all-trunks-busy* signals because it would have taken too long to obtain sufficient observations, because it is undesirable to artificially degrade the service in order to obtain sufficient observations and because it is assumed that the average subscriber does not recognize the difference between a *busy* and overflow signal. It is considered that the data, while collected for *busy* signals, accurately represent the situation with regard to overflow signals.

Beginning on December 22, 1943 and ending on February 29, 1944, a special record of 1,107 subscriber-dialed calls, where line *busies* were observed, was taken at the three New York City service observation bureaus. Up to a point, regular service observation practices were followed and the regular service observing data concerning the calls were entered on the service-observing records. The data concerning the line *busies* were entered on a special form. This form is shown below. Instructions for the observers accompanied these forms; these instructions follow the form.

Form S.O. 171

SPECIAL RECORDS—*BUSY* CALLS

Calling No. . . . . Date . . . .

Enter in space under attempt number, the cumulative seconds from the start of the original attempt to the start of the attempt indicated. In addition for the last attempt show disposition.

Attempt Number

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

Disposition of the call. . . . .

Data for attempts over 20 should be entered on the reverse side.

Special service observing form used to collect data concerning the dialing behavior of subscribers upon encountering a *busy*.

Instructions Applying to the Use of Form S.O. 171

These instructions apply to the use of Form S.O. 171 which has been developed in connection with a study of the behavior of customers upon encountering a *busy* signal.

This study will not include observations originating on P.B.X. trunks or on coin lines. On all other calls encountering a *busy* signal or an overflow signal the observer will hold the line in the observing position until one of the following conditions occurs:

(1) Call is disposed of by reaching the desired number.—Code OK

(2) 10 minutes have elapsed since the last attempt for the desired number.—Code AB
(3) Call is disposed of by being given to the operator.—Code PR
(4) Call is disposed of by receiving a "Don't answer" on an attempt to reach the desired number—Code DA

All attempts made during the period that an observation is ordinarily held will be entered on the service observing detail sheets in the regular way. In addition, these entries and entries showing any other attemps to reach the desired number together with the proper code listed above to show the final disposition of the call will be recorded on Form S.O. 171.

In order to minimize the number of cases not completed at the end of an observer's trick, no cases will be recorded on the special record on which the original *busy* signal is received after $\frac{1}{2}$ hour prior to the finish of any trick.

From the instructions it may be noted that observations originating on P.B.X. trunks or on coin lines were not included. The reason for this is, when a *busy* is observed on a call originating on a P.B.X. trunk the subsequent attempt might be made on one of the other P.B.X. trunks, thus the subsequent attempt would be missed. Also, at a P.B.X. two extensions may place calls, within a few seconds of each other, to the same *busy* line. The service observations on any one trunk might therefore be a mixture of attempts involving two or more calls. When a *busy* is observed on a call made from a coin line, the calling party will in many instances vacate the coin box in favor of someone else, and the subsequent attempt may then be made from another coin line. For these reasons the observations were restricted to business and residential individual lines and to two-party lines (12 observations were on two-party lines).

It may also be noted that the observers were instructed to hold the line in the observing position until ten minutes have elapsed since the last attempt for the desired number. This was a departure from regular service observing practices when a line is held until 1 minute has elapsed.

Table II is a tabulation of the data observed at the Manhattan Service Observing Bureau on Manhattan dial subscriber lines. The observations are arranged in the order of increasing magnitude of the time intervals between the start of the first attempt and the start of the second attempt. Of interest is observation number 197 where a subscriber made 25 attempts in about an hour.

Data similar to that observed on Manhattan dial subscriber lines were likewise observed on Bronx-Westchester and on Brooklyn-Queens dial subscriber lines.

Figure 1(a) shows graphically the data listed in Table II. This graph shows, by dots, the cumulative percentage of the 451 Manhattan observa-

TABLE II

RESULTS OF OBSERVATIONS ON 451 DIAL SUBSCRIBERS IN MANHATTAN

Seconds elapsing between start of previous attempt and start of attempt listed below:

Obser-vation No.	Attempt No.										Total Seconds	Disposition of the Call
	1	2	3	4	5	6	7	8	9	10		
1	0	13	24	13	11	20					81	O.K.
2	0*	16*	10*								26	AB.
3	0	16	48	54	82	108					308	O.K.
4	0	18									18	D.A.
5	0	19									19	PR.
6	0	19									19	O.K.
7	0*	20									20	O.K.
8	0	20									20	O.K.
9	0	20	64	189							273	PR.
10	0	21									21	AB.
11	0*	21									21	O.K.
12	0	21									21	PR.
13	0	21	208								229	O.K.
14	0	22	30	28							80	O.K.
15	0	22	22								44	O.K.
16	0	22	26	189	18	25					280	PR.
17	0	23									23	O.K.
18	0*	25									25	O.K.
19	0	25	28	33							86	O.K.
20	0	25	341	44							410	AB.
21	0	25									25	AB.
22	0	25	28	22	69	63	82	271			560	AB.
23	0	26	188								214	O.K.
24	0	27	35	35	29	38	36	42	43	53	338	O.K.
25	0	27									27	AB.
26	0	27									27	PR.
27	0	28									28	O.K.
28	0	28									28	AB.
29	0	29	110	22							161	PR.
30	0	30									30	AB.
31	0	30	31	23	20	20	86	20	54	26		
	51										361	AB.
32	0	30	440								470	O.K.
33	0	30									30	AB.
34	0	31	52	31	591						705	O.K.
35	0	31	45								76	AB.
36	0*	31									31	O.K.
37	0	31	105	66							202	O.K.
38	0	31									31	O.K.
39	0	31	98								129	O.K.
40	0	32									32	D.A.
41	0	32									32	AB.
42	0	32	32								64	O.K.
43	0	32									32	O.K.
44	0	33*									33	AB.
45	0	33	35	41	43	57					209	O.K.
46	0	35									35	O.K.
47	0	35	358								393	O.K.
48	0	36	88								124	O.K.
49	0	37	53	40							130	O.K.
50	0	39									39	O.K.
51	0	39									39	O.K.

*Overflow signal.

TABLE II (*Cont'd*)

Observation No.	Attempt No. 1	2	3	4	5	6	7	8	9	10	Total Seconds	Disposition of the Call
52	0	40	20	574							634	O.K.
53	0	40*	16	200	432						688	O.K.
54	0	40	409								449	O.K.
55	0	40									40	O.K.
56	0	41	45	55	52	27	25				245	AB.
57	0	41	45	84							170	O.K.
58	0	42	122	68							232	AB.
59	0	43	40	52	38	259					432	AB.
60	0	44									44	O.K.
61	0	46									46	O.K.
62	0	47	32	47	34	40	69				269	O.K.
63	0	47									47	O.K.
64	0	47	179	251							477	O.K.
65	0	48	64								112	AB.
66	0	49									49	AB.
67	0	49	51	57	62	71	60				350	O.K.
68	0	49	96	191							336	O.K.
69	0	50									50	O.K.
70	0	50									50	O.K.
71	0	50	85	151							286	O.K.
72	0	50									50	AB.
73	0	50									50	AB.
74	0	51									51	O.K.
75	0	51									51	AB.
76	0	52									52	O.K.
77	0	52	85	209							346	O.K.
78	0	53	195								248	O.K.
79	0	53									53	AB.
80	0	53									53	O.K.
81	0	55	43								98	AB.
82	0	55	43	27	170*						295	AB.
83	0	56	20	61	36	103					276	AB.
84	0	56	117	57							230	O.K.
85	0	56									56	O.K.
86	0	56	84								140	O.K.
87	0	57	74	81							212	O.K.
88	0	58									58	O.K.
89	0	58	139	84	163	62	127				633	O.K.
90	0	60									60	O.K.
91	0	60	139								199	O.K.
92	0	60									60	O.K.
93	0	60									60	AB.
94	0	61									61	O.K.
95	0	61									61	AB.
96	0	63									63	AB.
97	0	63	31	95	28	20					237	AB.
98	0	64	126	470	85	167					912	O.K.
99	0	64	61	67	84	67					343	O.K.
100	0	64	45	63	63	161					396	O.K.
101	0	65	482								547	O.K.
102	0	66	173	172							411	O.K.
103	0	66†	66	72							204	O.K.
104	0	66									66	AB.

*Overflow signal.
† Don't answer.

TABLE II (*Cont'd*)

Observation No.	Attempt No. 1	2	3	4	5	6	7	8	9	10	Total Seconds	Disposition of the Call
105	0	68	66								134	O.K.
106	0	68	330	380							778	O.K.
107	0	69									69	AB.
108	0	70									70	O.K.
109	0	71									71	AB.
110	0	71	95								166	O.K.
111	0	72	112								184	AB.
112	0	72									72	O.K.
113	0	74									74	O.K.
114	0	74	184	93							351	AB.
115	0	75									75	O.K.
116	0	75									75	O.K.
117	0	75	67	203							345	O.K.
118	0	76									76	O.K.
119	0	76									76	AB.
120	0*	77									77	O.K.
121	0	78									78	O.K.
122	0	78									78	O.K.
123	0	78	253	107	38						476	AB.
124	0	79	53								132	O.K.
125	0	80									80	O.K.
126	0	80	50								130	O.K.
127	0	80									80	AB.
128	0	80	117								197	O.K.
129	0	81									81	AB.
130	0	81									81	O.K.
131	0	83									83	O.K.
132	0	84									84	O.K.
133	0	85									85	O.K.
134	0	85	33	294	115						527	O.K.
135	0	88									88	AB.
136	0	88									88	O.K.
137	0*	89									89	O.K.
138	0	90	50	120							260	AB.
139	0	90									90	O.K.
140	0	90									90	AB.
141	0	90	51	39	46						226	AB.
142	0	91	78								169	O.K.
143	0	91									91	O.K.
144	0	91*	48								139	O.K.
145	0	91	116								207	O.K.
146	0*	91									91	O.K.
147	0*	92									92	O.K.
148	0*	92									92	O.K.
149	0	93									93	AB.
150	0	93	34	228	117						472	O.K.
151	0	94	94	75	91						354	AB.
152	0	95									95	AB.
153	0	95									95	O.K.
154	0	97	86	175							358	O.K.
155	0	97	143								240	O.K.
156	0	100									100	O.K.
157	0	100									100	O.K.
158	0	100									100	AB.
159	0	100									100	O.K.
160	0	102	115	198							415	O.K.

*Overflow signal.

TABLE II (*Cont'd*)

Observation No.	Attempt No.										Total Seconds	Disposition of the Call
	1	2	3	4	5	6	7	8	9	10		
161	0	102	80	96	152						430	O.K.
162	0	103									103	O.K.
163	0	104	17								121	O.K.
164	0	105									105	O.K.
165	0	105									105	O.K.
166	0	106	340								446	O.K.
167	0	108	98	140							346	O.K.
168	0	111									111	O.K.
169	0	111									111	AB.
170	0	111	94	125							330	O.K.
171	0	113									113	O.K.
172	0	114									114	O.K.
173	0	116									116	O.K.
174	0	116									116	O.K.
175	0	117									117	O.K.
176	0	120									120	O.K.
177	0	122									122	O.K.
178	0	124	131	209							464	O.K.
179	0	124									124	O.K.
180	0	125	354								479	O.K.
181	0	130									130	O.K.
182	0	130									130	O.K.
183	0	130	125								255	O.K.
184	0	130	56	101							287	O.K.
185	0	131	309								440	O.K.
186	0	134									134	O.K.
187	0	137	147	134	146						564	O.K.
188	0	139	125								264	AB.
189	0	139									139	AB.
190	0	139									139	O.K.
191	0	140	172	60							372	O.K.
192	0	140	400								540	A.B.
193	0	141									141	O.K.
194	0	143									143	O.K.
195	0	143	157								300	O.K.
196	0	144									144	O.K.
197	0	144	187	194	308	115	310	104	165	45		
	69	90	69	88	87	59	239	277	69	94		
	90	159	193	71	237						3,463	AB.
198	0	146									146	O.K.
199	0	146									146	O.K.
200	0	146	184	217							547	AB.
201	0	148									148	O.K.
202	0	149									149	O.K.
203	0	149	28	38	42	46					303	O.K.
204	0	149	121	84							354	A.B.
205	0	150									150	A.B.
206	0	150	26	142	119						437	A.B.
207	0	151	272								423	O.K.
208	0	152	90	95	89	79					505	O.K.
209	0*	155									155	O.K.
210	0	156									156	O.K.
211	0	156									156	O.K.
212	0	156	47	52	217						472	A.B.
213	0	160									160	A.B.
214	0	160									160	O.K.

*Overflow signal.

TABLE II (*Cont'd*)

Observation No.	Attempt No.										Total Seconds	Disposition of the Call
	1	2	3	4	5	6	7	8	9	10		
215	0	160									160	O.K.
216	0	160									160	O.K.
217	0	161									161	A.B.
218	0	164									164	O.K.
219	0	164									164	O.K.
220	0	165									165	A.B.
221	0	168									168	O.K.
222	0	169									169	O.K.
223	0	170									170	O.K.
224	0*	170									170	O.K.
225	0	171									171	O.K.
226	0	175									175	O.K.
227	0	179									179	O.K.
228	0	180									180	O.K.
229	0	181									181	O.K.
230	0	181	360								541	A.B.
231	0	182									182	O.K.
232	0	183									183	O.K.
233	0	183	312	33							528	P.R.
234	0	185									185	O.K.
235	0	186	251								437	A.B.
236	0	192	238								430	A.B.
237	0	195	477								672	O.K.
238	0	198									198	A.B.
239	0	202									202	O.K.
240	0	205	80								285	O.K.
241	0	208									208	O.K.
242	0	209									209	O.K.
243	0	209									209	O.K.
244	0	210									210	O.K.
245	0	214	50	33	29	34	79				439	O.K.
246	0	215	520								735	O.K.
247	0	215									215	A.B.
248	0	217									217	O.K.
249	0*	219									219	D.A.
250	0	219									219	O.K.
251	0	220	163	263	186	123	99	59	105		1,218	AB.
252	0	220	162								382	O.K.
253	0	222									222	O.K.
254	0	226									226	O.K.
255	0	228									228	O.K.
256	0	230									230	AB.
257	0	231	27								258	P.R.
258	0	232									232	O.K.
259	0	235									235	O.K.
260	0*	235									235	O.K.
261	0	238									238	O.K.
262	0	242									242	O.K.
263	0	245									245	O.K.
264	0	246									246	O.K.
265	0	252									252	AB.
266	0	252									252	AB.
267	0	258									258	O.K.
268	0	260	333								593	O.K.
269	0	267	193								460	AB.
270	0	272	219	88*							579	AB.
271	0	278									278	O.K.

*Overflow signal.

TABLE II (*Cont'd*)

Observation No.	Attempt No. 1	2	3	4	5	6	7	8	9	10	Total Seconds	Disposition of the Call
272	0	281									281	O.K.
273	0	287									287	O.K.
274	0	288									288	O.K.
275	0	289	256								545	O.K.
276	0	290									290	O.K.
277	0	296									296	O.K.
278	0	306									306	O.K.
279	0	319									319	O.K.
280	0	320									320	O.K.
281	0	320									320	AB.
282	0*	322									322	O.K.
283	0	331									331	O.K.
284	0	332									332	DA.
285	0	338									338	AB.
286	0	339									339	AB.
287	0	347									347	O.K.
288	0	351	454								805	O.K.
289	0	351									351	O.K.
290	0	363									363	O.K.
291	0	365									365	O.K.
292	0	369									369	DA.
293	0	376									376	O.K.
294	0	378									378	O.K.
295	0	382									382	O.K.
296	0	395									395	O.K.
297	0	398									398	O.K.
298	0	398									398	AB.
299	0	400									400	O.K.
300	0	402									402	O.K.
301	0	409									409	O.K.
302	0	416									416	O.K.
303	0	448									448	O.K.
304	0	449									449	O.K.
305	0	455									455	O.K.
306	0	473									473	O.K.
307	0	484									484	O.K.
308	0	484									484	A.B.
309	0	498									498	O.K.
310	0	505									505	O.K.
311	0	509									509	O.K.
312	0	510									510	A.B.
313	0	513									513	O.K.
314	0	526									526	O.K.
315	0	535	456	541							1,532	O.K.
316	0	543									543	O.K.
317	0	556	249								805	O.K.
318	0	561	389								950	O.K.
319	0	568									568	O.K.
320	0	569									569	O.K.
321	0	570									570	O.K.
322	0	586									586	O.K.
323	0	605	(over 600 seconds)								605	A.B.
324	0	624	(over 600 seconds)								624	A.B.
325	0	(At 30 seconds received on incoming call from the party desired)										A.B.
326–334	0*	(9 observations)										A.B.
335–451	0	(117 observations)										A.B.

*Overflow signal.

tions that equalled or exceeded particular time intervals between the starts of the first and second attempts. Figure 1(b) shows similar graphical data for 211 Bronx-Westchester observations and Fig. 1(c) shows similar graphical data for 445 Brooklyn-Queens observations. Each of these three graphs is compared with a composite curve for 1107 observations. This composite curve is developed from the data on Fig. 2(a).

Figure 2(a) shows, by dots, the cumulative percentage for 1107 observations, which are comprised of the 451 Manhattan, 211 Bronx-Westchester and 445 Brooklyn-Queens observations, that equalled or exceeded particular time intervals between the starts of the first and second attempts. A smooth curve was drawn through these plotted data. This curve is also shown on other figures, for the purpose of visual comparison of the various plots of data with the overall results.

Figure 2(b) shows a graph concerning 465 observations of the total 1107 observations. These are the cases where a *busy* was observed on a second attempt. (Of the 1107 total observations, 817 resulted in a second attempt within ten minutes and 290 were classified as abandoned. Of the 817 second attempts, 327 cases were able to complete their calls, 16 resulted in a don't answer, 9 were referred to an operator and 465 encountered a *busy*.) Figure 2(b) shows, by dots, the cumulative percentage of the 465 second attempts that equalled or exceeded particular time intervals between the starts of the second and third attempts. The graph of Fig. 2(b) does not differ significantly from the composite curve for 1107 observations. This feature indicates that, when observations concerning subscriber *busies* are made, it is not necessary to have the first observed attempts coincide with the first actual attempts. The observations can begin with any attempt.

Figures 3 and 4 are graphs similar to that shown on Fig. 2(a), the difference being in the graphical ordinates used in order to present additional pictorial representations of the data and to project the curve beyond the observed limits.

The percentage of subscribers who dial their calls again after encountering *busies* is estimated from Figs. 3 and 4 to be 90%. The data on Fig. 3 are projected to a time interval of 1,500 seconds (25 minutes). Judging by eye, beyond this point, it appears that the curve is asymptotic to the 10% horizontal line. This means that 10% of the subscribers abandon their calls and 90% try again. The part of the curve on Fig. 4 that projects beyond the limit of the observed data crosses the 10% line at 6,400 seconds, an interval of $1\frac{3}{4}$ hours. This seems to be a very long time for a subscriber to wait before redialing his call. It is unlikely that many attempts are made beyond this period.

Table III was prepared to determine the disposition of the calls on second attempts and to see if a correlation exists between certain time intervals,

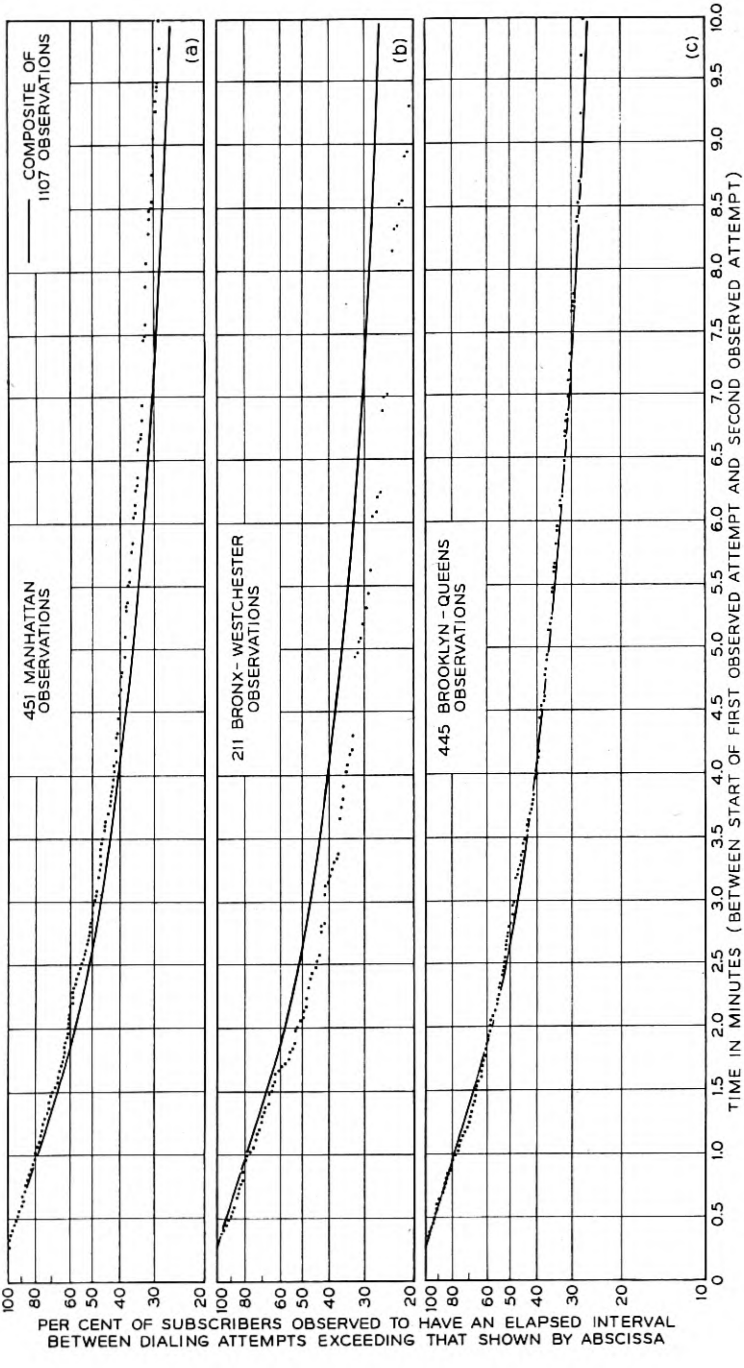

Fig. 1—Results of observations taken at three New York City service observing bureaus concerning the dialing behavior of subscribers upon encountering a *busy*.

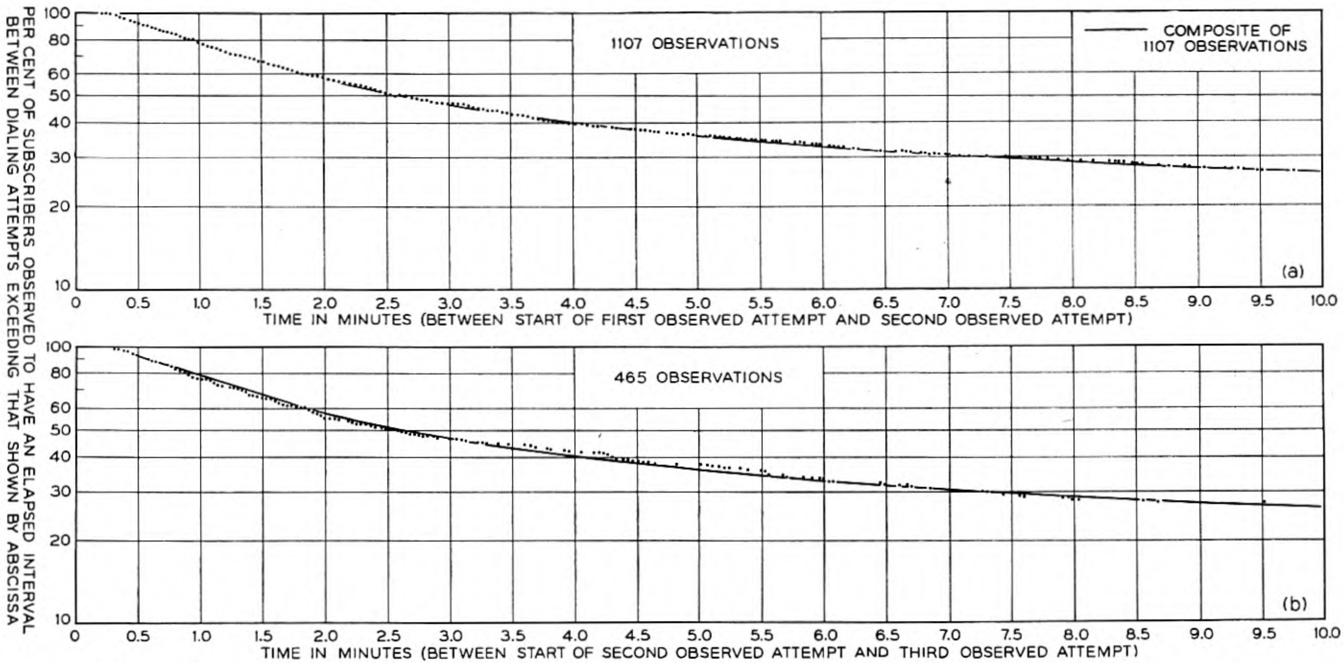

Fig. 2—Composite results of 1107 observations concerning the dialing behavior of subscribers upon encountering a *busy* on a first attempt and results of 465 observations concerning the dialing behavior of subscribers upon encountering a *busy* on a second attempt.

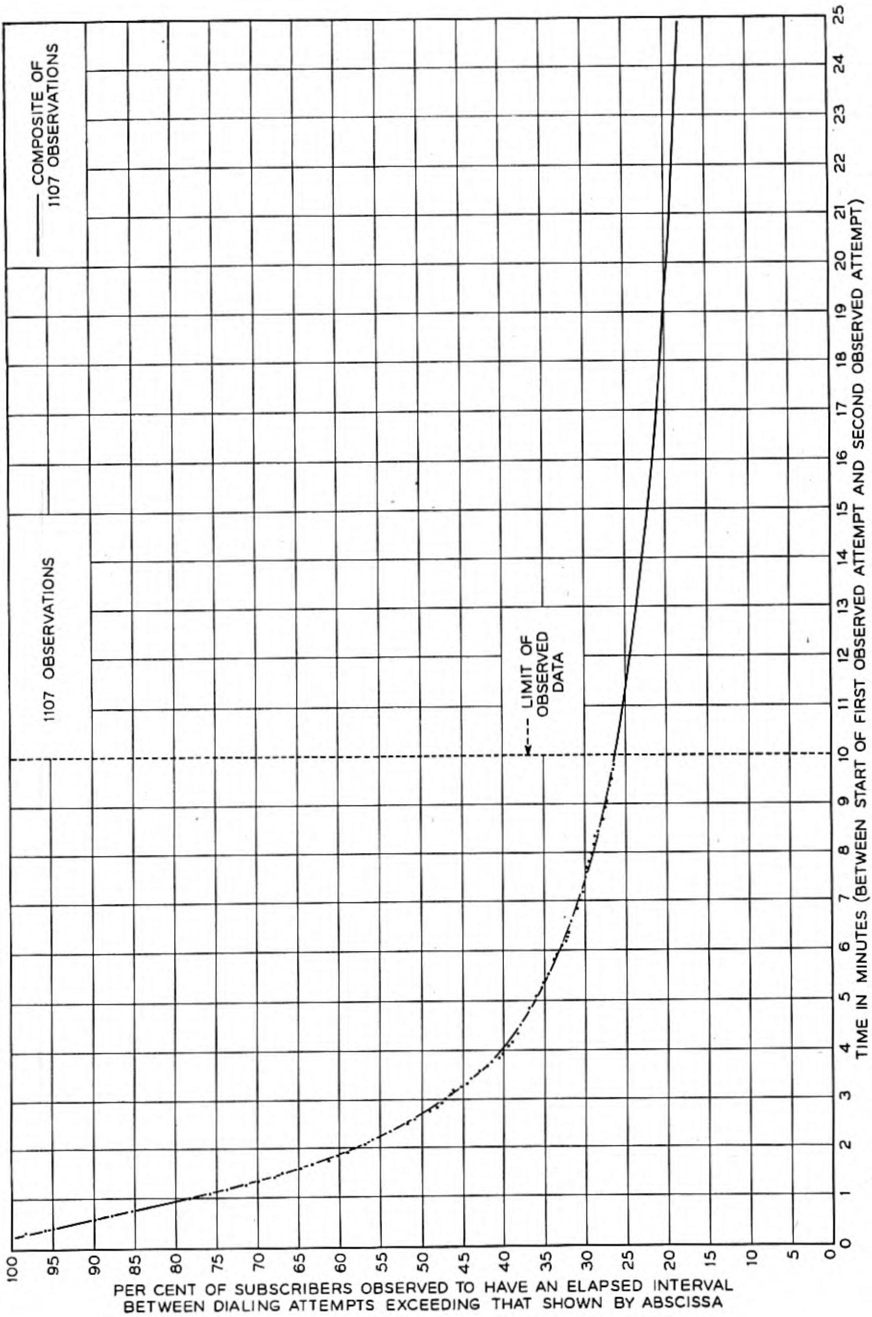

Fig. 3—Composite results of 1107 observations concerning the dialing behavior of subscribers upon encountering a *busy*.

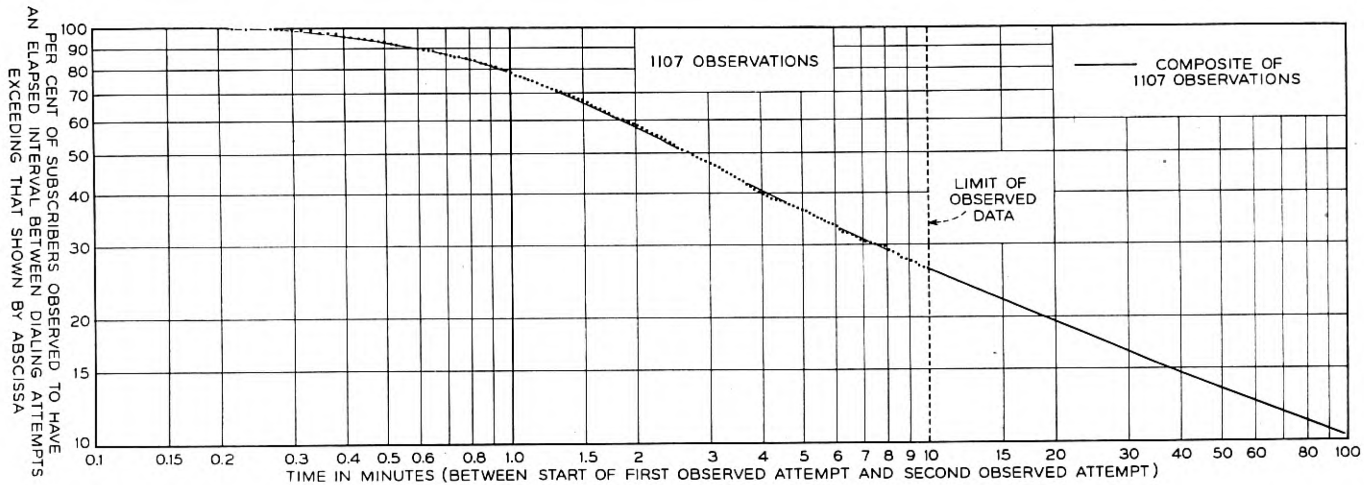

Fig. 4—Composite results of 1107 observations concerning the dialing behavior of subscribers upon encountering a *busy*.

namely, between the first and second attempts and between the second and third attempts. This table was developed by allocating the 817 observations where a second attempt occurred into 5 ranges of time intervals between the first and second attempts of about 163 observations each. For each range of time interval the number of calls that were respectively O.K., DA, PR and AB is listed. Where a third attempt occurred, the numbers of calls are tabulated by ranges of time intervals between the second and third attempts. The ranges of time intervals are the same as

TABLE III

DISPOSITION OF SECOND ATTEMPTS AND CORRELATION OF TIME INTERVALS BETWEEN DATA CONCERNING 817 OBSERVATIONS HAVING A SECOND ATTEMPT

Range of Time Intervals in Seconds Between the First and Second Attempts	Total Number of Observed Second Attempts	Disposition of the Second Attempt: Number of Second Attempts that				Correlation of Time Intervals Between Attempts: Number of Second Attempts Each of Which Resulted in a *Busy* and which was Followed by a Third Attempt Within the Range of Seconds Listed in the Column Headings Below				
		Were OK	Were DA	Were PR	Were AB	0–45	46–78	79–130	131–226	227–600
0– 45	164	36	7	5	26	43*	17	12	8	10
46– 78	164	44	1	2	24	14	26*	28	13	12
79–130	164	71	2	2	21	6	14	24*	11	13
131–226	162	83	2	0	27	5	9	5	15*	16
227–600	163	93	4	0	28	6	0	4	5	23*
	817	327	16	9	126	74	66	73	52	74

* The asterisk marks the items that had the same range of time intervals between the first and second attempts and between the second and third attempts.

those used between the first and second attempts in order to see if a correlation exists. The significant facts concerning these data are:—

1. The degree of success in obtaining an O.K. call was better for those subscribers who waited longer before making a subsequent attempt. Only 22% of the subscribers who waited from 0 to 45 seconds were successful as against 57% who waited from 227 to 600 seconds.
2. The number of calls referred to the operator or where don't answers occurred are not significant to the problem in hand.
3. The incidence of abandoned calls appears to be uniform for the five ranges of time intervals. This means that the 90% figure estimated from Fig. 3 can be considered to apply with equal effect to all subscribers without regard to the previous time interval between dialing attempts.
4. The correlation data indicate a tendency for subscribers to establish a tempo or pace which they follow when redialing their calls. If this tempo did not exist the items on Table III that are marked with asterisks would not be larger than the surrounding items.

It was previously indicated that no observations were taken on P.B.X. and coin lines. An earlier attempt to collect data concerning the behavior of subscribers when encountering *busies* produced data that showed fewer subsequent attempts than was believed to be the case. The differences between the earlier data, which included a high proportion of observations on P.B.X. and coin lines, and the data developed herein are believed to be fully accounted for and it is believed that the P.B.X. and coin lines have the same basic characteristics regarding dialing behavior upon encountering *busies* as have the subscribers who were observed. No significant differences between the results for residential and business offices were noted. From these indirect facts, it is concluded that no significant differences exist between classes of subscribers.

Effect on the Trunk Plant

As explained earlier, neither the Poisson nor the Erlang B formula gives an accurate picture of the facts when trunk shortages occur on trunk groups handling subscriber-dialed calls. In both formulae it is assumed that only one attempt is made per call. In the case of the Poisson formula, the call is assumed to be held by the dial equipment until a trunk becomes available or until the subscriber hangs up, and in the case of the Erlang B formula, the call is assumed to clear out. The data developed from the service observations, concerning the dialing behavior of subscribers when encountering *busies*, indicate that subscribers usually make many subsequent attempts when a *busy* is encountered. Also the dial equipment with which we are familiar clears out the calls by giving an *all-trunks-busy* signal. In order to determine what a trunk capacity table might be like that takes into account the habits of subscribers and the limitations of the dial equipment a study based on simulated traffic was made. This study consisted of 150 CCS (hundred call seconds per hour) of traffic offered to a trunk group varying from 5 to 12 trunks. This study utilized the data developed from the service observations.

A study based on simulated traffic is a method used to study the capacities of trunking arrangements where a formula is not available. This type of study is based on the idea that calls are placed at random, that holding times of the calls follow an exponential law, and that these characteristics can be simulated by random numbers drawn from an appropriate source.

The study of 150 CCS of simulated traffic was based on 1,000 calls offered to a trunk group during a ten-hour period. The average holding time per call was 150 seconds, with the total holding time being 150,000 seconds or 41.66667 hours. Sub-divisions of an hour were expressed in decimal terms, the smallest division being a hundred-thousandth part. Three sets of random numbers were used for the following purposes:

1. To determine at what time in the ten-hour period a particular call is offered to the trunk group.
2. To furnish the holding time of a particular call.
3. To define for each call the pattern of resubmission of the call to the trunk group should an *all-trunks-busy* be encountered by the call.

In each instance the numbers were taken from the tail-end portions of successive entries of 19 significant figures of e^x (Tables of the Exponential Function—WPA—1939). The numbers drawn and their functions in the study are as follows:

A set of 1,000 six-digit numbers was taken from the last six digits of entries of e^x from $x = 0.4000$ to $x = 0.4999$. These 1,000 six-digit numbers were arranged in numerical order to give the placing time of 1,000 simulated calls. The first digit in every number was used to represent the hour and the last five digits the hundred-thousands part of the hour when a particular call was placed. The randomness of this particular draw was checked by determining the differences between successive placement times and then arranging the differences in numerical order. The results were plotted on a cumulative basis on Fig. 5, where a visual comparison can be made with theoretical results.

A set of 1,000 seven-digit random numbers between 0,000,000 to 4,166,667 inclusive were taken from the last seven digits of entries of e^x from $x = 0.5000$ to $x = 0.7344$. Numbers above 4,166,667 were disregarded. These seven-digit numbers when arranged in numerical order accounted for the total holding time of all the calls. The difference between successive numbers arranged in numerical order, furnished 1,000 individual holding times.

A third set of 1,000 random numbers were taken from two sources in the e^x tables. These 1,000 numbers contained a variable number of digits. These numbers were for use when calls encountered *all trunks busies* in order to determine which calls were to be resubmitted and to determine the time interval for resubmitting a call. Previously, it was estimated from Fig. 3, that 90% of the subscribers after encountering a *busy* redial their call. This estimate was used by assigning to the numerals 1 to 9 in the third set of random numbers the characteristic that a call may make a subsequent attempt if it encounters an *all trunks busy* and by assigning to the numeral 0 the characteristic that the call drops out if it encounters an *all trunks busy*. About 10% of the 1,000 numbers show a numeral 0 in the first place and hence no further digits are needed because the call drops out. The remaining 90% of the numbers show numerals from 1 to 9 in the first place and hence may make a second attempt. If an *all trunks busy* is encountered on the second attempt, a numeral from 1 to 9 in the second place determines that a third attempt may be made while the numeral 0 determines that the call drops out. This process is repeated for each place of each number in the third set of 1,000 random numbers until the numeral 0 appears. The number of consecutive places showing only numerals from 1 to 9, indicates the total number of attempts that a particular call might make before it drops out. Thus for a particular number the numerals might be 4720. In this case, three subsequent attempts can be made. Another number might be 834650. In this case, five subsequent attempts can be made.

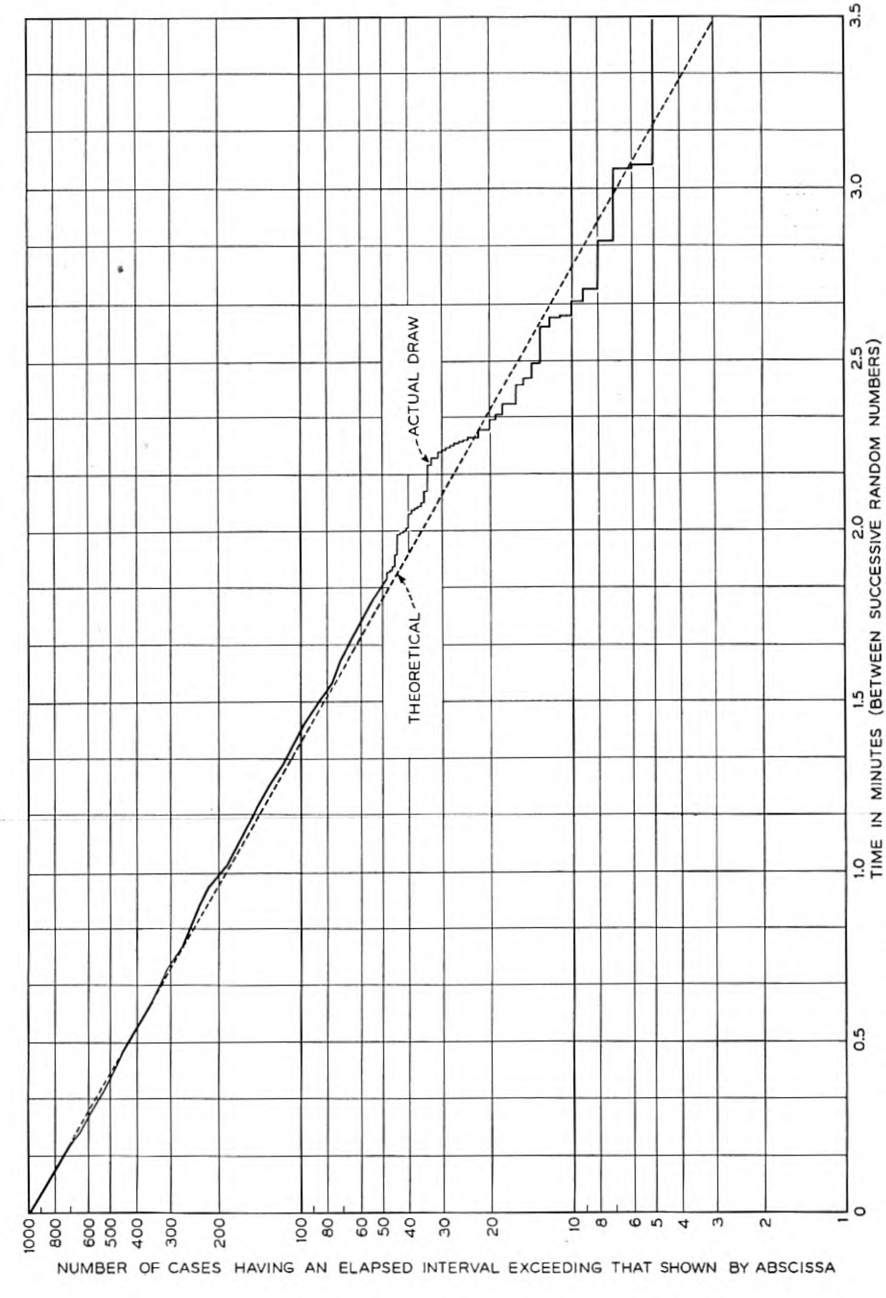

Fig. 5—Check of the randomness of numbers drawn that represent the time of placement of calls.

The effect of using numerals in this way is that 90% of the calls encountering *all trunks busies* appear as subsequent attempts.

The numeral in the first place of each of the third set of random numbers was used to establish the time interval for resubmitting each call. The time intervals were developed from the data on Fig. 4 by dividing the vertical scale into 10% bands. The time interval corresponding to the midpoint of each band was used as applicable to the 10% of the calls that fell within that band. The midpoint values, the corresponding time intervals, and the random numerals used are as follows:

TABLE No. IV

Midpoint Values of the 10% Bands of figure 4	Corresponding Time Intervals in Seconds	Equivalent Hundred-Thousandth Part of an Hour	Assignment of Random Numerals
a	*b*	$c = b \div .036$	*d*
95	25	700	9
85	46	1,300	8
75	67	1,900	7
65	93	2,600	6
55	132	3,700	5
45	195	5,400	4
35	320	8,900	3
25	665	18,500	2
15	2,250	62,500	1
5	Infinite	Call drops out	0

Based on the results indicated by Table III, that subscribers tend to make repetitious attempts at a uniform pace or tempo, the time interval determined by the numeral in the first place of a particular number of the third set of random numbers was repeated each time that a particular call was resubmitted.

The results of the study of simulated traffic are as follows:

TABLE V

Trunks Provided	Attempts (Calls Offered Plus All Subsequent Attempts)	Overflows (Calls Encountering *All Trunks Busies*)	Ratios of Overflows to Attempts	Calls Handled	Calls Abandoned	Approx. CCS Handled
a	*b*	*c*	$d = c \div b$	$e = b - c$	$f = 1000 - e$	$g = .150xe$
5	1,658	720	.4343	938	62	141
6	1,287	319	.2479	968	32	145
7	1,147	155	.1351	992	8	149
8	1,071	75	.0700	996	4	149
9	1,027	28	.0273	999	1	150
10	1,011	12	.0119	999	1	150
11	1,005	5	.0050	1,000	0	150
12	1,000	0	.0000	1,000	0	150

The ratios of overflows to attempts compared with theoretical results for the Poisson and Erlang B formulae for 150 CCS of offered traffic are as follows:

Table VI

Trunks Provided	Study of Simulated Traffic: Ratios of Overflows to Attempts	Theoretical Results	
		Erlang B: Ratio of Calls Lost to Calls Offered	Poisson: Ratio of Calls Delayed to Calls Offered
5	.4343	.2139	.4037
6	.2479	.1293	.2414
7	.1351	.0715	.1288
8	.0700	.0359	.0617
9	.0273	.0163	.0268
10	.0119	.0068	.0106
11	.0050	.0026	.0038
12	.0000	.0009	.0013

The ratios of overflows to attempts are apparently very close to the Poisson results. No further conclusion should be drawn from this, at this time, without further study.

Summary

Data concerning the dialing behavior of subscribers who encounter *busies* have been obtained for New York City subscribers. These data indicate quantitatively: (1) how soon after obtaining a *busy*, a subscriber redials his call; (2) what percentage of subscribers make subsequent attempts; and (3) the pattern of time intervals between successive subsequent attempts. These data appear to have direct application in the development of trunk capacity tables for trunks handling subscriber-dialed traffic when trunk shortages occur.

Acknowledgments

The writer gratefully acknowledges the help of Mr. H. P. Penny in planning the method for taking the service observations and in taking the Manhattan and Bronx-Westchester observations. The help of Mr. R. A. Colbeth is acknowledged in taking the Brooklyn-Queens observations.

Spectra of Quantized Signals

By W. R. BENNETT

1. Discussion of Problem and Results Presented

SIGNALS which are quantized both in time of occurrence and in magnitude are in fact quite old in the communications art. Printing telegraph is an outstanding example. Here, time is divided into equal divisions, and the number of magnitudes to be distinguished in any one interval is usually no more than two, corresponding to the closed or open positions of a sending switch. It is only in recent years, however, that the development of high speed electronic devices has progressed sufficiently to enable quantizing techniques to be applied to rapidly changing signals such as produced by speech, music, or television. Quantizing of time, or time division, has found application as a means of multiplexing telephone channels.[1] The method consists of connecting the different channels to the line in sequence by fast moving switches synchronized at the transmitting and receiving ends. In this way a transmission medium capable of handling a much wider band of frequencies than required for one telephone channel can be used simultaneously by a group of channels without mutual interference. The plan is the same as that used in multiplex telegraphy. The difference is that ordinary rotating machinery suffices at the relatively low speeds employed by the latter, while the high speeds needed for time division multiplex telephony can be realized only by practically inertialess electron streams. Also the widths of frequency band required for multiplex telephony are enormously greater than needed for the telegraph, and in fact have become technically feasible only with the development of wide-band radio and cable transmission systems. As far as any one channel is concerned the result is the same as in telegraphy, namely that signals are received at discrete or quantized times. In the limiting case when many channels are sent the speech voltage from one channel is practically constant during the brief switch closure and, in effect, we can send only one magnitude for each contact or quantum of time. The more familiar word "sampling" will be used here interchangeably with the rather formidable term "quantizing of time".

Quantizing the magnitude of speech signals is a fairly recent innovation. Here we do not permit a selection from a continuous range of magnitudes but only certain discrete ones. This means that the original speech signal is to be replaced by a wave constructed of quantized values selected on a minimum error basis from the discrete set available. Clearly if we assign the quantum values with sufficiently close spacing we may make the quantized wave indistinguishable by the ear from the original. The purpose of quantization of magnitudes is to suppress the effects of interference in the transmission medium. By the use of precise receiving instruments we can restore the received quanta without any effect from superposed interference provided the interference does not exceed half the difference between adjacent steps.

By combining quantization of magnitude and time, we make it possible to code the speech signals, since transmission now consists of sending one of a discrete set of magnitudes for each distinct time interval.[2,3,4,5,6,7] The maximum advantage over interference is obtained by expressing each discrete signal magnitude in binary notation in which the only symbols used are 0 and 1. The number which is written as 4 in decimal notation is then represented by 100, 8 by 1000, 16 by 10,000; etc. In general, if we have N digit positions in the binary system, we can construct 2^N different numbers. If we need no more than 2^N different discrete magnitudes for speech transmission, complete information can be sent by a sequence of N on-or-off pulses during each sampling interval. Actually a total of $2^N!$ different coding plans (sets of one-to-one correspondences between signal magnitudes and on-or-off sequences) is possible. The straightforward binary number system is taken as a representative example convenient for either theoretical discussion or practical instrumentation. We assume that absence of a pulse represents the symbol 0 and presence of a pulse represents the symbol 1. The receiver then need only distinguish between two conditions: no transmitted signal and full strength transmitted signal. By spacing the repeaters at intervals such that interference does not reach half the full strength signal at the receiver, we can transmit the signal an indefinitely great distance without any increment in distortion over that originally introduced by the quantizing itself. The latter can be made negligible by using a sufficient number of steps.

To determine the number of quantized steps required to transmit specific signals, we require a knowledge of the relation between distortion and step size. This problem is the subject of the present paper.* We divide the problem into two parts: (1) quantizing the magnitude only and (2) combined quantizing of magnitude and time. The first part can be treated by a simple model: the "staircase transducer", which is a device having the instantaneous ouput vs. input curve shown by Fig. 1. Signals impressed on the stair-

* Other features of the quantizing and coding theory are discussed in forthcoming papers by Messrs. C. E. Shannon, J. R. Pierce, and B. M. Oliver.

case transducer are sorted into voltage slices (the treads of the staircase), and all signals within plus or minus half a step of the midvalue of a slice are replaced in the output by the midvalue. The corresponding output when the input is a smoothly varying function of time is illustrated in Fig. 2. The output remains constant while the input signal remains within the boundaries of a tread and changes abruptly by one full step when the signal crosses the boundary. It is not within the scope of the present paper to discuss the internal mechanism of a staircase transducer, which may have many different physical embodiments. We are concerned rather with the distortion produced by such a device when operating perfectly.

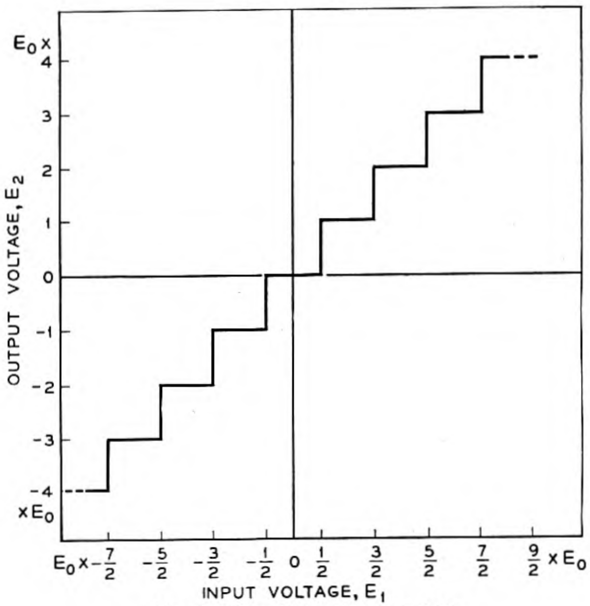

Fig. 1—Quantizing characteristic.

The distortion or error consists of the difference between the input and output signals. The maximum instantaneous value of distortion is half of one step, and the total range of variation is from minus half a step to plus half a step. The error as a function of input signal voltage is plotted in Fig. 3 and a typical variation with time is indicated in Fig. 2. If there is a large number of small steps, the error signal resembles a series of straight lines with varying slopes, but nearly always extending over the vertical interval between minus and plus half a step. The exceptional cases occur when the signal goes through a maximum or minimum within a step. The limiting condition of closely spaced steps enables us to derive quite simply an approximate value for the mean square error, which will later be shown to be sufficiently accurate in most cases of practical importance. This approximation consists of calculating the mean square value of a straight line going from minus half a step to plus half a step with arbitrary slope. If

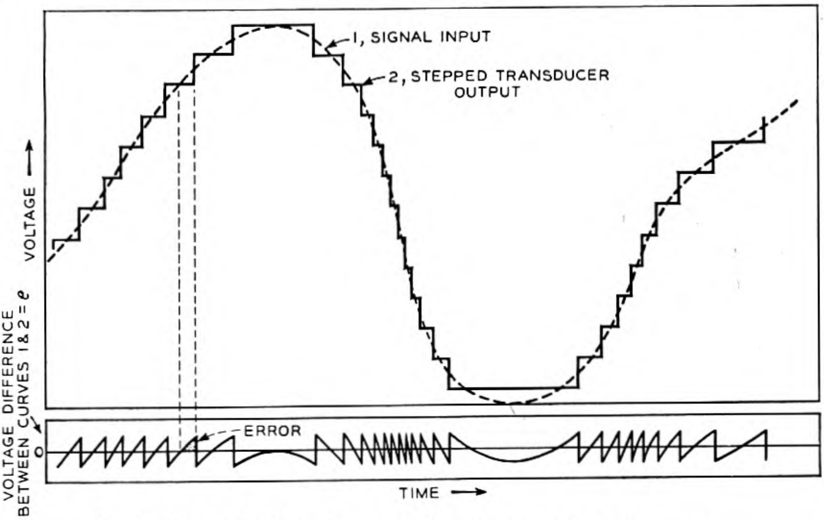

Fig. 2—A quantized signal wave and the corresponding error wave.

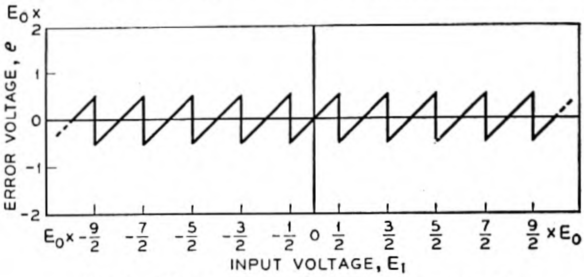

Fig. 3—Characteristic of the errors in quantizing.

E_0 is the voltage corresponding to one step, and s is the slope, the equation of the typical line is:

$$\epsilon = st, \qquad -\frac{E_0}{2s} < t < \frac{E_0}{2s} \tag{1.0}$$

where ϵ is the error voltage and t is the time referred to the midpoint as origin. Then the mean square error is

$$\overline{\epsilon^2} = \frac{s}{E_0} \int_{-E_0/2s}^{E_0/2s} \epsilon^2 \, dt = \frac{E_0^2}{12}, \tag{1.1}$$

or one twelfth the square of the step size.

Not all the distortion falls within the signal band. The distortion may be considered to result from a modulation process consisting of the application of the component frequencies of the original signal to the non-linear staircase characteristic. High order modulation products may have frequencies quite remote from those in the original signal and these can be excluded by a filter passing only the signal band. It becomes of importance, therefore, to calculate the spectrum of the error wave. This we shall do in the next section for a generalized signal using the method of correlation, which is based on the fact that the power spectrum of a wave is the Fourier cosine transform of the correlation function. The result is then applied to a particular kind of signal, namely one having energy uniformly distributed throughout a definite frequency band and with the phases of the components randomly distributed. This is a particularly convenient type of signal because it in effect averages over a large number of possible discrete frequency components within the band. Single or double-frequency signal waves are awkward for analytical purposes because of the ragged nature of the spectra produced. The amplitudes of particular harmonics or cross-products of discrete frequency components are found to oscillate violently with magnitude of input. The use of a large number of input components smooths out the irregularities.

The type of spectra obtained is shown in Fig. 4. Anticipating binary coding, we have shown results in terms of the number of binary digits used. The number of different magnitudes available are 16, 32, 64, 128, and 256 for $N = 4, 5, 6, 7$ and 8 digits, respectively. Here a word of explanation is needed with respect to the placing of the scale of quantized voltages. A signal with a continuous distribution of components along the frequency scale is theoretically capable of assuming indefinitely great values of instantaneous voltage at infrequent instants of time. An actual quantizer (staircase transducer) has a finite overload value which must not be exceeded and hence can have only a finite number of steps. This difficulty is resolved here by the experimentally observed fact that thermal noise, which has the type of spectrum we have assumed for our signal, has never been observed to exceed appreciably a voltage four times its root-mean-square value. Hence we have placed the root-mean-square value of the input signal at one-fourth the overload input to the staircase. This fixes the relation between step size and the total number of steps. In the actual calculation the number of steps is taken as infinite; the effect of the assumed additional steps beyond 2^N is negligible because of the rarity of excursion into this range.

The curves of Fig. 4 are drawn for the case in which the signal band starts at zero frequency. The original signal band width is represented by one unit on the horizontal scale. The relatively wide spread of the distortion spectrum is clearly shown. As the number of digits (or steps) is increased

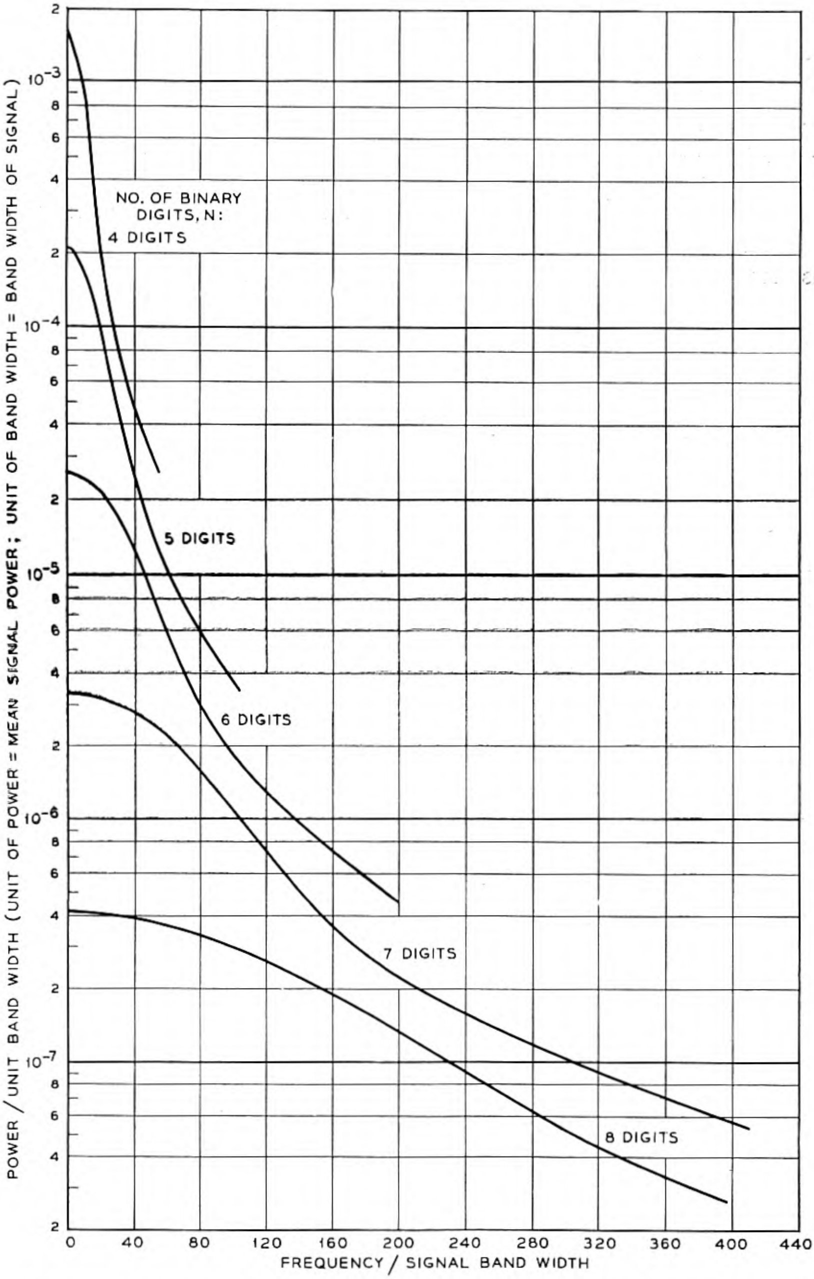

Fig. 4—Spectrum of distortion from quantizing the magnitudes of a random noise wave. Full load on the quantizer is reached by peaks 12 db above the r.m.s. value of input.

the spectrum becomes flatter over a wider range, but with a smaller maximum density. The area under each curve represents the total mean power in the corresponding error wave and is found to agree quite accurately with the approximate result of Eq. (1.1). The distortion power falling in the signal band is represented by the area included under the curve from zero to unit abscissa.

Quantizing the magnitude only is not a technically attractive method of transmission because of the wide frequency band required to preserve the discrete values of the quanta. Thus in a 128-step system, a full load sinusoidal signal passes through 64 different steps each quarter cycle and hence would require transmitting 256 successively different magnitudes during each period of the signal frequency. We therefore consider the second problem—that of sampling the quantized magnitudes.

The theory of periodic sampling of signals is a limiting case of commutator modulation theory as previously shown by the author.[1] We may think of a periodically closed switch in series with the line and source as producing a multiplication of the signal by a switching function. The switching function has a finite value during the time of switch closure and is zero at other times. It may be expanded in a Fourier series containing a term of zero frequency, the repetition frequency of switch closure, and all harmonics of the latter. Multiplication of the signal by the Fourier series representing the constant component of the switching function gives a term proportional to the signal itself. Multiplication of the signal by the fundamental component of the switching function gives upper and lower sidebands on the repetition frequency. Likewise multiplication by the harmonics gives sidebands on each harmonic. The signal is separable from the sidebands on a frequency basis if the signal band does not overlap the lower sideband on the repetition frequency. This leads to the condition for no distortion in time division: the highest signal frequency must be less than one-half the repetition frequency.

To apply the above theory to instantaneous sampling we let the duration of switch closure in one period approach zero. We then approach the condition of one signal value in each period, so that the repetition frequency now becomes the sampling frequency. Clearly the sampling frequency must slightly exceed twice the highest signal frequency. We also note that as the contact time tends toward zero, the switching function approaches a periodically repeated impulse. The important terms of the Fourier series representing the switching function accordingly become a set of harmonics of equal amplitude with a constant component equal to half the amplitude of the typical harmonic. On multiplication of this series by the signal, we get a set of sidebands of equal amplitude including the one corresponding to the original signal itself, the sideband on zero frequency.

These results may be applied to the staircase transducer. The output may be resolved into the input signal plus the error. The sampling frequency is assumed to exceed its minimum required value of twice the top signal frequency. The component of the output that is equal to the original signal can therefore be separated at the receiver by a filter passing the original signal band. A similar statement cannot be made for the error component, for it has been found to extend over a vastly greater range than the original signal. To calculate the total distortion received in the signal band, we can multiply the distortion spectrum by the switching function and sum up all sideband contributions to the original signal band. Each har-

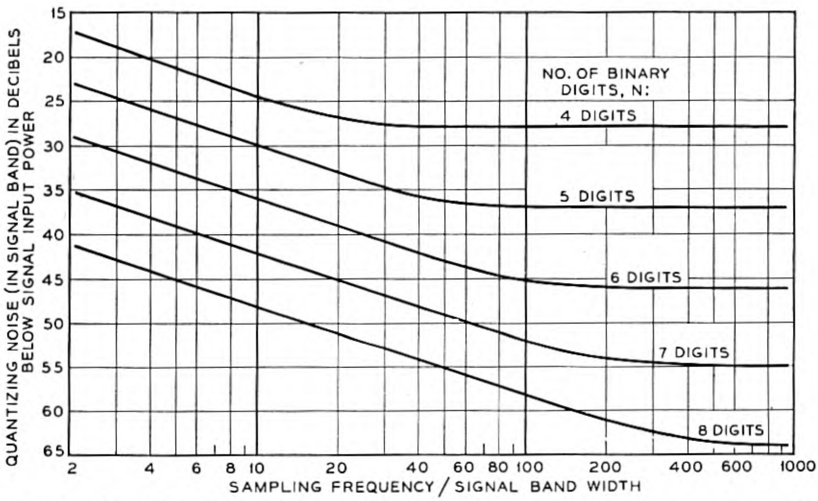

Fig. 5—Total distortion in signal band from quantizing and sampling a random noise wave. Full load on the quantizer is 12 db above the r.m.s. value of input.

monic of the switching function makes such contributions by beating with a band of the error spectrum above and below the frequency of the harmonic. These contributions add as power when the sampling frequency is independent of the individual frequencies contained in the signal. The total error power accepted by the signal band filter decreases as the sampling frequency is increased because each harmonic of the sampling frequency is thereby pushed upward into a less dense portion of the error spectrum. In the limit as the sampling frequency is made indefinitely large, we return to the non-sampled case, that of the staircase transducer only.

Figure 5 shows the calculated curves of distortion in the signal band plotted as a function of ratio of sampling frequency to signal band width. The curves have downward slopes approaching asymptotes corresponding to the area from zero to unity under the corresponding curves of Fig. 4.

The initial points at the minimum sampling rate are determined on the other hand by the total area under the curves of Fig. 4, since the accepted sidebands on the harmonics in this case exactly fill out the entire error spectrum. These initial points are therefore given quite accurately by Eq. (1.1), which, as pointed out before, is a good approximation for the total areas. We can also give a direct demonstration of the applicability of Eq. (1.1) to the initial points of the curves of Fig. 5 by means of the following theorem:

Theorem I. The mean square value of the response of an ideal low-pass filter to a train of unit impulses multiplied by instantaneous samples occurring at double the cutoff frequency is equal to the mean square value of the samples provided no harmonic of the sampling frequency is equal to twice the frequency of one component or equal to the sum or difference of two component frequencies of the sampled signal. Proof of the theorem is given in Appendix I. To apply it here we resolve the input into two components: the true signal and the error. The former is reproduced with fidelity in the output because it contains only frequencies below half the sampling rate. The error component in the output represents the response of the low-pass filter to the error samples. Except for very special types of signals, the error samples are uniformly distributed throughout the range from minus half a step to plus half a step. Calculation of the mean square value of such a distribution gives Eq. (1.1).

We have tacitly assumed above that the sampled values applied to the filter in the output of the system are infinitesimally narrow pulses of height proportional to the samples. In actual systems it is found advantageous to hold the sampled values constant in the individual receiving channels until the next sample is received. This means that the input to the channel filter is a succession of rectangular pulses of heights proportional to the samples. The resulting magnitude of recovered signal is much larger than would be obtained if very short pulses of the same heights were used; stretching the pulses in time produces in effect an amplification. The amplification is obtained, however, at the expense of a variation of channel transmission with signal frequency. Infinitesimally short pulses have a flat frequency spectrum, while pulses of finite duration do not. The frequency characteristic introduced by lengthening the pulses is easily calculated by determining the steady state admittance function of a network which converts impulses to the actual pulses used. The general formula for this admittance when a unit impulse input is converted into an output pulse $g(t)$ is easily shown to be:

$$Y(i\omega) = f_s \int_{-\infty}^{\infty} g(t)e^{-i\omega t}\, dt \tag{1.2}$$

where f_s is the repetition frequency and ω is the angular signal frequency.

We shall call this Theorem II and give the proof in Appendix II. This relation is similar to that found in television and telephotography for the "aperture effect", or variation of transmission with frequency caused by the finite size of the scanning aperture. The pulse shape $g(t)$ is analogous to a variation in aperture height $g(x)$, where x is distance along the line of scanning. Hence it has become customary to use the term "aperture effect" in the theory of restoring signals from samples. The aperture effect associated with rectangular pulses lasting from one sample to the next amounts to an amplitude reduction of $\pi/2$ or 3.9 db at the top signal frequency (one half the sampling rate) compared to a signal of zero frequency. There is also a constant delay introduced equal to half the sampling period. The latter does not cause any distortion and the amplitude effect can be corrected by properly designed equalizing networks.

The fact that many pulse spectra can be simply expressed in terms of a flat spectrum associated with sharp pulses and an aperture effect caused by the particular shape of pulse used does not appear to have been recognized in the recent literature, although applications were made by Nyquist in a fundamental paper[8] of 1928. Premature introduction of a specific finite pulse not only complicates the work, but also restricts the generality of the results.

Distortion caused by quantizing errors produces much the same sort of effects as an independent source of noise. The reason for this is that the spectrum of the distortion in the receiving filter output is practically independent of that of the signal over a wide range of signal magnitudes. Even when the signal is weak so that only a few quantizing steps are operated, there is usually enough residual noise on actual systems to determine the quantizing noise and mask the relation between it and the signal. Eq. (1.1) yields a simple rule enabling one to estimate the magnitude of the quantizing noise with respect to a full load sine wave test tone. Let the full load test tone have peak voltage E; its mean square value is then $E^2/2$. The total range of the quantizer must be $2E$ because the test signal swings between $-E$ and $+E$. The ratio $2E/E_0 = r$ is a convenient one to use in specifying the quantizing; it is the ratio of the total voltage range to the range occupied by one step. The ratio of mean square signal to mean square quantizing noise voltage is

$$\frac{E^2/2}{E_0^2/12} = \frac{6E^2}{4E^2/r^2} = \frac{3r^2}{2} \tag{1.3}$$

Actual systems fail to reproduce the full band $f_s/2$ because of the finite frequency range needed for transition from pass-band to cutoff. If we introduce a factor κ to represent the ratio of equivalent rectangular noise band to $f_s/2$, the actual received noise power is multiplied by κ. Then the signal-to-noise ratio in db for a full load test tone is

$$D = 10 \log_{10} \frac{3r^2}{2\kappa} \text{ db} \tag{1.4}$$

In practical applications the value of κ is about 3/4 which gives the convenient rule:

$$D = 20 \log_{10} r + 3 \text{ db} \tag{1.5}$$

In other words, we add 3 db to the ratio expressed in db of peak-to-peak quantizing range to the range occupied by one step. For various numbers of binary digits the values of D are:

TABLE I

Number of Digits	D
3	21
4	27
5	33
6	39
7	45
8	51

From Table I we can make a quick estimate of the number of digits required for a particular signal transmission system provided that we have some idea of the required signal-to-noise ratio for a full load test tone. The latter ratio may be expressed in terms of the full load test tone which the system is required to handle and the maximum permissible unweighted noise power at the same level point. Since quantizing noise is uniformly distributed throughout the signal band, its interfering effect on speech or other program material is probably similar to that of thermal noise with the same mean power. Requirements given in terms of noise meter readings must be corrected by the proper weighting factor before applying the table. If the signal transmitted is itself a multiplex signal with channels allotted on a frequency division basis, the noise power falling in each channel is the same fraction of the total noise power as the band width occupied by the signal is of the total band width of the system.

We have thus far considered only the case in which the quantized steps are equal. In actual systems designed for transmission of speech it is found advantageous to taper the steps in such a way that finer divisions are available for weak signals. For a given number of total steps this means that coarser quantization applies near the peaks of large signals, but the larger absolute errors are tolerable here because they are small relative to the bigger signal values. Tapered quantizing is equivalent to inserting complementary nonlinear transducers in the signal branch before and after the quantizer. In the usual case, the transducer ahead of the quantizer is of the "compressing" type in which the loss increases as the signal increases. If the full load signal just covers all the linear quantizing steps, a weak signal gets a bigger share of the steps than it would if the transducer were linear. The transducer after the quantizer must be of the "expanding" type which gives decreased loss to the large signals to make the overall combination linear.

On the basis of the theory so far discussed, we can say that the error spectrum out of the linear quantizer is virtually the same whether or not the signal input is compressed. The operation of the expandor then magnifies the errors produced when the signal is large. When weak signals are applied, the mean square error is given by Eq. (1.1), as before, but when the signal is increased an increment in noise occurs. The mean square value of noise voltage under load may be computed from the probability density of the signal values and the output-vs-input characteristic of the expandor, or its inverse, the compressor. A first order approximation, valid when the steps are not too far apart, replaces (1.1) by:

$$\overline{\epsilon^2} = \frac{E_0^2}{12} \int_{Q_2}^{Q_1} \frac{p_1(E_1)\, dE_1}{[F'(E_1)]^2} \tag{1.6}$$

where Q_1 and Q_2 are the minimum and maximum values of the input signal voltage E_1, $p_1(E_1)$ is the probability density function of the input voltage, and $F'(E_1)$ is the slope of $F(E_1)$, the compression characteristic.

Some experimental results obtained with a laboratory model of a quantizer are given in Figs. 6–9. Figs. 6–7 show measurements on the third harmonic associated with 6-digit quantizing. As mentioned before, the amplitude of any one harmonic oscillates with load. The calculated curves shown were obtained by straightforward Fourier analysis. In the measurements it was convenient to spot only the successive nulls and peaks.

In Fig. 6 the bias was set to correspond to the stair-case curve of Fig. 1, while in Fig. 7 the origin is moved to the point $(E_0/2, E_0/2)$, i.e., to the middle of a riser instead of a tread. The peaks of ratio of harmonic to fundamental decrease steadily as the amplitude of the signal is increased to full load, which is just opposite to the usual behavior of a communication system. It is difficult to extrapolate experience with other systems to specify quality in terms of this type of harmonic distortion.

Figure 8 shows measurements of the total distortion power falling in the signal band when the signal is itself a flat band of thermal noise. The technique of making such measurements has been described in earlier articles.[9,10] Measurements are shown for quantizing with both equal and tapered steps. The particular taper used is indicated by the expandor characteristic of Fig. 9. The compression curve is found by interchanging horizontal and vertical scales. The measurements were made on a quantizer with 32, 64, and 128 steps, and a sampling rate of 8,000 cycles per sec-

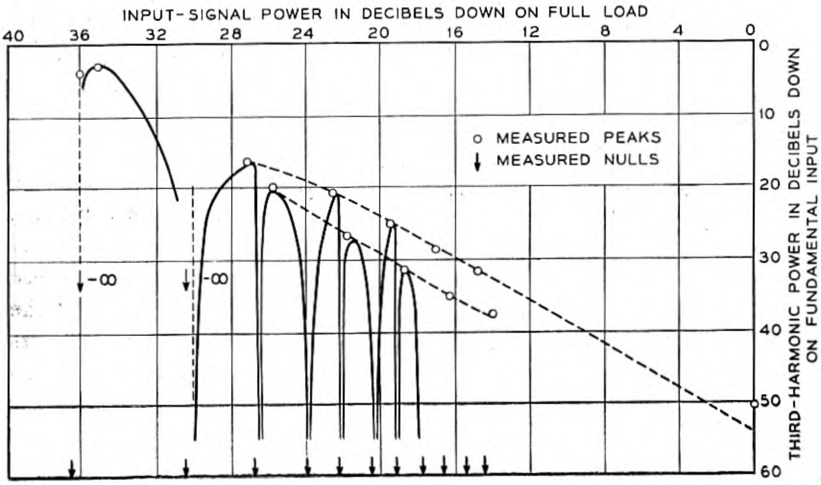

Fig. 6—Third harmonic in 64-step quantized output with bias at mid tread. The smooth curves represent computed values.

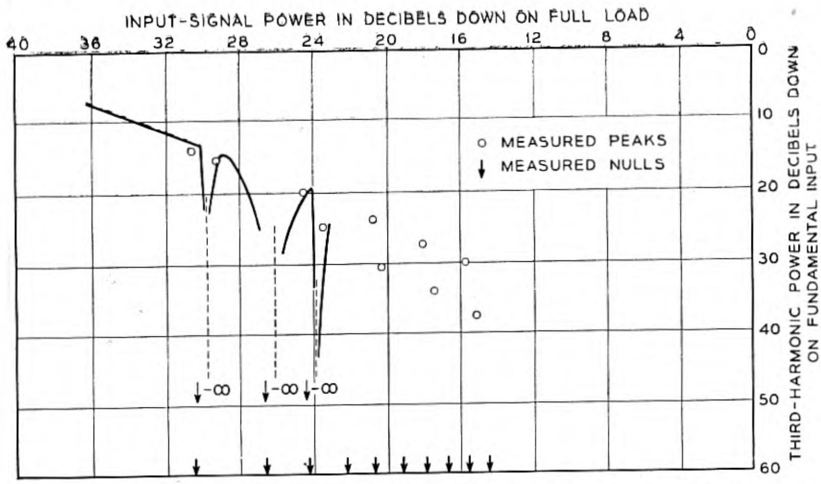

Fig. 7—Third harmonic in 64-step quantized output with bias at mid-riser. The smooth curves represent computed values.

ond. The applied signal was confined to a range below 4,000 cycles per second. With equal steps the distortion power is practically independent of load as shown by the db-for-db straight lines. With tapered steps, the distortion is less for weak signals, and only slightly greater for large signals.

The vertical line designated "full load random noise input" represents the value of noise signal power at which peaks begin to exceed the quantizing

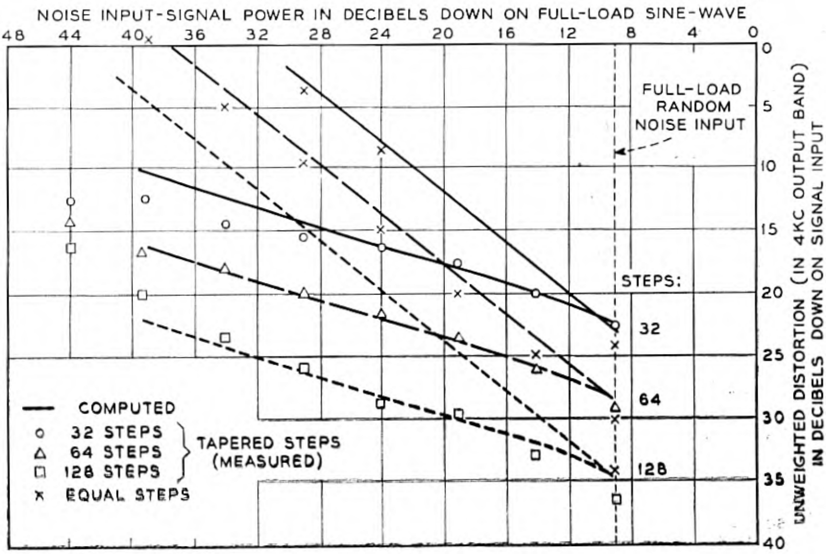

Fig. 8—Total distortion in signal band from quantizing with equal and tapered steps.

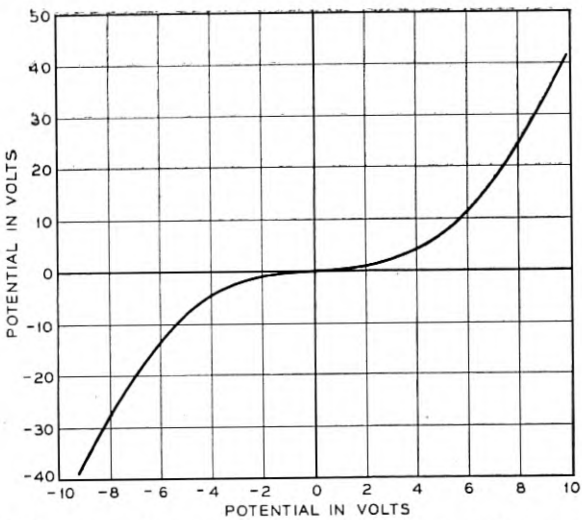

Fig. 9—Expanding characteristic applied to noise in tapered steps of Fig. (8).

range. This occurs when the rms value of input is 9 db below the rms value of the sine wave which fully loads the quantizer.

Flatness of the distortion spectrum with frequency within the signal band is demonstrated by Fig. 10. Two kinds of input were used here—a flat band of thermal noise and a set of 16 sine waves with frequencies distributed throughout the band. Results in the two cases were practically the same. The theoretical levels of distortion power for the band widths of the measuring filters (95 cps) are shown by the horizontal lines.

In the experimental results given here use has been made of laboratory studies by Messrs. A. E. Johanson, W. A. Klute, and L. A. Meacham.

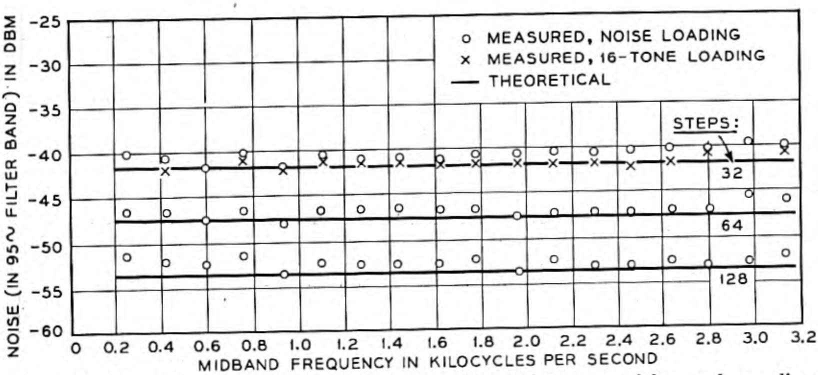

Fig. 10—Spectral density of distortion in signal band from quantizing and sampling. The quantizing steps were equal and the quantizer was fully loaded by a random noise or 16-tone input signal with mean power = −2.5 dbm.

2. Theoretical Analysis

The correlation theorem discovered by N. Wiener[11] may be stated as follows: Let ψ_τ represent the average value of the product $I(t)I(t + \tau)$, where $I(t)$ is the value of a variable such as current or voltage at time t, and $I(t + \tau)$ is the value at a time τ seconds later. Mathematically:

$$\psi_\tau = \overline{I(t)I(t + \tau)} = \lim_{T \to \infty} \frac{1}{T} \int_0^T I(t)I(t + \tau)\, dt \qquad (2.0)$$

From analogy with statistical theory, ψ_τ is called the correlation of $I(t)$ with itself, or the autocorrelation function of the signal. Since we shall not deal here with the correlation of two signals, we shall shorten our terms and call ψ_τ simply the correlation of $I(t)$. Let $w_f\, df$ represent the mean power in the output of an ideal bandpass filter of width df centered at f. We assume that the ideal filter is designed to work between resistances of one ohm each and that the input signal $I(t)$ is delivered to the filter from a source with internal resistance of one ohm. (The use of unit resistances does not restrict the generality of the results, since equivalent transmission performance of any linear electrical circuit is obtained by multiplying all impedances by a constant factor. All voltages are multiplied and all currents divided by the same factor. By assuming unit values of resistance we are able to use

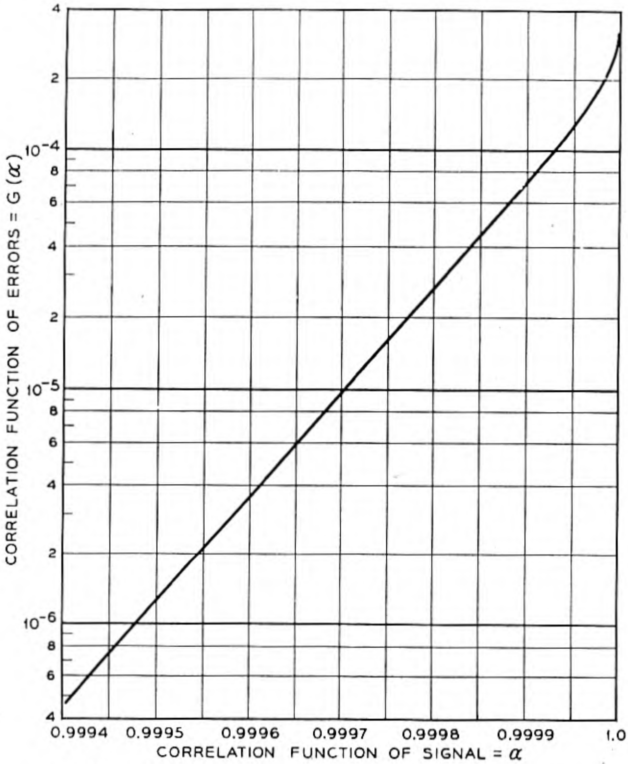

Fig. 11—Correlation function of 7-digit quantizing errors.

squared values of voltages and currents to represent power.) The theorem states that w_f and ψ are related by the equation:

$$w_f = 4 \int_0^\infty \psi_\tau \cos 2\pi f\tau \, d\tau \tag{2.1}$$

Proof may be found in the references cited. When the signal contains periodic components, the integral in (2.1) becomes divergent in the ordinary or Riemann sense, but this difficulty may be overcome by either applying the theory of divergent integrals or replacing Riemann by Stieltjes integration. We shall not require these modifications here because we shall base our analysis on signals with a continuous spectrum. We note that ψ_0 is the mean square value of the signal itself. We also point out that the inversion formula for the Fourier integral enables us to express ψ_τ in terms of w_f, thus:

$$\psi_\tau = \int_0^\infty w_f \cos 2\pi\tau f \, df \tag{2.2}$$

It also may be shown that the ratio ψ_τ/ψ_0 cannot have values outside the interval from -1 to $+1$.

The correlation theorem furnishes a powerful analytical tool for the solution of modulation problems because the calculation of the average ψ_τ is often a straightforward process, while direct calculation of w_f may be a very devious one. Once ψ_τ has been obtained, Eq. (2.1) brings the highly developed theory of Fourier integrals to bear on the computation of w_f.

We shall give the derivation of w_f for quantizing noise making use of the correlation function. In the analysis we shall apply a number of other needed theorems with appropriate references given for proof.

Our first problem is that of calculating the spectrum of the output of the staircase transducer, Fig. 1, when the spectrum of the input signal is given. Let w_f represent the power spectrum of the input signal and ψ_τ the autocorrelation function. The two quantities are related by (2.1) and it is sufficient to express our results in terms of either one. If the instantaneous value of the input signal is represented by E_1, and that of the output by E_2, the staircase function may be defined mathematically by:

$$E_2 = mE_0, \qquad \frac{2m-1}{2} E_0 < E_1 < \frac{2m+1}{2} E_0, \qquad m = 0, \pm 1, \pm 2, \cdots \tag{2.3}$$

The error is the difference between E_1 and E_2 and may be written as

$$\epsilon(t) = E_1 - E_2 = E_1 - mE_0, \quad \frac{2m-1}{2} E_0 < E_1 < \frac{2m-1}{2} E_0 \tag{2.4}$$

The error characteristic is plotted in Fig. 3.

One approach depends on a knowledge of the probability density function $p(V_1, V_2)$ of the variables $V_1 = E_1$ at time t and $V_2 = E_2$ at time $t + \tau$. The definition of this function is that $p(V_1, V_2)\, dV_1\, dV_2$ is the probability that V_1 and V_2 lie in a rectangle of dimensions dV_1 and dV_2 centered on the point V_1, V_2 of the V_1V_2-plane. The function $p(V_1, V_2)$ has been calculated for certain types of signals and in theory could be computed for any signal by standard methods. If it is assumed known, we may determine the correlation function of the error. Let

$$F(V_1, V_2) = \epsilon(t)\epsilon(t + \tau) = (V_1 - mE_0)(V_2 - nE_0),$$

$$\frac{2m - 1}{2} E_0 < V_1 < \frac{2m + 1}{2} E_0, \quad \frac{2n - 1}{2} E_0 < V_2 < \frac{2n + 1}{2} E_0, \quad (2.5)$$

$$m, n = 0, \pm 1, \pm 2, \cdots$$

Eq. (2.5) defines $F(V_1, V_2)$ as a definite constant value in each square of width E_0 in the V_1V_2-plane. By elementary statistical theory, the correlation function ξ_τ of the error wave is now

$$\xi_\tau = \overline{F(V_1, V_2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(V_1, V_2)p(V_1, V_2)\, dV_1\, dV_2 \quad (2.6)$$

The correlation may therefore be calculated since F and p are known functions. The power spectrum Ω_f of the error wave is then equal to the right-hand member of (2.1) with ξ_τ substituted for ψ_τ.

We are interested in the case in which the signal voltage has a smoothly varying spectrum over a specified band. This is a property of a random noise function which has a normal distribution of instantaneous voltages. The two-dimensional probability density function of such a wave is known[12]. It is

$$p(V_1, V_2) = \frac{1}{2\pi\sqrt{\psi_0^2 - \psi_\tau^2}} \exp\left[\frac{\psi_0(V_1^2 + V_2^2) - 2\psi_\tau V_1 V_2}{2(\psi_0^2 - \psi_\tau^2)}\right]. \quad (2.7)$$

By inserting this value and that of $F(V_1, V_2)$ from (2.5) in (2.6), making the change of variable:

$$\left.\begin{aligned} V_1 - mE_0 &= E_0x/2 \\ V_2 - nE_0 &= E_0y/2 \end{aligned}\right\} \quad (2.8)$$

and adopting the notation,

$$k = E_0^2/\psi_0, \ \alpha = \psi_\tau/\psi_0, \ G(\alpha) = \xi_\tau/\psi_0, \quad (2.9)$$

we obtain the following integral determining ξ_τ,

$$G(\alpha) = \frac{k^2}{32\pi(1 - \alpha^2)^{1/2}} \cdot \int_{-1}^{1}\int_{-1}^{1} xyH(x, y) \exp \frac{-k(x^2 + y^2 - 2\alpha xy)}{8(1 - \alpha^2)}\, dx\, dy \quad (2.10)$$

$$H(x, y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \cdot \exp \frac{-k[m^2 + m(x - \alpha y) + n^2 + n(y - \alpha x) - 2\alpha mn]}{2(1 - \alpha^2)} \quad (2.11)$$

The power density spectrum of the errors is, from (2.1),

$$\begin{aligned}\Omega_f &= 4\int_0^\infty \xi_\tau \cos 2\pi f\tau\, d\tau \\ &= 4\psi_0 \int_0^\infty G(\alpha)\cos 2\pi f\tau\, d\tau \qquad (2.12)\end{aligned}$$

If the signal band is flat from $f = 0$ to $f = f_0$, with no energy outside this band,

$$\alpha = \frac{1}{f_0}\int_0^{f_0} \cos 2\pi\tau f\, df = \frac{\sin 2\pi f_0 \tau}{2\pi f_0 \tau} \qquad (2.13)$$

Letting $\gamma = f/f_0$,

$$\Omega_0(\gamma) = \frac{f_0\,\Omega_f}{\psi_0} = \frac{2}{\pi}\int_0^\infty G\left(\frac{\sin z}{z}\right)\cos \gamma z\, dz, \qquad (2.14)$$

To complete the calculation, we must evaluate the integral (2.10). The first step is to transform the double summation (2.11) into products of single sums by the change of indices:

$$\begin{pmatrix} m + n = m' \\ m - n = n' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} m = \dfrac{m' + n'}{2} \\ n = \dfrac{m' - n'}{2} \end{pmatrix} \qquad (2.15)$$

The rearrangement is permissible because the double series is absolutely convergent. The new indices m' and n' also run from minus to plus infinity, but must be either both even or both odd because $m' \pm n'$ is even. On dropping the primes after the substitution is completed, we find

$$\begin{aligned} H(x, y) = &\sum_{m=-\infty}^{\infty} \exp \frac{-k[2m(x + y) + 4m^2]}{4(1 + \alpha)} \sum_{n=-\infty}^{\infty} \\ &\cdot \exp \frac{-k[2n(x - y) + 4n^2]}{4(1 - \alpha)} + \sum_{m=-\infty}^{\infty} \\ &\cdot \exp \frac{-k[(2m + 1)(x + y) + (2m + 1)^2]}{4(1 + \alpha)} \sum_{n=-\infty}^{\infty} \qquad (2.16) \\ &\cdot \exp \frac{-k[2n + 1)(x - y) + (2n + 1)^2]}{4(1 - \alpha)} \end{aligned}$$

A further simplification results from a change of the variables of integration to eliminate the terms in xy. This is done by setting

$$\begin{pmatrix} x = u + v \\ y = u - v \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u = (x+y)/2 \\ v = (x-y)/2 \end{pmatrix} \qquad (2.17)$$

By calculating the Jacobian of the transformation, we find $dx\,dy = 2\,du\,dv$. The region of integration in the uv-plane is a rhombus bounded by the lines $u \pm v = \pm 1$. We then have:

$$G(\alpha) = \frac{k^2}{16\pi(1-\alpha^2)^{1/2}}\left[\int_{-1}^{0}\int_{-1-u}^{1+u} dv + \int_0^1 du \int_{u-1}^{1-u} dv\right](u^2 - v^2)$$
$$\exp\left[-\frac{k}{4}\left(\frac{u^2}{1+\alpha} + \frac{v^2}{1-\alpha}\right]\sum_{m=-\infty}^{\infty}\exp\frac{-2mk(2u+2m)}{4(1+\alpha)}$$
$$\sum_{n=-\infty}^{\infty}\exp\frac{-2nk(2v+2n}{4(1-\alpha)} + \sum_{m=-\infty}^{\infty}\exp\frac{-(2m+1)k(2u+2m+1)}{4(1+\alpha)}$$
$$\sum_{n=-\infty}^{\infty}\exp\frac{-(2n+1)k(2v+2n+1)}{4(1-\alpha)} \qquad (2.18)$$

If we substitute $u = -x$ in the first double integral, $m = -m'$ in the first series, and $m = -m' - 1$ in the third series, we see that the two double integrals are equal. We therefore drop the first double integral and multiply the second by two. The inner integral may then be split into parts with limits from $v = 0$ to $v = 1 - u$ and $v = u - 1$ to $v = 0$. Substituting $v = -y$ in the second part and treating the series as before, we find that the two parts give equal contributions, so that the bracketed integral terms become

$$4\int_0^1 du \int_0^{1-u} dv$$

applied to the integrand.

The series in (2.18) may be written as Theta Functions, and the imaginary transformation of Jacobi then used as an aid in reduction. We may proceed in a more direct manner, however, by applying Poisson's Summation Formula:[13]

$$\sum_{n=-\infty}^{\infty}\varphi(2\pi n) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\tau)e^{-im\tau}\,d\tau \qquad (2.19)$$

We thereby show that

$$\sum_{m=-\infty}^{\infty}\exp\left[-am(x+2m)\right] =$$
$$\sqrt{\frac{\pi}{2a}}\,e^{ax^2/8}\left[1 + 2\sum_{m=1}^{\infty}e^{-m^2/\pi^2 2a}\cos\frac{m\pi x}{2}\right] \qquad (2.20)$$

$$\sum_{m=-\infty}^{\infty}\exp\left[-a(2m+1)(x+2m+1)\right]$$
$$= \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{ax^2/4}\left[1 + 2\sum_{m=1}^{\infty}(-)^m e^{-m^2\pi^2/4a}\cos\frac{m\pi x}{2}\right] \qquad (2.21)$$

When the series in (2.18) of type corresponding to the left-hand members of (2.20) and (2.21) are replaced by the equivalent righthand members, positive exponents containing the squared variables of integration are introduced which cancel the negative exponents already present in the integrand. The resulting integral may be written:

$$G(\alpha) = \frac{k}{4}\int_0^1 du \int_0^{1-u} (u^2 - v^2)[f_1(1+\alpha, u)f_1(1-\alpha, v) + f_2(1+\alpha, u)f_2(1-\alpha, v)]\, dv, \quad (2.22)$$

where

$$f_1(a, x) = 1 + 2\sum_{m=1}^{\infty} \exp\frac{-m^2\pi^2 a}{2k}\cos\frac{m\pi x}{2} \quad (2.23)$$

$$f_2(a, x) = 1 + 2\sum_{m=1}^{\infty} (-)^m \exp\frac{-m^2\pi^2 a}{2k}\cos\frac{m\pi x}{2} \quad (2.24)$$

The integrations may now be performed without difficulty. The complete result, which as we shall immediately show is hardly ever necessary to use in full is:

$$\begin{aligned} G(\alpha) = {} & \frac{k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-\frac{4n^2\pi^2}{k}\right)\sinh\frac{4n^2\pi^2\alpha}{k} \\ & + \frac{k}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m \neq n)\frac{1}{(m^2-n^2)}\exp\frac{-4(m^2+n^2)\pi^2}{k} \\ & \sinh\frac{4(m^2-n^2)\pi^2\alpha}{k} - \frac{k}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m \neq n)\frac{1}{(m-\frac{1}{2})^2-(n-\frac{1}{2})^2} \\ & \exp\frac{-4[(m-\frac{1}{2})^2+(n-\frac{1}{2})^2]\pi^2}{k}\sinh\frac{4[(m-\frac{1}{2})^2-(n-\frac{1^2}{2})]\pi^2\alpha}{k} \end{aligned} \quad (2.25)$$

An alternative derivation of (2.25), subsequently suggested by Mr. S. O. Rice, is based on the fact that $\epsilon(t)$ as defined by (2.4) or Fig. 3 is a periodic function of E_1 which can be expanded in a Fourier series with period E_0. Substituting the series in (2.5) leads to an expression for $\epsilon(t)\ \epsilon\ (t+\tau)$ as the product of two Fourier series. After proof that it is permissible to write this product as a double series and to calculate the average sum as the sum of the averages of the individual terms the problem is reduced to a double series in which the typical term is proportional to the average value of exp $i(uV_1 + vV_2)$ where u and v are constants depending on the position of the term in the series. Rice has shown[12] that the average value of such a term is $\exp[-(u^2 + v^2)\psi_0/2 - uv\psi_\tau]$. Summation of these terms leads again to (2.25).

From the defining equation (2.9) we note that k is a small quantity when more than a very few steps are used in the quantizer so that exponentials with exponent containing the factor $-1/k$ are very small except when the factor is multiplied by a number near zero. It will be seen that this can only happen in the first series and then only when α approaches the value unity. We recall that α lies in the range -1 to $+1$ and it is apparent from (2.25) that $G(\alpha)$ is an odd function of α. We thus need consider only positive values of α very slightly less than unity. Only the component of the $\sinh$ with positive exponent is then significant, and we write the very accurate approximation for $G(\alpha)$:

$$G(\alpha) \doteq \frac{k}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \frac{-4n^2\pi^2(1-\alpha)}{k} \tag{2.26}$$

A typical curve of $G(\alpha)$ vs. α for a fixed value of k is shown in Fig. 11. The rapidity with which it falls away at the left of the point $\alpha = 1$ is such that the curve can only be plotted by greatly expanding the scale of α in this region. The physical significance of the spike-shaped curve is that $G(\alpha)$ is a measure of the correlation of the errors as a function of the correlation of the applied signal. When there are many steps there is virtually no correlation between errors in successive samples except when there is complete correlation of successive signal values.

Use of the approximation (2.26) enables us to derive a convenient formula for the spectral density of the errors in a flat band input signal. Substituting (2.26) in (2.14) we obtain:

$$\Omega_0(\gamma) \doteq \frac{k}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} \exp\left[\frac{-4n^2\pi^2}{k}\left(1 - \frac{\sin z}{z}\right)\right] \cos \gamma z \, dz \tag{2.27}$$

The integrand is negligible except when z is near zero, and in this region we may replace $(\sin z)/z$ by the first two terms of its power series expansion. We then find

$$\begin{aligned} \Omega_0(\gamma) &\doteq \frac{k}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} \exp\left(\frac{-2n^2\pi^2 z^2}{3k}\right) \cos \gamma z \, dz \\ &= \frac{k}{2\pi^3} \sqrt{\frac{3k}{2\pi}} \sum_{n=1}^{\infty} \frac{1}{n^3} \exp\left(\frac{-3k\gamma^2}{8n^2\pi^2}\right). \end{aligned} \tag{2.28}$$

Only one set of calculations from the infinite series need be made since we may define a function of one variable

$$B(z) = \sum_{n=1}^{\infty} \frac{e^{-z/n^2}}{n^3}. \tag{2.29}$$

Then

$$\Omega_0(\gamma) \doteq \frac{k}{2\pi^3} \sqrt{\frac{3k}{2\pi}}\, B\left(\frac{3k\gamma^2}{8\pi^2}\right). \tag{2.30}$$

The curves of Fig. (4) were obtained in this way. The relation between k and the number of digits N is based on the assumption of the rms value of signal reaching one-fourth the instantaneous overload voltage of the quantizer. Since zero signal voltage is in the middle of the quantizing range $2^N E_0$, the overload signal measured from zero is $2^{N-1}E_0$. The mean square signal input is ψ_0. Therefore

$$2^{N-1} E_0 = 4\sqrt{\psi_0} \tag{2.31}$$

or from (2.9)

$$k = 1/4^{N-3} \tag{2.32}$$

We thus have obtained the spectrum of the quantizing errors without sampling. To apply our results to the sampling case we sum up all contributions from each harmonic of the sampling rate beating with the noise spectrum from quantizing only. The resulting power spectrum is given by

$$A_f = \Omega_f + \sum_{n=1}^{\infty} (\Omega_{nf_s-f} + \Omega_{nf_s+f}), \qquad 0 \leq f \leq f_s/2. \tag{2.33}$$

If y is the ratio of sampling frequency to signal band width and $A_0(y)$ is the ratio of quantizing power received in the signal band to the applied signal power,

$$A_0(y) = \Omega_0(1) + \sum_{n=1}^{\infty} [\Omega_0(ny + 1) + \Omega_0(ny - 1)]. \tag{2.34}$$

This is the equation used in calculating the curves of Fig. (5).

APPENDIX I

Relation Between Mean Squares of Signal and Its Samples

We have already shown that there is a unique relationship between a signal occupying the band of all frequencies less than f_c, and the sampled values of the signal taken at a rate $f_s = 2f_c$. If we are given the signal wave, we can obviously determine the samples; and if we are given the samples, we can determine the signal wave since it is the response of an ideal low-pass filter of cutoff frequency f_c to unit impulses multiplied by the samples. If we apply samples of a signal containing components of frequency greater than f_c, the output of the filter is a new signal with frequencies confined to the band from zero to f_c and yielding the same sampled values as the original wideband signal.

We now consider the problem of determining the mean square value of the samples of an arbitrary function $f(t)$. Let the samples be taken at $t = nT$, $n = 0, \pm 1, \pm 2, \cdots$, where $T = 1/2f_c = 1/f_s$.

We may write an expression for the squared samples as a limit of the product of the squared signal and a periodic switching function of infinitesimal contact time, thus

$$f^2(nt_0) = \lim_{\tau \to 0} f^2(t) S(\tau, t) \tag{I—1}$$

where:

$$S(\tau, t) = \begin{pmatrix} 1, & -\tau/2 < t < \tau/2 \\ 0, & \tau/2 < t < T - \tau/2 \end{pmatrix} \tag{I—2}$$

$$S(\tau, t + T) = S(\tau, t), n = 0, \pm 1, \pm 2, \cdots \tag{I—3}$$

By straightforward Fourier series expansion:

$$S(\tau, t) = \frac{\tau}{T} + \sum_{m=1}^{\infty} \frac{2 \sin m\pi\tau/T}{m\pi} \cos 2m\pi f_s t. \tag{I—4}$$

The mean square value of the samples is the limit of the average value of f^2S taken over the contact intervals of duration τ. The average value of f^2S taken over all time, including the blank intervals, is in the limit a fraction τ/T of the average over the contact intervals only. Therefore

$$\overline{f^2(nt_0)} = \lim_{\tau \to 0} \overline{\frac{T}{\tau} f^2(t) S(\tau, t)}$$

$$= \overline{\lim_{\tau \to 0} f^2(t) + \sum_{m=1}^{\infty} \frac{2T \sin m\pi\tau/T}{m\pi\tau} f^2(t) \cos 2m\pi f_s t} \tag{I—5}$$

$$= \overline{f^2(t)} + \lim_{\tau \to 0} \sum_{m=1}^{\infty} \frac{2T \sin m\pi\tau/T}{m\pi\tau} \overline{f^2(t) \cos 2m\pi f_s t}.$$

Now the long time average value of $f^2(t) \cos 2m\pi f_s t$ must vanish unless $f^2(t)$ contains a component of frequency mf_s. This could not happen except where $f(t)$ itself contains a component of frequency $mf_s/2$ or two components f_1 and f_2 such that

$$| f_1 \pm f_2 | = mf_s \tag{I—6}$$

When no such relation of dependency exists:

$$\overline{f^2(nt_0)} = \overline{f^2(t)}. \tag{I—7}$$

As pointed out before if $f(t)$ contains no frequencies above f_c, the response of the ideal low-pass filter to the samples is $f(t)$, and $f(nt_0)$ represents the samples of $f(t)$. If $f(t)$ does contain frequencies exceeding f_c, the response of the filter is $\phi(t)$, where $\phi(t)$ is wholly confined to the band 0 to f_c and yields the same samples as $f(t)$, i.e.,

$$\phi(nt_0) = f(nt_0), n = 0, \pm 1, \pm 2, \cdots \tag{I—8}$$

Eq. (I—7) applied to $\phi(t)$ gives the result:

$$\overline{\phi^2(nt_0)} = \overline{\phi^2(t)}. \tag{I—9}$$

By combining (I—8) and (I—9), we obtain

$$\overline{f^2(nt_0)} = \overline{\phi^2(t)}. \tag{I—10}$$

APPENDIX II

Fundamental Theorem on Aperture Effect in Sampling

If we sample the wave $Q \cos qt$ at a rate f_s, and multiply each sample by a short rectangular pulse of unit height and duration τ centered at the sampling instants, we obtain by reference to Eq. (I–4) replacing $2\pi f_s$ by ω_s,

$$F(t) = Q \cos qt\, S(\tau, t) = \frac{\tau}{T} Q \cos qt + Q \sum_{m=1}^{\infty} \frac{\sin m\pi\tau/T}{m\pi} [\cos (m\omega_s + q)t + \cos (m\omega_s - q)t]. \tag{II—1}$$

The fact that pulse modulation is similar to the more familiar carrier modulation processes is brought out by this equation; the sampling frequency is in fact the carrier. The writer has found that the method of calculation he published in 1933,[14] in which the signal and carrier frequencies are taken as independent variables, is ideally suited for calculations of pulse-modulated spectra. Artificial and cumbersome devices such as assuming the signal and sampling frequencies to be harmonics of a common frequency are thereby avoided.

A unit impulse $\delta(t)$ has zero duration and unit area; hence we may write:

$$\delta(t) = \lim_{\tau \to 0} \frac{S(\tau, t)}{\tau}. \tag{II—2}$$

A train of samples in which each sample is multiplied by a unit impulse may therefore be written as

$$\sum_{n=-\infty}^{\infty} Q \cos qt\, \delta(t - t_n) = \lim_{\tau \to 0} \left[\frac{Q}{T} \cos qt + Q \sum_{m=1}^{\infty} \frac{\sin m\pi\tau/T}{m\pi\tau} [\cos (m\omega_s + qt) + \cos (m\omega_s - q)t] \right]. \tag{II—3}$$

Suppose we apply the train of waves (II–3) to a linear electrical network which delivers the response $g(t)$ when the input is a unit impulse $\delta(t)$. The steady state admittance of the network is given by[15]

$$Y_0(i\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t}\, dt \tag{II—4}$$

and the response of the network to (II–3) is therefore:

$$\begin{aligned} I(t) = &\frac{Q}{T} \mid Y_0(iq) \mid \cos [qt + ph\, Y_0(iq)] \\ &+ \frac{Q}{T} \sum_{m=1}^{\infty} (\mid Y_0(im\omega_s + iq) \mid \cos [(m\omega_s + q)t \\ &+ ph\, Y_0(im\omega_s + iq)] + \mid Y_0(im\omega_s - iq) \mid \cos [(m\omega_s - q)t \\ &+ ph\, Y_0(im\omega_s - iq)]). \end{aligned} \qquad \text{(II—5)}$$

But $I(t)$ evidently represents a train of pulses in which the pulse occurring at $t = nT$ is equal to the *nth* sample multiplied by $g(t - nT)$. We have thus obtained the spectrum of a set of samples in which the pulse representing a unit sample is the generalized wave form $g(t)$. Furthermore if the signal frequency q is less than $\omega_s/2$, an ideal low-pass filter with cutoff at $\omega_s/2$ responds only to the first component of (II—5).

The "aperture effect" or variation of transfer admittance with signal frequency is thus given by

$$Y(iq) = \frac{1}{T} Y_0(iq) = f_s Y_0(iq). \qquad \text{(II—6)}$$

This is Theorem II. Since the system is linear when the signal frequency does not exceed half the sampling frequency, the principle of superposition may be applied to composite signals. In the case of distortion from quantizing errors the aperture effect applies to the error component delivered by the low-pass output filter. For an imperfect low-pass filter in the output we multiply the aperture admittance function by the actual transfer admittance of the filter.

A theorem equivalent to the above has been derived by a different method in a recent paper[16] published after completion of the above work.

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Analysis and Performance of Waveguide-Hybrid Rings for Microwaves

By H. T. BUDENBOM

This paper presents an analytical treatment of waveguide hybrid rings for microwaves, considered as re-entrant transmission lines. The resulting lines are transformed into equivalent "T" or "lattice" network sections, and determinantal methods are applied in analyzing these equivalent network assemblies for their transmission properties. Some experimental results obtained from a carefully constructed sample of each of two specific types are given. A satisfactory agreement is obtained between the values predicted by theory and experimental results.

INTRODUCTION

IN A recent paper[1], Mr. W. A. Tyrrell has described two general types of waveguide or waveguide/coaxial structures whose properties include bridge or null balance characteristics analogous to those of the hybrid coil common in voice-frequency communication practice. One type, the hybrid junction, is a particular orthogonal junction of four rectangular waveguides. Certain properties of the hybrid junction, notably its impedance characteristics, have been the subject of a British publication[2]. The present paper presents a method for detailed analysis of the other general structure described by Tyrrell, the hybrid ring. This latter structure is essentially an annular ring or annulus of waveguide, at present usually an integral number of quarter wavelengths in circumference, and fitted with an appropriate number of series or shunt branch taps. In this article, phrases such as "quarter wavelength," etc., describing tap spacing or mean annulus perimeter, refer to wavelength in the guide, not to free space wavelength.

The method of analysis employed herein is essentially to treat the tapped annulus as a re-entrant transmission line. Certain circuit equivalences and quarter wave impedance transformations were used by Tyrrell in his paper to develop, with the aid of the reciprocity theorem, many basic properties of hybrid circles and hybrid junctions. In the present paper "T" or "lattice" equivalents (neglecting dissipation) are developed for each section of the annulus, and the method of determinants is applied.

The hybrid junction (known also as the "magic tee") came into use in the newer radars in the latter part of the war. One of its uses, that of providing

[1] "Hybrid Circuits for Microwaves," W. A. Tyrrell, *Proc. I. R. E.*, November 1947.
[2] "The Theory and Experimental Behaviour of Right-Angled Junctions in Rectangular-Section Wave Guides," *I. E. E. Jour.*, September 1946, p. 177.

as outputs the sum and the difference of two input voltages*, is shown on Fig. 1. Matching stubs at the crossing, as indicated, are required to reduce standing waves to a reasonable value. The corresponding type of hybrid ring for providing sum and difference outputs is likewise shown, to-

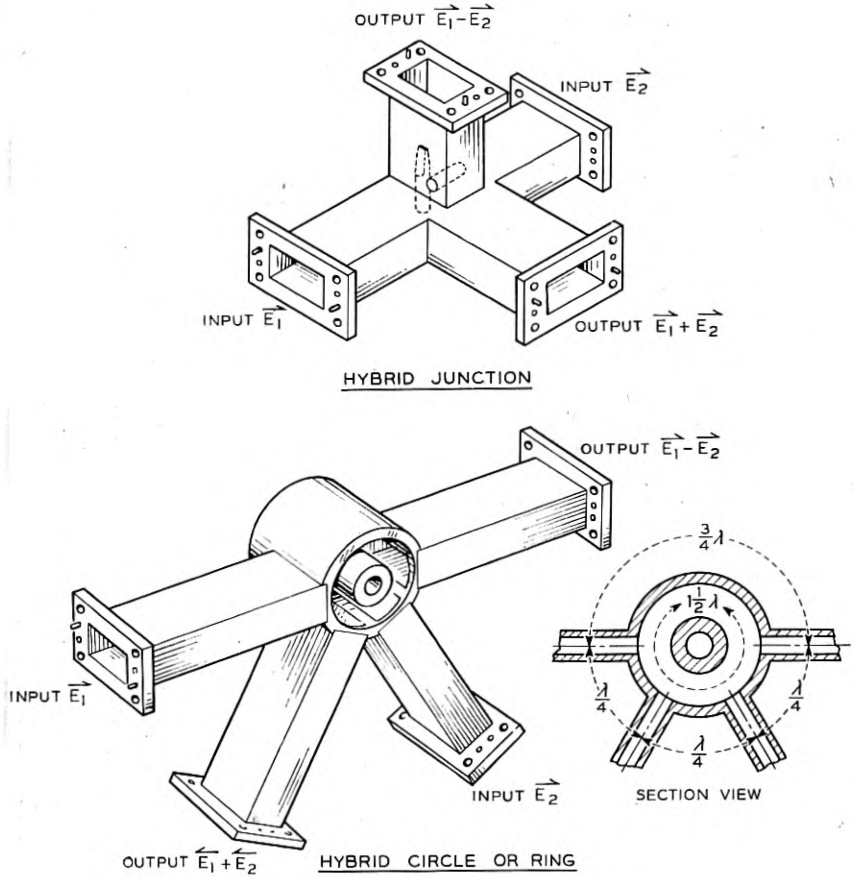

Fig. 1—Hybrid junction and hybrid circle or ring.

gether with a diagram dimensioned in terms of wavelength. Since the path lengths from each input to the output between them are equal, this output gives their sum; the path lengths to the remaining outputs differ by one half wavelength, consequently this output feeds out the difference of the two in-

* More exactly, of two input powers.

puts. No matching stubs are required to achieve a fairly good standing wave ratio; however, the bandwidth over which the ring operates differentially is inherently narrower than that of the junction. The rings have considerably higher power capacity.

The use of hybrids, both junctions and circles, has been noted, as applied to both duplexer and mixer design[3].

There follows a circuit analysis of hybrid rings, primarily of the series type. The method used is to consider the annulus as a continuous line closed on itself. The sections between series taps are then treated as being made up of integral single or multiple quarter wave line sections. Equivalent T or lattice sections are derived for 1, 2, 3, and 4 quarter-wavelength sections, ignoring line dissipation. These equivalences are used to draw equivalent mesh networks. The mesh networks are then solved by determinantal methods. To study some effects of frequency shift off the design center, where the mean periphery of the ring departs from an exact integral number of quarter-wavelengths, the increments in the element values for a quarter-wave equivalent T section are calculated and utilized. The example studied is a ring of $1\frac{1}{2}\ \lambda$ mean perimeter with 3 and 4 taps.

The general procedure neglects possible fringing effects at the junctions. It also neglects the fact that each tap embraces a length of ring which is distinctly more than a small fraction of a wavelength. Nevertheless, the results appear in every case to give a good first approximation. The writer is indebted to Messrs. J. T. Caulfield and J. F. P. Martin for checking the calculations.

Throughout the analysis Z_0 represents guide impedance and $\overline{Z}$ represents annulus impedance. It will be noted that the analytical match condition listed is $\sqrt{2}\ \overline{Z} = Z_0$ for the $1\frac{1}{2}\ \lambda$ rings.

The variation of the method necessary to treat the case of shunt taps is indicated.

I. Circuit Analysis

The rings studied herein are of the series type. This type is the one which results when waveguide is bent in the H plane, into a circle, and tap connections are made to the broad outer face. This type of ring is used, for example, in the "rat race" plumbing.

Such rings may be considered on the basis that the annular slot is a transmission line, whose characteristic impedance will here be called $\overline{Z}$ and propa-

[3] E. G. Schneider, *Proc. I. R. E.*,—August 1946, p. 528 et seq.—see page 550 et seq. and Figs. 40, 42 and 47.

gation constant P. The transmission line is closed upon itself. Series connections are made by the waveguide connections. The waveguide outlets are assumed, by virtue of their lengths and/or terminations, to present wave-

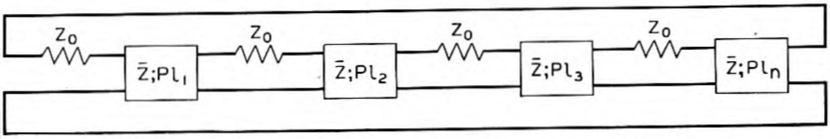

Fig. 2a—Series type hybrid ring as re-entrant transmission line.

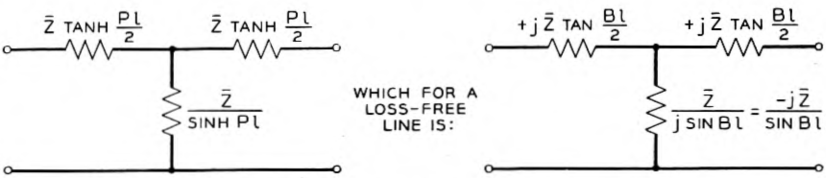

Fig. 2b—T Network equivalent to a line section.

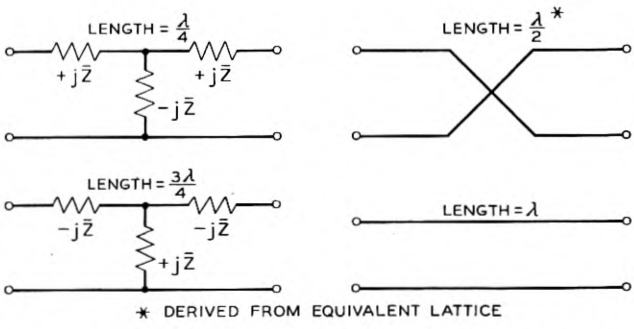

Fig. 2c—Networks equivalent to particular lengths of loss—free line.

* For this case the T becomes indeterminate. However, the needed equivalence can be proved by using the equivalent lattice. If we call Z_a and Z_b the respective series and shunt arms of the T, then the equivalent lattice has series arms $= Z_a$ and diagonal arms $Z_a + 2Z_B$.

guide characteristic impedance to the ring; this will herein be called Z_0. Diagrammatically, the situation is as in Fig. 2a. In the course of the following, the line sections will be replaced by equivalent networks, assumed non-dissipative.

II. Equivalent Line Sections

The first method following evaluates the line sections between outlets. The second views each line section as made up of the necessary number of quarter-wave sections, each represented by its equivalent T.

Method 1—The equivalent T for a recurrent structure[4] of constants $\overline{Z}$ (characteristic impedance) and P $(=A + jB)$ propagation constant per unit length is as shown in Fig. 2b.

There result the equivalences sketched in Fig. 2c.

Once the circuit is diagrammed using the above equivalences, it can be reduced to simpler form by successive combinations of Ts, by well known formulae.

Method 2—Determinants. We now consider the line to be máde up of the appropriate number of quarter-wave sections, with series taps. Thus we will have Fig. 3.

The shunt impedances are identical; call each Y. The series impedances are made identical by first assuming a tap at each quarter-wave junction;

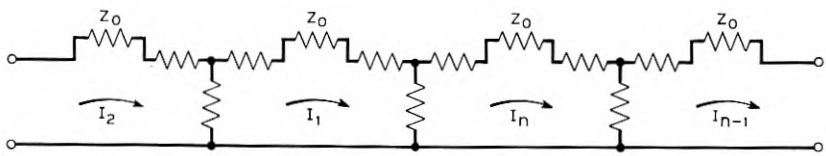

Fig. 3—Re-entrant line as succession of equivalent (quarter wave) T networks and series taps.

call each series leg S. Then the (skew symmetrical) circuit determinant for the case where $N = 10$, (or a $2\frac{1}{2}$ wavelength ring) is

$$D_{10} = \begin{vmatrix} (S+2Y) & -Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Y \\ -Y & (S+2Y) & -Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -Y & (S+2Y) & -Y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -Y & (S+2Y) & -Y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Y & (S+2Y) & -Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -Y & (S+2Y) & -Y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -Y & (S+2Y) & -Y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -Y & (S+2Y) & -Y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Y & (S+2Y) & -Y \\ -Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Y & (S+2Y) \end{vmatrix} \quad \text{II—2.1}$$

Now, for the case of an exact integral number of quarter wavelengths around the ring, all $Y_{1-n} = -j\overline{Z}$ and all $S_{1-n} = Z_0 + 2j\overline{Z}$, so all $S + 2Y = Z_0$.

[4] K. S. Johnson, "Transmission Circuits for Telephone Communication" Book published by D Van Nostrand Co., New York, N. Y.

The determinant then becomes

$$D'_{10} = \begin{vmatrix} Z_0 & +j\bar{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +j\bar{Z} & E_1 \\ +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_2 \\ 0 & +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 & 0 & 0 & 0 & 0 & E_3 \\ 0 & 0 & +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 & 0 & 0 & 0 & E_4 \\ 0 & 0 & 0 & +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 & 0 & 0 & E_5 \\ 0 & 0 & 0 & 0 & j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 & 0 & E_6 \\ 0 & 0 & 0 & 0 & 0 & +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 & E_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & E_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +j\bar{Z} & Z_0 & +j\bar{Z} & E_9 \\ +j\bar{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +j\bar{Z} & Z_0 & E_{10} \end{vmatrix} \qquad \text{II—2.2}$$

For a $1\frac{1}{2}\ \lambda$ ring, $n = 6$ and the system shrinks to

$$D_6 = \begin{vmatrix} Z_0 & +j\bar{Z} & 0 & 0 & 0 & +j\bar{Z} \\ +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 & 0 \\ 0 & +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 \\ 0 & 0 & +j\bar{Z} & Z_0 & +j\bar{Z} & 0 \\ 0 & 0 & 0 & +j\bar{Z} & Z_0 & +j\bar{Z} \\ +j\bar{Z} & 0 & 0 & 0 & +j\bar{Z} & Z_0 \end{vmatrix} \qquad \text{II—2.3}$$

For the study of the effects occurring if we move off the design center, we can modify the individual Ts to a length $\ell = \dfrac{\lambda}{4} \pm \dfrac{\lambda}{N}$. Each series arm, assuming no line dissipation, and N large so $\dfrac{\lambda}{N} \ll \lambda$, is:

$$j\bar{Z} \tan\left[\frac{2\pi}{\lambda}\left(\frac{\ell}{2}\right)\right] = j\bar{Z} \tan\left[\frac{2\pi}{2\lambda}\left(\frac{\lambda}{4} \pm \frac{\lambda}{N}\right)\right] = j\bar{Z} \tan\left[\frac{\pi}{4} \pm \frac{\pi}{N}\right]$$
$$= j\bar{Z}\,\frac{1 \pm \tan \pi/N}{1 \mp \tan \pi/N} \doteq j\bar{Z}(1 \pm 2\pi/N) = j\bar{Z}(1 \pm \Delta) \qquad \text{II—2.4}$$

Similarly, each shunt arm is:

$$\frac{\bar{Z}}{j \sin\left[\dfrac{2\pi}{\lambda}(\ell)\right]} = \frac{\bar{Z}}{j \sin\left[\dfrac{2\pi}{\lambda}\left(\dfrac{\ell}{4} \pm \dfrac{\lambda}{N}\right)\right]} = \frac{\bar{Z}}{j \sin\left[\dfrac{\pi}{2} \pm \dfrac{2\pi}{N}\right]}$$
$$= \frac{-j\bar{Z}}{\sin \dfrac{\pi}{2} \cos \dfrac{2\pi}{N}} = \frac{-j\bar{Z}}{\cos \Delta} \doteq \frac{-j\bar{Z}}{1 - \Delta^2/2} \doteq -j\bar{Z}\left(1 + \frac{\Delta^2}{2}\right) \doteq -j\bar{Z} \qquad \text{II—2.5}$$

So the shunt arm is, to a first approximation, not affected by a small shift off design center. Our shunts Y thus remain $-j\bar{Z}$ and

$$S + 2Y = Z_0 + 2j\bar{Z} \pm 2j\Delta\bar{Z} - 2j\bar{Z} = Z_0 \pm 2j\Delta\bar{Z}.$$

Therefore, determinants II—2.2 and II—2.3 can be used by merely considering $\bar{Z}_0 + j2\Delta\bar{Z}$ as a special value of Z_0.

As is well known,[5,6] the current solutions are obtained by writing in an external column the driving voltages, opposite their associated meshes, as is done at the right of II—2.2. In the present case the number of driving voltages is usually one, never more than two; so the column will be zeros, save for one (or two) meshes. The current in any mesh is a fraction having D as denominator, and as numerator the minor formed from D by substitut-

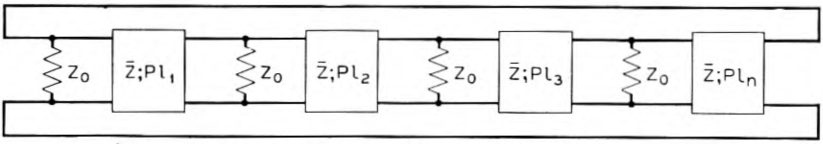

Fig. 4a—Re-entrant line with shunt taps.

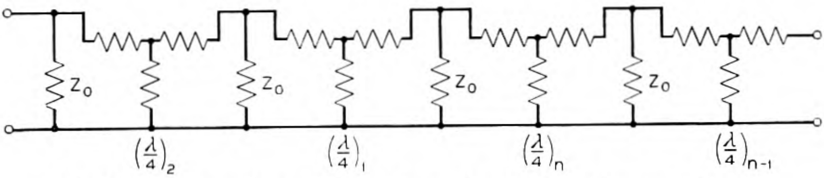

Fig. 4b—Re-entrant line with shunt taps—T networks as line equivalents.

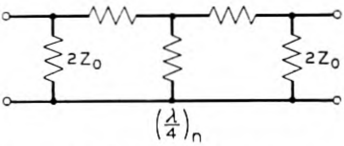

Fig. 4c—Element typical single shunt tapped section.

ing the e.m.f. column in the column corresponding to the mesh where the current is desired, i.e., column n if I_n is desired.

Since D is common to all mesh current expressions, questions of relative power division between branches or of null balance can be handled by operations performed entirely with the numerator minors.

Some slight advantage in evaluating the numerator minors is gained by proceeding where possible so as to make I_1 or I_n the desired current.

[5] E. A. Guillemin, "Communication Networks," Vols. I and II. Books published by John Wiley and Sons Inc., New York, N. Y.

[6] L. Silberstein, "Synopsis of Applicable Mathematics." Book published by D. Van Nostrand Co., New York, N. Y.

Alternatively, meshes where $Z_0 = 0$ can be chosen. Another needed quantity is driving point impedance. Since $I_n = \frac{E \cdot d_n}{D}$, then $\frac{1}{Z_{D.P.}} = \frac{d_n}{D}$ and $Z_{DP} = \frac{D}{d_n}$. The resulting impedance will include an extra Z_0, the generator impedance, to which we must match.

It may be of interest to show how the reentrant transmission line analysis can be extended to the case of hybrid rings involving shunt taps. For the reiterative shunt case we have the conditions illustrated in Fig. 4a. With substitution of quarter-wave equivalences Fig. 4a becomes Fig. 4b. Clearly determinants analogous to II—2.1 et seq. can be written for this structure. Alternatively we can split each Z_0 into two parallel impedances, each $2Z_0$, yielding a typical symmetrical section which can be reduced to a simple T or π by well known transformation methods[4] as shown in Fig. 4c.

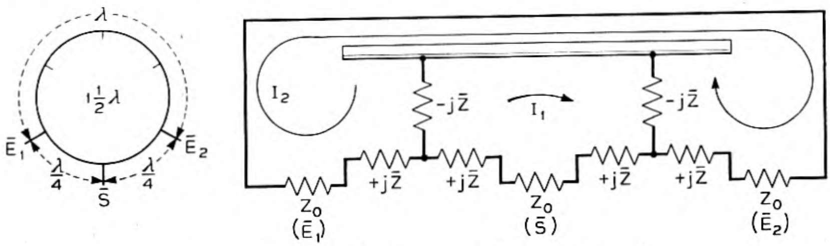

Fig. 5—1½ λ ring—3 arm—equivalent mesh circuit.

III. Detailed Analysis of Specific Cases of Series Type Rings

Case A. 1½ λ Ring—3 Arm—As Power Divider—Two Way

This is most simply analyzed using equivalents from Method 1. The equivalent circuit is shown in Fig. 5. It is immediately clear that:

a. Power fed in at $\bar{S}$ will divide equally between $\bar{E}_1$ and $\bar{E}_2$.

b. Although $\bar{E}_1$ and $\bar{E}_2$ are in proper wavelength relationship for isolation relative to each other, they are effectively in series and there will not be cancellation. The particular wavelength spacing is thus a necessary but not sufficient condition.

c. With a voltage E at $(\bar{S})$ we will have

$$E = I_1 Z_0 \qquad - I_2(-2j\bar{Z})$$

$$o = -I_1(-2j\bar{Z}) + I_2(2Z_0)$$

or

$$\begin{array}{c|cc|} E & Z_0 & +2j\bar{Z} \\ & & \\ o & +2j\bar{Z} & 2Z_0 \end{array} \qquad \text{III—1}$$

[4] loc. cit. page 282.

so

$$\frac{I_1}{E} = \frac{2Z_0}{2Z_0^2 + 4\overline{Z}^2}$$

and the mesh impedance at S is

$$\frac{Z_0^2 + 2\overline{Z}^2}{Z_0} = Z_0 + \frac{2\overline{Z}^2}{Z_0}.$$

Therefore an impedance match is secured if

$$\sqrt{2}\,\overline{Z} = Z_0. \qquad \text{III—2}$$

d. For a voltage e at $\bar{E}_1$, the current at $\bar{E}_2$ may be obtained from

$$\begin{array}{c|cc|} e & 2Z_0 & -2j\bar{Z} \\ o & -2j\bar{Z} & Z_0 \end{array}$$

and is

$$\frac{eZ_0}{2Z_0^2 + 4\bar{Z}^2}. \qquad \text{III—3}$$

Under the impedance match condition $\sqrt{2}\bar{Z} = Z_0$, this is $e/4Z_0$ which is just half the current which could be drawn through a load Z_0 connected to a source Z_0 with internal voltage e. Therefore, the "loss" from $\bar{E}_1$ to $\bar{E}_2$ is 6 db.

Case B. $1\frac{1}{2}\,\lambda$ Ring—4 Arms—As Power Divider and Null Device

As power divider—Two Way (Fig. 6). Using the determinantal method, let $\bar{E}_1$ be in mesh 1. Then D is in mesh 4, $\bar{E}_2$ in mesh 5 and $\overline{S}$ in mesh 6. $Z_0 = 0$ for meshes 2 and 3. Then the determinant of II—2.3 and its minor for mesh 5 ($\bar{E}_2$ in Fig. 6) with voltage applied at $\bar{E}_1$, are respectively, from II—2.3:

$$D'_6 = \begin{vmatrix} Z_0 & +j\bar{Z} & 0 & 0 & 0 & j\bar{Z} \\ +j\bar{Z} & 0 & +j\bar{Z} & 0 & 0 & 0 \\ 0 & +j\bar{Z} & 0 & +j\bar{Z} & 0 & 0 \\ 0 & 0 & +j\bar{Z} & Z_0 & +j\bar{Z} & 0 \\ 0 & 0 & 0 & +j\bar{Z} & Z_0 & +j\bar{Z} \\ +j\bar{Z} & 0 & 0 & 0 & +j\bar{Z} & Z_0 \end{vmatrix} \qquad \text{III—3}$$

and

$$d'_5 = \begin{vmatrix} +j\bar{Z} & 0 & +j\bar{Z} & 0 & 0 \\ 0 & +j\bar{Z} & 0 & +j\bar{Z} & 0 \\ 0 & 0 & +j\bar{Z} & Z_0 & 0 \\ 0 & 0 & 0 & +j\bar{Z} & +j\bar{Z} \\ +j\bar{Z} & 0 & 0 & +j\bar{Z} & Z_0 \end{vmatrix} \qquad \text{III—4}$$

Upon expansion d_5' is found to be 0. Therefore, by adding outlet $\overline{D}$, we have isolated branch $\bar{E}_1$ from branch $\bar{E}_2$. (Compare with case A.) Since it is well known for this structure that $\bar{D}$ is isolated from input at $\bar{S}$, it must follow that an input at S will still divide equally betwen $\bar{E}_1$ and $\bar{E}_2$.

As Null Device ($1\frac{1}{2}\lambda$—4 Arms). We now associate $\bar{S}$ with mesh 1, $\bar{E}_1$ with mesh 2, $\bar{D}$ with mesh 5, $\bar{E}_2$ with mesh 6 (see Fig. 7) which leads to:

$$D_6 = \begin{vmatrix} Z_0 & +j\bar{Z} & 0 & 0 & 0 & +j\bar{Z} \\ +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 & 0 \\ 0 & +j\bar{Z} & 0 & +j\bar{Z} & 0 & 0 \\ 0 & 0 & +j\bar{Z} & 0 & +j\bar{Z} & 0 \\ 0 & 0 & 0 & +j\bar{Z} & Z_0 & +j\bar{Z} \\ +j\bar{Z} & 0 & 0 & 0 & +j\bar{Z} & Z_0 \end{vmatrix} \qquad \text{III—3.5}$$

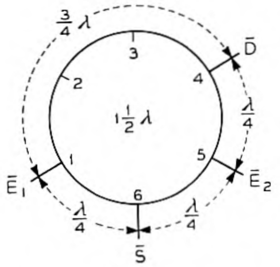

Fig .6

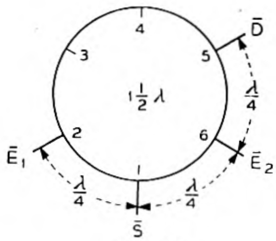

Fig. 7

Fig. 6—1½ λ ring—4 arm—tap spacing and identification for power division analysis by determinants.

Fig. 7—1½ λ ring—4 arm—tap spacing and identification for determinantal analysis as null device.

With voltage applied at $\bar{S}$, mesh 1, the minor for current at $\bar{D}$, mesh 5 is:

$$d_{5(\bar{S})} = + \begin{vmatrix} +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 \\ 0 & +j\bar{Z} & 0 & +j\bar{Z} & 0 \\ 0 & 0 & +j\bar{Z} & 0 & 0 \\ 0 & 0 & 0 & +j\bar{Z} & +j\bar{Z} \\ +j\bar{Z} & 0 & 0 & 0 & Z_0 \end{vmatrix} \qquad \text{III—5.1}$$

where the $5(\bar{S})$ indicates that the voltage is at $\bar{S}$ and the current is sought at mesh 5. Corresponding minors for the current in mesh $5(\bar{D})$, due to voltages at $\bar{E}_1$ and $\bar{E}_2$ are:

$$d_{5(\bar{E}1)} = - \begin{vmatrix} Z_0 & +j\bar{Z} & 0 & 0 & +j\bar{Z} \\ 0 & +j\bar{Z} & 0 & +j\bar{Z} & 0 \\ 0 & 0 & +j\bar{Z} & 0 & 0 \\ 0 & 0 & 0 & +j\bar{Z} & +j\bar{Z} \\ +j\bar{Z} & 0 & 0 & 0 & Z_0 \end{vmatrix} \qquad \text{III—5.2}$$

$$d_{5(\bar{E}2)} = - \begin{vmatrix} Z_0 & +j\bar{Z} & 0 & 0 & +j\bar{Z} \\ +j\bar{Z} & Z_0 & +j\bar{Z} & 0 & 0 \\ 0 & +j\bar{Z} & 0 & +j\bar{Z} & 0 \\ 0 & 0 & +j\bar{Z} & 0 & 0 \\ 0 & 0 & 0 & +j\bar{Z} & +j\bar{Z} \end{vmatrix} \qquad \text{III—5.3}$$

Evaluating, we find $d_{5(\bar{S})} = 0$, showing $\bar{D}$ is isolated from $\bar{S}$. The other two expansions give

$$d_{5(\bar{E}_1)} = (+j\bar{Z})[2\bar{Z}^4 - \bar{Z}^2 Z_0^2]$$

and

$$d_{5(\bar{E}_2)} = (+j\bar{Z})[\bar{Z}^2 Z_0^2 - 2\bar{Z}^4]. \qquad \text{III—5.4}$$

So the difference between the voltages at $\bar{E}_1$ and $\bar{E}_2$ is transmitted to $\bar{D}$. Hereafter we will operate on a single voltage at $\bar{S}$. Note, incidentally, that if the $\bar{S}$ arm were not terminated the Z_0 in column 1, row 1 of $d_{5(\bar{E}_1)}$ and $d_{5(\bar{E}_2)}$ would be zero, in which case

$$d_{5(\bar{E}_1)} = -(+j\bar{Z})(2\bar{Z}^4) \text{ and } d_{5(\bar{E}_2)} = +(+j\bar{Z})(2\bar{Z}^4) \qquad \text{III—5.5}$$

To study frequency shift on current from $\bar{S}$ at $\bar{D}$ we can write III—5.1 as

$$d_{5(\Delta)} = \begin{vmatrix} +j\bar{Z} & Z_0+2j\Delta\bar{Z} & +j\bar{Z} & 0 & 0 \\ 0 & +j\bar{Z} & 2j\Delta\bar{Z} & +j\bar{Z} & 0 \\ 0 & 0 & +j\bar{Z} & 2j\Delta\bar{Z} & 0 \\ 0 & 0 & 0 & +j\bar{Z} & +j\bar{Z} \\ +j\bar{Z} & 0 & 0 & 0 & Z_0+2j\Delta\bar{Z} \end{vmatrix} \qquad \text{III—6}$$

$$= 2\bar{Z}^2(-\bar{Z}^2 j\Delta\bar{Z} + 2Z_0\Delta\bar{Z}^2 + 4j\Delta\bar{Z}^3) = -2\bar{Z}^4\cdot\Delta\bar{Z}. \qquad \text{III—7.1}$$

Match Condition. The impedance match condition is readily shown to be $\sqrt{2}\bar{Z} = Z_0$ as for the three-arm $1\frac{1}{2}\lambda$ ring.

Construction of Test Samples

From the drawing of the hybrid circle (Fig. 1), it will be seen that the multiple soldering of guides into the ring can present difficulty in fabrication, especially where numerous branches are required. In addition, early measurements indicated the necessity of accurate dimensions, both linear and angular. As a consequence, the experimental hybrid circles which were used in the measurements reported herein were milled from brass cylinders. Figure 8 shows a 4-branch ring opened so that interior detail can be seen. This form of experimental construction enables dimensions to be held to average values of about half a thousandth of an inch and ten minutes of arc. The mating surfaces are flat to within this tolerance. However, no currents resulting from the field tend to flow in the direction crossing the gap and no loss ensues from this source. These mechanical tolerances are essential only to a basic experiment of the nature here described; larger tolerances could undoubtedly be specified in practice.

Experimental Results

There follows a tabulation of some experimental data on samples of the specific series types analyzed. The attenuation figures are probably good to ± .25 db up to 10 db, to ± .5 db up to 50 db. The SWR figures may not be better than ± .2 db.

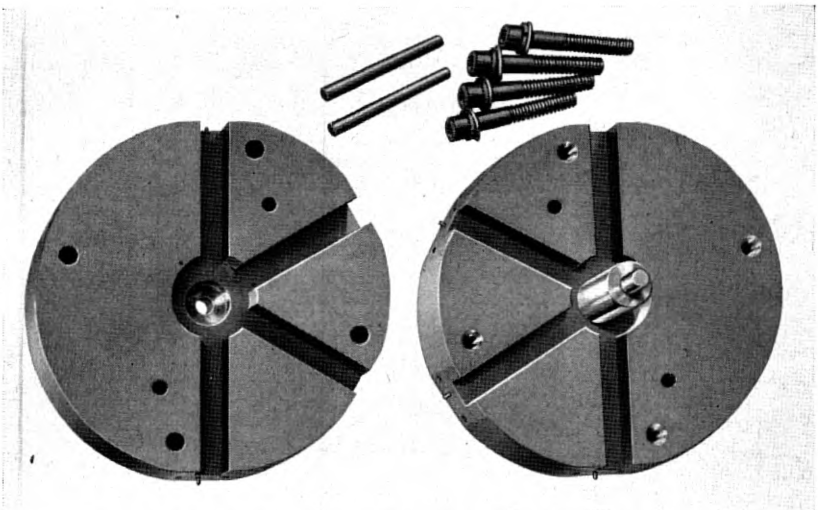

Fig. 8—1½ λ ring—4 arm—photograph of machined test sample.

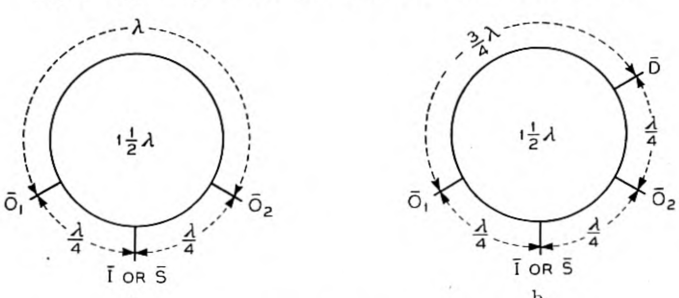

Fig. 9a—1½ λ ring—3 arm—as power divider.
Fig. 9b—1½ λ ring—4 arm—as power divider and null device.

The data are for structures built in terms of .900 inch by .400 inch rectangular guide size (inside) and the test wavelengths* are in the 3-centimeter region. The design wavelength* is λ_0, the test wavelength λ^*.

Case A: $1\frac{1}{2}\lambda$; Three Arms; Impedance Match $\sqrt{2}\bar{Z} = Z_0$; Reference Fig. 9a:

* These are space wavelengths.

	Experimental values at % $(\lambda - \lambda_0)/\lambda_0$				
	−6%	−3%	0	+3%	+6%
Power Division. Input at $\bar{I}$. Relative power output at $\bar{O}_1$ or O_2 in db (approx. constant over band)	←	←	$-3.7(\bar{O}_1)$ $-3.5(\bar{O}_2)$	→	→
Transmission Loss (Isolation) between $\bar{O}_1$ and $\bar{O}_2$ in db (approx. constant over band). $\bar{I}$ terminated.	←	←	6.0	→	→
Standing Wave Ratio (SWR) in db at $\bar{I}$. Outlets $\bar{O}_1$ and $\bar{O}_2$ terminated.	1.10	2.32	.84	1.50	1.20

Case B: $1\frac{1}{2}\lambda$; Four Arms; Impedance Match $\sqrt{2}\bar{Z} = Z_0$; Reference Fig. 9b:

	Experimental values at $100(\lambda - \lambda_0)/\lambda_0$				
	−6%	−3%	0	+3%	+6%
Power Division. Input at $\bar{I}$. Relative power output at $\bar{O}_1$ or $\bar{O}_2$ in db (approx. constant over band).	←	←	$-3.5(\bar{O}_1)$ $-3.5(\bar{O}_2)$	→	→
Transmission Loss (Isolation) between $\bar{O}_1$ and $\bar{O}_2$ in db $\bar{I}$ and $\bar{D}$ terminated.	20.3		48.5		19.7
Transmission Loss (Isolation) between $\bar{S}$ and $\bar{D}$ in db $\bar{O}_1$ and $\bar{O}_2$ terminated.	24.0		47.7		22.2
Standing Wave Ratio (SWR) in db at $\bar{I}$ $(\bar{S})$. Outlets at $\bar{O}_1$, $\bar{O}_2$ and $\bar{D}$ terminated.	3.50	1.20	.66	.77	2.20

Comparison Between Theory and Experiment

From the experimental results, we can now cite in support of the theory the following areas of agreement between theory and experiment, at the design wavelength:

Ring Type and Property	Theory	Experiment
Case A: $1\frac{1}{2}\lambda$; *Three Arms*		
Relative power at $\bar{O}_1$ and $\bar{O}_2$ for input at $\bar{I}$.	−3 db	−3.6 db
Impedance match (SWR)	0 db	.84 db
Observed center wavelength versus mean annulus perimeter guide wavelength	Agreement to about 1%	
Transmission loss (Isolation) from $\bar{O}_1$ to $\bar{O}_2$. $\bar{I}$ terminated.	6.0 db	6.0 db
Case B: $1\frac{1}{2}\lambda$; *Four Arms*		
Relative power at $\bar{O}_1$ and $\bar{O}_2$ for input at $\bar{I}$	−3 db	−3.5 db
Impedance Match (SWR)	0 db	.66 db
Transmission Loss (Isolation) $\bar{S}$ to $\bar{D}$. $\bar{O}_1$ and $\bar{O}_2$ terminated	Conjugacy	47.7 db
Transmission Loss (Isolation) $\bar{O}_1$ to $\bar{O}_2$. $\bar{D}$ and $\bar{I}$ terminated	Conjugacy	48.5 db

Conclusion

It is concluded that the theory developed provides calculated results in satisfactory accord with experiment.

It will be recalled that the approximation was initially made that the line sections were loss free. The theory could doubtless be extended to include dissipation by retaining a small real component in the propagation constant P of Fig. 2b. No doubt this real component could, in turn, be included to adequate accuracy in the equivalences of Fig. 2c by the addition of real components in the series arms only. That is, the series arm for a $\lambda/4$ section would be $\bar{Z}\,(r + j1) = r\,\bar{Z} + j\,\bar{Z}$ where $r \ll 1$. Since such terms appear as part of the $(S + 2\,Y)$'s in the basic determinant II—1, which is the same as in series with the Z_0's in determinant II—2, the inclusion of dissipation would appear to be formally straightforward.

Methods of Electromagnetic Field Analysis*

By S. A. SCHELKUNOFF

This paper presents a discussion of ideas involved in various mathematical methods of electromagnetic field analysis and of the inter-relations between these ideas. It stresses the points of contact between circuit and field theories and their mutually complementary character. While the field theory focuses our attention on the electromagnetic state as a function of position in space, the generalized circuit theory is preoccupied with the electromagnetic state as a function of time. The points of contact between the field and circuit theories are many. Thus, Maxwell's equations are identical with Kirchhoff's equations (really Lagrange-Maxwell equations) of certain three-dimensional networks in which only the adjacent meshes are coupled. The integral equations for the electrical current in conductors embedded in dielectric media are also Kirchhoff equations of certain networks containing infinitely many meshes with a coupling between every two meshes.

From the point of view of electrical performance the difference between a physical network of lumped elements and a continuous network, such as a resonator, is due to a certain difference in the distribution of the zeros and poles of associated impedance functions in the complex impedance plane. Similarly, the difference between ordinary transmission lines and wave guides is due to a difference in the distribution of natural propagation constants.

The paper ends with a general discussion of the discontinuities in wave guides, idealized boundary conditions for simplification of electromagnetic problems, and the analytical character of field vectors regarded as functions of the complex oscillation constant.

IN THE last few years engineering applications of electromagnetic field theory have been greatly expanded. Field theory has become essential for the solution of many practical problems and in planning engineering experiments. New applications have influenced the theory itself and have led to new conceptions. The chasm between the circuit theory of low frequency electrical phenomena and the field theory of high-frequency phenomena has disappeared. The two theories have met in wave guides and their merger has become essential. This paper is a discussion of the essential ideas underlying various mathematical methods of analysis of electromagnetic oscillations and waves in the light of new applications and of the merger of the originally distinct circuit and field theories.

Circuit Theory

Circuit theory is a mathematical method and it should not be confused with circuits. Empty space is neither a circuit nor a network; but as we shall soon see, for the purposes of analysis the empty space can be treated as a network. It is perfectly true that until recently circuit theory was con-

* This paper was originally delivered as a lecture at a meeting sponsored by the Basic Science Group of the American Institute of Electrical Engineers, April 12, 1945.

cerned almost exclusively with aggregates of "circuit elements" interconnected in various ways. It is also true that the most familiar form of circuit equations is that which is similar to Kirchhoff's equations for the steady current flow in networks of conducting rods, published[1] in April 1845.

This form is applicable only to circuits. However, the application of these "Kirchhoff equations" to alternating currents, natural as it may seem to us now, was not obvious one hundred years ago. The first equation for a simple circuit consisting of a capacitor, an inductor, and a resistor in series was published in 1853 by Lord Kelvin.[2] Interestingly enough his approach is based on the ideas applicable both to conventional circuits and to high-frequency resonators. If q is the electric charge on one plate of the capacitor, the energy stored in the capacitor is $q^2/2C$, where the coefficient C depends on the geometry of the capacitor. The magnetic energy of the circuit is $\frac{1}{2} L\dot{q}^2$, where $\dot{q}$ is the time rate of change of the charge, that is, the current in the circuit, and L is a coefficient depending on the geometry of the circuit. The rate of energy transformation into heat is $R\dot{q}^2$, where R is a coefficient depending on the geometry of the conductors (and of course on their resistivity). The law of conservation of energy demands that

$$\frac{d}{dt}[q^2/2C + \tfrac{1}{2}L\dot{q}^2] = -R\dot{q}^2. \tag{1}$$

When the differentiation is performed and $\dot{q}$ is cancelled, the usual form of the equation is obtained. The coefficients of proportionality, that is, the inductance L, the capacitance C, and the resistance R sum up and stress the really important electrical characteristics of the circuit; the details of the construction of the circuit are suppressed.

It was Maxwell who formulated the general equations for electric networks by extending the application of a method developed by Lagrange for mechanical systems. This Maxwell did in his last two lectures. In the words of his student, J. H. Fleming:[3] "Maxwell, by a process of extraordinary ingenuity, extended this reasoning (the method of Lagrange) from materio-motive forces, masses, velocities and kinetic energies of gross matter to the electromotive forces, quantities, currents, and electrokinetic energies of electrical matter, and in so doing obtained a similar equation of great generality for attacking electrical problems."

Before discussing the Lagrange-Maxwell method more completely, let us see if we can construct a network whose electrical properties would be the same as those of a continuous medium.

[1] *Annalen der Physik.*
[2] *Philosophical Magazine.*
[3] *Philosophical Magazine*, 1885.

Natural Network Models of Continuous Media and Maxwell's Differential Equations

Transmission line theory represents a well known example of the application of circuit theory to continuous systems. Two-wire transmission lines are subdivided into infinitesimal sections by planes perpendicular to the lines. Each section is replaced by a capacitor whose capacitance is so chosen that, for a given voltage across the transmission line, the electric charges on the plates of the capacitor are correspondingly equal to the charges on the sections of the wires constituting the line. The leads connecting the terminals of these capacitors are then assumed to possess an inductance and a resistance but no capacitance. Thus the electric flux or displacement is "swept" into tiny capacitors, and the magnetic flux or displacement into tiny inductors.

This representation is good only at low frequencies because it depends on the assumption that the electric displacement is only in one direction, namely at right angles to the transmission line. In effect, this representation neglects the capacitance between different parts of the same conductor and includes only the capacitance between the opposite segments of different conductors. That is, while we have recognized that the inductance and capacitance are distributed in the direction parallel to the transmission line, we have ignored the fact that they are also distributed at right angles to the line. In the general representation we should subdivide the medium into infinitesimal blocks and devise a three-dimensional network lattice of infinitely small meshes, Fig. 1. The displacement current can be swept equally into tiny capacitors. If the medium is dissipative, the resistors may be inserted in parallel with the capacitors to take care of the conduction currents in the medium. The magnetic flux is swept equally into tiny coils in the corners of each mesh. However, the resulting network is not homogeneous. Besides meshes of type A consisting of four capacitors and four inductors, it contains meshes of type B consisting of inductors only; and yet we started with a homogeneous medium. Gabriel Kron solved the difficulty by introducing ideal transformers (with one-to-one turn ratio) with their windings in series with the coils at the opposite corners of each A-mesh. These transformers do not affect the electrical performance of the A-meshes but introduce infinite impedance into B-meshes and thus effectively eliminate them.

As a matter of fact, such transformers should properly be included in the network representations of two-wire lines. In fact, by implication they *are* included as soon as we state that the direct and return currents in the line are equal and opposite. Without an infinite impedance to currents flowing in the same direction we cannot have the balance. Pursuing the matter further, we should say that all this is in accord with physical facts. The inductance per unit length of an *infinitely long* isolated wire is infinite. The mutual inductance between two parallel wires is also infinite. The two wires are the "windings" of an ideal transformer and a finite impedance is presented only to equal and opposite currents. In the case of wires of finite length the essentially three-dimensional character of the structure manifests itself, and other modes of propagation have to be considered.

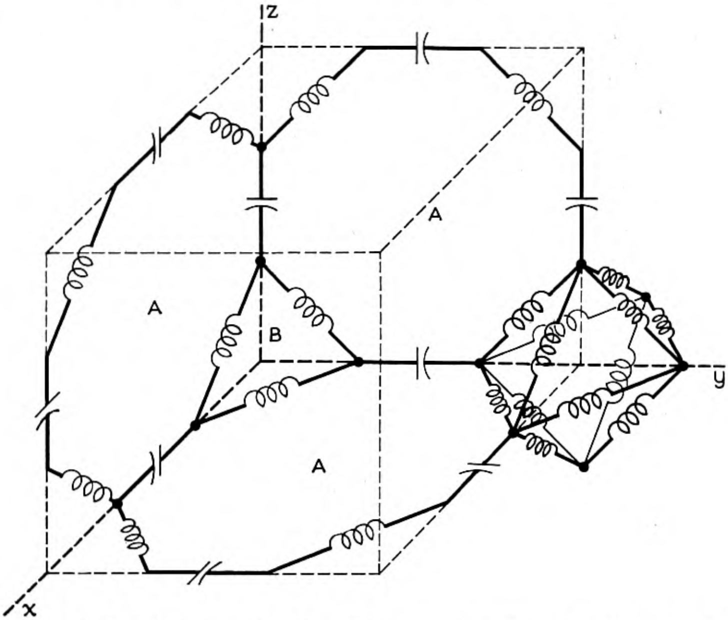

Fig. 1—Typical equivalent meshes in a circuit representation of continuous media.

It is evident that the homogeneity of the medium is not a prerequisite for the existence of its network model. Having the values of L and C at our disposal, we can choose them to reflect the dependence of the permeability μ and the dielectric constant ϵ on position.

If we divide the medium into small blocks of volume $\Delta x \, \Delta y \, \Delta z$, the capacitance C_x of the typical capacitor in those branches of the network which are parallel to the x-axis is $C_x = \epsilon \, \Delta y \, \Delta z / \Delta x$, where ϵ is the dielectric constant. The conductance in parallel with this is $G_x = g \Delta y \Delta z / \Delta x$. The inductance of the typical coil in the xy-plane is $L_{xy} = \mu \, \Delta x \Delta y / 4 \Delta z$. The voltages across the capacitors are $E_x \Delta x$, $E_y \Delta y$, $E_z \Delta z$, where E_x, E_y, E_z are the electric intensities, that is, the voltages per unit length in the respective directions. The currents in the coils situated in the xy-plane are equal to $H_z \Delta z$; similarly the currents in the other coils are $H_x \Delta x$ and $H_y \Delta y$. It is to be noted that the capacitors are associated with the corresponding longitudinal components of the electric field while the inductors go with the transverse components of the magnetic field. Applying Kirchhoff's laws to the network in Fig. 1, we should and do obtain Maxwell's field equations. Similarly, we can construct network lattices in the patterns of other coordinate systems, cylindrical and spherical, for example.

Among the obvious conclusions to be drawn from this analysis of the network structure of the medium supporting the electromagnetic field is the validity of certain general network theorems such as the Reciprocity Theorem and Thevenin's Theorem.

Reduced Network Models and Integral Equations of Lorentz Type

So far we have been concerned with the electromagnetic field in its entirety. In order to visualize the medium as a three-dimensional network we have selected the most direct course: We have subdivided the medium into blocks of displacement current, compressed them into capacitors, and eliminated displacement currents from the rest of space; similarly, we have

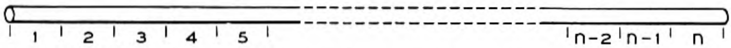

Fig. 2—Subdivision of a straight antenna for its representation by a reduced network with n meshes.

swept the magnetic flux into neat little packages. But this is not the only course open to us. We can suppress the medium just as completely as we normally do in the analysis of elementary networks. In order to illustrate this method let us consider a doublet antenna, Fig. 2. We shall divide it into n sections. The current and charge in any one section exert forces on the charge in any other section. We can regard each section of the antenna as a mesh of a network in which every mesh is coupled to every other mesh. In each mesh the voltage which is necessary to compensate for the electromotive force of self-induction of the mesh itself, for the resistance of the mesh (or rather for the internal impedance of the wire), and for the voltages induced from all the other meshes, is the impressed voltage. The equations assume the following form:

$$
\begin{aligned}
Z_{11}I_1 + Z_{12}I_2 + Z_{13}I_3 + \cdots + Z_{1n}I_n &= V_1, \\
Z_{21}I_1 + Z_{22}I_2 + Z_{23}I_3 + \cdots + Z_{2n}I_n &= V_2, \\
Z_{n1}I_1 + Z_{n2}I_2 + Z_{n3}I_3 + \cdots + Z_{nn}I_n &= V_n,
\end{aligned} \tag{2}
$$

where the I's are the currents in the various sections of the antenna and the V's are the *impressed* voltages. The Z's are the self-impedances and the mutual impedances, and are calculated from the law of force between two charged particles. In a transmitting antenna the impressed voltage is zero everywhere except in a restricted region. In the receiving antenna the voltage is impressed on all sections; but one section, the "load," has a very different self-impedance from the remaining sections.

When n is finite, our equations are approximate. If we make n infinite and introduce the impressed electric intensity, that is, the impressed voltage per unit length, we convert equations (2) into a single integral equation. More generally we may have to consider the transverse dimensions of the antenna and divide the entire surface of the antenna into elementary surface elements, each of which will represent *two* meshes in our network. We have to have two meshes for each surface element because the current may in general change its direction from point to point and in order to specify it completely we must consider two components of the current. These may be taken as tangential to some Gaussian coordinate lines drawn on the surface of the antenna. The exact network equations will appear as a system of two integral equations involving double integrals.

In this discussion, we have assumed that the medium outside the antenna is homogeneous. No difficulty is presented by the simultaneous inclusion of a transmitting and a receiving antenna. The two form just one network and the voltages impressed on the various meshes of the receiving antenna represent simply the coupling between these meshes and the meshes of the transmitting antenna. All the mutual impedances are calculable from the general equation,

$$E = -\mu \frac{\partial A}{\partial t} - \text{grad } V, \tag{3}$$

representing the force per unit charge due to a given moving charge. If we so desire, we can take equation (3) with the explicit expressions for A and V in terms of electric current and charge as the fundamental equations of electromagnetic theory and dispense with Maxwell's differential equations altogether. This course is feasible but inexpedient. Actual applications of this equation turn out to be much too complicated in the great majority of practical problems. It is only when we already know the current and charge distribution that (3) becomes really useful. Thus in the accepted development of electromagnetic theory (3) is subordinated to Maxwell's equations and derived from them.

Normalized Network Model and Lagrange-Maxwell Electrodynamical Equations

Let us now return to the ideas of Lagrange as applied to electromagnetics. In dynamics the Lagrange equations are formulated in terms of the kinetic energy T expressed as a function of velocities, potential energy U expressed as a function of coordinates, and a dissipation function F expressed as a function of velocities. In network theory T is the magnetic energy expressed as a function of currents, U is the electric energy expressed in terms of charges, and F is the dissipation function in terms of currents. Lagrange-Maxwell equations are then written in the following form

$$\frac{d}{dt}\left[\frac{\partial}{\partial I_n}(T - U)\right] - \frac{\partial}{\partial q_n}(T - U) + \frac{\partial F}{\partial I_n} = V_n, \tag{4}$$

where I_n is the typical mesh current, q_n is its time integral, and V_n is the *impressed* electromotive force, that is, the electromotive force not accounted for by the magnetic induction and the charges in the network. The various functions in the equation are

$$T = \Sigma_m \Sigma_n \tfrac{1}{2} L_{mn} I_m I_n, \qquad U = \Sigma_m \Sigma_n \frac{q_m q_n}{2C_{mn}}, \qquad F = \Sigma_m \Sigma_n \tfrac{1}{2} R_{mn} I_m I_n, \tag{5}$$

where L_{mn} is the mutual inductance between two typical meshes (the self-inductance if $m = n$), C_{mn} is the mutual capacitance and R_{mn} is the mutual resistance. The mesh currents are introduced in order to insure that the total current either entering or leaving a typical junction of the network elements is zero. If we perform the differentiations indicated in equation (4), we shall obtain the network equations in their usual form.

Let us now suppose that $F = 0$ and $V_n = 0$. In higher algebra it is shown that by a linear transformation two quadratic functions, T and U for example, can be reduced to normal forms in which there are no mutual terms

$$T = \Sigma_n \tfrac{1}{2} L_n \hat{I}_n^2, \qquad U = \Sigma_n \hat{q}_n^2 / 2C_n. \tag{6}$$

In this case equations (4) will assume the following simple form

$$L_n \frac{d\hat{I}_n}{dt} + \frac{\hat{q}_n}{C_n} = 0. \tag{7}$$

It is as if we had a certain number of isolated single-mesh circuits. Equations (7) represent the *normal modes of oscillation* of the network.

Take the simple case of two identical coupled circuits, Fig. 3. The network equations are

$$L \frac{d^2 I_1}{dt^2} + \frac{I_1}{C} - M \frac{d^2 I_2}{dt^2} = 0, \qquad -M \frac{d^2 I_1}{dt^2} + \frac{I_2}{C} + L \frac{d^2 I_2}{dt^2} = 0. \quad (8)$$

It is evident by inspection that there are two possible modes of oscillation. In one mode $I_1 = I_2$ and in the other $I_1 = -I_2$. The natural frequency of the first mode is $\omega_1 = 1/\sqrt{(L - M)C}$ and that of the second mode $\omega_2 = 1/\sqrt{(L + M)C}$. The magnetic energy function is

$$\begin{aligned} T &= \tfrac{1}{2}LI_1^2 - MI_1 I_2 + \tfrac{1}{2}LI_2^2 \\ &= \tfrac{1}{2}(L - M)\left[\frac{I_1 + I_2}{\sqrt{2}}\right]^2 + \tfrac{1}{2}(L + M)\left[\frac{I_1 - I_2}{\sqrt{2}}\right]^2. \end{aligned} \quad (9)$$

Thus the sum and the difference of the currents in the two meshes oscillate independently.

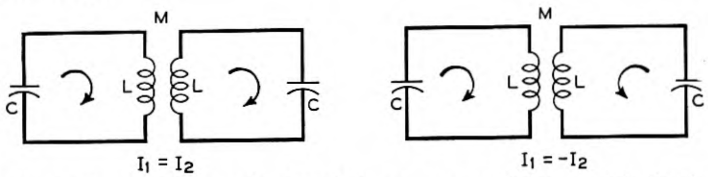

Fig. 3—Two possible modes of oscillation in a symmetric two-mesh circuit.

More generally a network with n meshes possesses n independent modes of oscillation. In each mode the ratios of the mesh currents $I_1, I_2, \cdots I_n$ are prescribed by the network parameters and the connections of the network elements, but the relative strength of the oscillation remains arbitrary. When we pass to networks with distributed parameters such as sections of transmission lines and cavity resonators, we find merely that the number of independent modes of oscillation is infinite. In the case of a nondissipative uniform transmission line with both ends shorted, the natural frequencies of the various oscillation modes are proportional to the sequence of integers: 1, 2, 3, ...The current distribution for the n-th mode is given by $\sin(n\pi x/\ell)$, where ℓ is the length of the section; but the actual amplitude remains arbitrary. For the gravest mode ($n = 1$) the middle part of the line section behaves as a capacitor and the ends as inductors. For the higher modes the line is subdivided into sections, some of which act primarily as capacitors and others as inductors.

In the case of cavity resonators of some simple shapes, such as parallelopipedal, cylindrical and spherical, the determination of the oscillation modes is a fairly simple problem. The dynamical equations of the resonator

(Maxwell's field equations) are partial differential equations. Their solutions would normally involve arbitrary functions; but since the tangential electric intensity vanishes at the conducting boundary of the resonator, the solutions assume a much less arbitrary form involving only an infinite set of arbitrary constants. Particular solutions are sought in the form of products of three functions, each depending on only one coordinate. For parallelopipedal cavity resonators the various components of electric and magnetic intensity are assumed in the form $X(x)\ Y(y)\ Z(z)$. By substituting in Maxwell's equations it is found—very fortunately indeed—that X, Y, Z may be obtained as solutions of ordinary differential equations. The boundary conditions at the boundaries of the box $x = 0,a$; $y = 0,b$; $z = o,c$ are easy to satisfy because we have to work with only one of these three functions at a time.

In general, however, the problem of calculating oscillation modes is by no means simple; but once these modes have been determined, the problem of forced oscillations as well as free oscillations is practically solved. For instance, a small loop inside a resonator is coupled to the various modes and the coupling coefficients can be determined by evaluating the flux linkages.

Every physical circuit possesses an infinite number of degrees of freedom and circuits with a finite number of degrees of freedom are abstractions. If we take special measures to concentrate magnetic energy as much as possible in a few regions of the medium and electric energy in a few other regions, we shall have a physical network in which a finite number of oscillation modes will be well separated on the frequency scale from all the rest. If we are concerned only with the frequencies comparable to the natural frequencies of this cluster of modes, we can ignore all the higher modes and for our purposes we may regard the network as a finite network. At these frequencies the infinitely small meshes into which we could subdivide the individual "inductors" (regions of magnetic energy concentration) and "capacitors" (regions of electric energy concentration) will oscillate in unison in groups.

Briefly we can summarize the above methods of analysis as follows: The medium supporting the electromagnetic field may be regarded as a three-dimensional network of infinitely small meshes in which every mesh is coupled only to the adjacent mesh. Circuit equations applied to this network lead to Maxwell's differential equations. In contrast with this "*natural network model of the medium*" we can construct a *reduced network model* in which only the conductors of the medium are subdivided into meshes. The medium surrounding the conductors is concealed in the mutual impedances of the constituent meshes. Every mesh is coupled to every other mesh and the mutual impedance (or the coupling factor) is determined from the law of force exerted by a moving charge on a stationary charge. This approach leads to one or two integral equations which can be approximated by a system of linear algebraic equations. While the latter may seem much simpler than the differential equations obtained from the natural network model, in reality their solution would often constitute a much more difficult analytical problem. The natural network model in which each mesh is coupled only to the adjacent meshes is in harmony with the idea of continuous propagation of electromagnetic disturbances; while the reduced network model conforms to the action at a distance philosophy. The difference is merely in the language and ideas and not in substance.

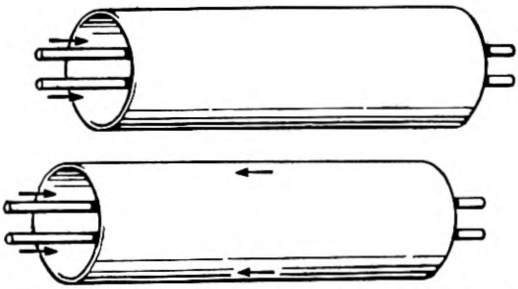

Fig. 4—Two possible modes of propagation in a symmetrically shielded parallel pair.*

Finally, the third method is based on the idea that at certain frequencies, called the natural frequencies, various parts of a closed system oscillate in phase or 180° out of phase, that the most general natural oscillation is the sum of such oscillations, and that the most general forced oscillation can be expressed in terms of fields associated with the natural modes of oscillation. We may call this the *normalized network model of the electromagnetic field*. Thus far we have described it with reference to closed systems or cavity resonators. In effect we have assumed that the amounts of magnetic and electric energy are finite or else we could not talk about T and U functions. The method can be extended to open systems of wave guides.

Modes of Transmission

Let us begin with a coaxial transmission line. Everyone is familiar with the particular mode of transmission in which equal and opposite currents flow in the two conductors. The circuit is completed through the dielectric where the displacement current flows from one conductor to the other. Next, consider a shielded parallel pair. If the structure is symmetric, we shall recognize at once two modes of transmission, Fig. 4. In one mode, the balanced mode, the currents in the wires are equal and opposite; there are

* In the upper part of this figure one of the directional arrows should be reversed.

also equal and opposite currents in the shield which, however, are not equal to the corresponding currents in the wires. In the other mode, the currents in the wires are equal and similarly directed, the return path being through the shield; this mode is similar to the coaxial mode since the wires act in parallel, effectively as one conductor. In the case of n wires there are n distinct modes of transmission. Each mode is characterized by the ratio of currents in the wires and by the field pattern that goes with it.

In all these modes the longitudinal current paths are conductive; but there is no reason whatsoever why the circuit closure should not take place through the dielectric. Even in those modes of transmission in which all longitudinal current paths are conductive, we have to depend on the dielectric for completion of the circuit; this should prepare us for the idea that conductors are not essential for wave transmission. If we include the dielectric, the number of possible longitudinal tubes of flow becomes infinite and so does the number of possible transmission modes; but as the cross-section of each individual tube decreases the longitudinal capacitance also decreases, and these modes will participate in the transfer of power over substantial distances only at correspondingly higher frequencies. It is not merely that at low frequencies the longitudinal impedance becomes very high; it is capacitive and causes high attenuation. The effect is analogous to the attenuation in high-pass filters below the cutoff.

The mathematical analysis which lends quantitative substance to these ideas is similar to that involved in the cavity resonator problem. Once all the modes of transmission have been found, the next problem is that of the excitation of these modes by a given source, that is, of coupling of the source to various modes.

To summarize: A physical transmission line or a wave guide has always an infinite number of transmission modes either independent or substantially independent of each other. It is as if we had a system of single-mode transmission lines without couplers. For each transmission mode the structure behaves as a high-pass filter. If n is the number of conductors, there are $n - 1$ transmission modes with the cutoff frequency equal to zero. Since the lowest non-zero cutoff frequency corresponds to a wavelength comparable to the transverse dimensions of the guide, it is clear that in systems with two or more conductors we have a certain finite number of transmission modes which are well separated on the frequency scale from all the rest. For this reason we may ignore all the higher modes when we are concerned with transmission of low frequencies only, by "low" meaning the frequencies well below the frequency equal to the velocity of light divided by the largest transverse dimension of the transmission line.

Analysis of waves in free space proceeds along similar lines. An electric "dipole" is the source of the simplest spherical electromagnetic wave. We may picture this dipole as a pair of small spheres connected by a thin rod. Under the influence of an impressed force the charge is made to surge back and forth between the spheres. We cannot have a simple source like a uniformly expanding and contracting sphere as in the case of sound waves. The electric charge is conserved, and the only way we can alter the charge in one place is to transfer it to some other place. A more symmetrical dipole would be a single sphere on the surface of which the charge is made to move back and forth between two hemispheres. Let us call these hemispheres respectively the "northern" and the "southern". When the positive charge accumulates on the northern hemisphere, the radial displacement current flows outwards from it. At the same time an equal radial displacement current flows toward the southern hemisphere. The situation is analogous to the balanced mode of transmission along parallel wires, with the two half spaces acting as "the wires". The distance along the line is the distance from the dipole. The radial transmission line is capacitively loaded but the series capacitance increases as the square of the radius and therefore the capacitive series admittance decreases as the reciprocal of the square of the radius. Hence, at some distance from the dipole, the wave propagation will be quite unimpeded just as in ordinary transmission lines free from loading. Near the dipole the series capacitance is high, and the power carried by the wave in comparison with the energy stored is small.

In the next spherical mode of transmission the polar regions of the spherical generator are similarly charged while the opposite charge is concentrated in the equatorial zone. The zonal character of the radial current distribution persists at all distances from the generator. As might be expected the reactive field in the vicinity of a small "tripole" generator is even stronger than in the case of the dipole source.

The sequence of zonal modes of transmission can be continued indefinitely. Next we could imagine tubular modes in which the space surrounding the generator is subdivided into conical tubes with the radial current in adjacent tubes flowing in opposite directions. This picture is essentially physical; but it corresponds very closely to the mathematical expansion of the general solution of Maxwell's equations in spherical harmonics.

Field Representation in Terms of Fields of Special Types

From the mathematical point of view the method which we have just been considering is based on the idea of representation of the general field in terms of particular fields having certain relatively simple properties. The method is analogous to that employed in circuit theory when the response to the general electromotive force is expressed in terms of responses to the unit step function, or the unit impulse function, or the steady state responses at various frequencies.

There are numerous variations of the same general idea, some of which are more suitable to one class of problems and others to another class. If the distribution of electric charge and current is known, then in many cases (but not in all) it is best to subdivide it into small volume elements. Except for a possible static electric charge distribution, the elements will be dipoles. The entire field can thus be regarded as the resultant of spherical waves generated by dipoles of given moment and position. To simplify the integration involved in this method certain auxiliary functions, called the retarded potentials, are introduced. One should not try to ascribe to these auxiliary mathematical functions any physical significance and one should always remember that on certain occasions potential functions, other than the retarded potentials, turn out to be more useful. We should also keep in mind that, in order to apply this method, we have to know the complete distribution of electric conduction currents and as a general rule we do not have this information. Consider, for instance, the problem of electromagnetic shielding. The current in the coil is given; but that in the shield has to be determined. There are methods for calculating the induced current; but these methods give at the same time the shielding effectiveness, and that without employing retarded potentials. It is in approximate studies of radiation patterns of antennas and antenna arrays that the retarded potential method is displayed to the best advantage.

The retarded potentials are based on representation of fields in terms of spherical coordinates; that is, in terms of fields associated with hypothetical *point sources* at the origin of the coordinate system. General fields can also be expressed in terms of cylindrical coordinates and, consequently, in terms of fields associated with hypothetical *line* sources situated along the axis of the coordinate system. Likewise, fields can be expressed in cartesian coordinates; that is, in terms of "plane waves". All such representations have useful applications. The current in the coil is given.

Discontinuities

In the analysis of the various transmission modes for a given wave guide it is assumed at first that the boundaries of the wave guide are analytic functions of the coordinates. Any discontinuity or irregularity has to be treated separately, simply because there is nothing in the analytic part of the wave guide to suggest that a discontinuity might occur, or to prescribe the properties of this discontinuity. Discontinuities may be accidental, unavoidable or intentional. A kink in a wire is an example of an accidental discontinuity. Open air wire lines have to be supported on poles which, together with the insulators, constitute unavoidable discontinuities. The beginning and the end of a line are always present. Usually these latter discontinuities are simply unavoidable; but, in radio, at least one discontinuity, the antenna, is made to serve a useful purpose. It is clear that the generator and the load connected by a two-wire line, Fig. 5, are dipoles which will generate spherical waves as well as the wave guided by the transmission line. At low frequencies the length of the dipoles is so small compared with the wavelength that the field does not reach out into

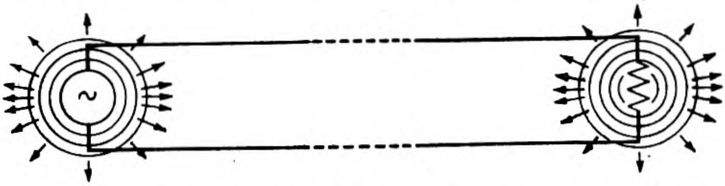

Fig. 5—Formation of spherical waves at the ends of a long pair of parallel wires.

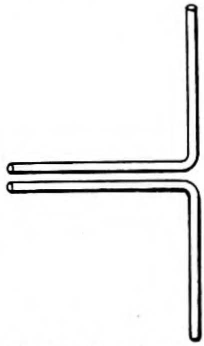

Fig. 6—An antenna.

the region where the radial capacitance becomes negligible and where the spherical wave starts carrying off all the energy that gets there. Spherical waves generated at the beginning and the end of the transmission line are practically stationary waves and constitute merely local reactive reservoirs of energy. The energy is withdrawn from the generator or the transmission line during one half of the cycle only to be returned during the other half. At low frequencies the energy thus exchanged back and forth is so small that normally we don't even think about it. The antenna, Fig. 6, is designed to be a more efficient transformer of the plane wave guided by the parallel pair into the spherical wave which will carry off power to distant points.

Quite frequently discontinuities are introduced intentionally in order to discriminate against some frequencies. A capacitor in parallel with the wave guide or an inductor in series with it will favor transmission of low frequencies at the expense of high frequencies. These discontinuities are deliberately designed to be sufficiently large to produce noticeable effect. A frequency filter is a more elaborate structure made up of capacitors and inductors designed to achieve desired frequency discrimination.

Discontinuities in high-frequency wave guides are also either accidental, unavoidable or intentional. The principal difference is in the order of magnitude—any irregularity of apparently small physical dimensions may represent a large virtual reservoir of energy. Among the simplest types of intentional discontinuities in wave guides are "irises", Fig. 7. Local fields are created in the vicinity of the irises. Under the influence of a wave traveling along the guide, electric charge and current are induced in the metal partition. On either side of the partition the complete field is the result of the superposition of fields representing various transmission modes. The cutoff frequencies of these modes may be arranged in an

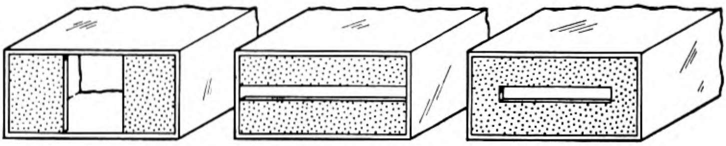

Fig. 7—Inductive, capacitive, and resonant irises.

increasing sequence. If the operating frequency is between the lowest cutoff frequency and the next higher, the propagation constants of all modes except the dominant are real and the corresponding fields will not extend very far from the iris. During one-half cycle the local field withdraws energy from the dominant wave—this being the only source of energy —and during the remaining half this energy is returned. The local field acts as a *virtual source* of power—"virtual" since it operates on borrowed power. On account of symmetry the dominant waves generated by this virtual source and traveling in opposite directions will be of equal intensities. The *scattered wave* traveling toward the source of the incident wave is called *the wave reflected from the iris*; on the other side the scattered and incident waves merge into the *transmitted wave*. The storage of energy in the local field depends on the frequency—hence, the frequency selectivity.

In the case shown in Fig. (7a) the flow of current in the partition is unimpeded and there is no tendency for any local concentration of charge in the partition; the local field is largely magnetic and the iris represents an inductive reactance. Since any variation of the magnetic field with time always creates an electric field, there will be some capacitance in parallel with the inductance. The same idea may be expressed by saying that the inductance of the iris is not quite independent of the frequency. This lack of constancy is not peculiar to ultra-high frequencies; it is true of coils at low frequencies. Likewise, even at very low frequencies the inductance varies with the frequency because of skin effect.

In the iris shown in Fig. (7b) there are alternating charge concentrations on the upper and lower partitions. The local field is largely electric and the iris is capacitive. A feeble magnetic field associated with charging current is unavoidable, of course; this is also true of capacitors at low frequencies but this time the effect is greater. Finally, an iris of the type shown in Fig. (7c) may be designed to behave as an antiresonant circuit.

In that frequency range in which only the dominant wave is an effective carrier of power to great distances, any discontinuity will behave as a reactive T or Π-network—assuming that observations are made at some distance from the iris where the local field is too feeble to count. This could not be otherwise since there are three parameters at our disposal: two reflection coefficients for waves traveling in opposite directions and one transmission coefficient across the discontinuity. The Reciprocity Theorem requires that the transmission coefficients in the two directions be equal. These three parameters determine the ratios of the reactance elements of the equivalent T or Π-network to the characteristic impedance of the guide.

If the operating frequency exceeds the second cutoff frequency, other waves besides the dominant become effective carriers of power and the equivalent network for the iris becomes more complicated. The iris behaves not only as a dissipative impedance to the dominant wave but also as a negative resistance, to one or more higher order waves.

Boundaries

So far we have paid little attention to the boundaries of the electromagnetic field. Strictly speaking, in any actual situation the field always extends to infinity; the only boundaries there are, are the geometric boundaries between media with different electromagnetic properties. This means that we should solve electromagnetic equations for each homogeneous region, or region with analytically varying properties, and then match the solutions at the boundaries. In many cases, however, this procedure would be very complicated and quite unnecessary. In the case of a cylindrical metal tube with a dipole as a source of power the exact solution may be represented as a particularly formidable integral; but experimentally we would not be able to detect any difference between the "exact" solution and a much simpler approximate solution.

In the case of rectangular tubes we don't even know how to obtain the "exact" solution in any form; but good approximate solutions are exceedingly simple. The word "exact" is in quotation marks because there can be no really exact solutions of actual physical problems. In the first place the properties of materials are not known exactly; the boundaries between media do not exist in the exact sense of the term; and we just don't know the exact laws of nature. All we really want of any solution is to be accurate enough for some particular purpose. And here is where the idea of idealized boundaries helps in the formulation of simplified, clear-cut mathematical problems. The idea lends flesh and blood to idealized mathematical boundary conditions. *Perfect conductors* have long been mentioned in literature as idealizations of good conductors; but other types of boundaries are of much more recent origin. Perfect conductors are *boundaries of zero surface impedance*; they support electric currents of finite strength when the tangential electric intensity is zero. At these boundaries the tangential magnetic intensity is different from zero. The natural counterpart is a *boundary of infinite impedance* at which the tangential magnetic intensity vanishes but the tangential electric intensity does not. The further generalization is a boundary with a given finite surface impedance which is defined as the ratio of two mutually perpendicular tangential components of the electric and magnetic intensity. The boundary may be isotropic, with its *surface impedance* the same in all directions; likewise, the boundary may be aelotropic. The surface impedance is defined as the ratio of the tangential components of E and H. Since it is necessary to adopt a convention regarding "positive directions" of E and H, these are so chosen that a right-handed screw will advance into the boundary if its handle is turned through 90° from the positive direction of E to coincide with the positive direction of H. In accordance with this convention the positive real part of the surface impedance is associated with an average flow of power into the boundary—that is, with a *passive boundary*. An *active boundary* is a boundary with a negative surface resistance; such boundaries may be used to represent idealized generators of electromagnetic waves and to eliminate from explicit consideration the internal mechanisms of these generators.

Field Equations

Thus far I have tried to present the ideas behind the physical and mathematical analysis of electromagnetic transmission phenomena. These are broader than the electromagnetic laws themselves and, with some superficial modifications, would apply to sound waves, for instance. There are two fundamental equations of transmission of an electromagnetic state, expressing Faraday's law of induction of an electromotive force by a magnetic displacement current and Ampère-Maxwell's law of induction of a magnetomotive force by an electric current. In their most general mathematical form the equations are

$$\oint E_s \, ds = -\frac{\partial}{\partial t} \iint \mu H_n \, dS,$$
$$\oint H_s \, ds = \iint \rho v_n \, dS + \iint g E_n \, dS + \frac{\partial}{\partial t} \iint \epsilon E_n \, dS, \qquad (10)$$

where the subscript s indicates components tangential to a closed path of integration and the subscript n designates components normal to any surface bounded by this closed path. Thus on the left we have "sums" of infinitesimal emf's and mmf's as we travel round some closed curve either on the surface of a wire or just in free space, and on the right we have total magnetic and electric currents linked with this curve. According to our present physical conceptions the magnetic current is always a displacement current defined as the time rate of change of magnetic flux or "displacement". Not that there is anything inconceivable about an actual flow of magnetic charge; it is simply that so far there has been no satisfactory evidence of its existence. In the mathematical analysis it has long been a custom to consider magnetic charges of opposite signs as if they existed; but this is merely for convenience.

The electric current, on the other hand, consists of three components: the *convection current* whose density is the product of the electric charge density ρ and the velocity v; the *conduction current* whose density is proportional to the electric intensity (the gE term in the above equation) and the *displacement current* defined as the time rate of change of the electric displacement. Strictly speaking, the conduction current is a convection current but of such a kind that it would be extremely awkward to think of it in terms of charged particles and their velocities.

At the same time the statistical result of the irregular movements of these particles can be expressed, for purposes of transmission of an electromagnetic state, as a continuous movement of charge encountering some resistance. There are, of course, such phenomena as resistance noise which are thus automatically excluded from consideration.

In general to these electromagnetic transmission equations we should add the dynamical equations of motion of electric charge; this is essential when dealing with vacuum tubes. But, in considering passive transmission systems, we either omit the convection current altogether, or else assume that the velocities of the charged particles are specified, and that the forces which they exert on each other are completely neutralized by the forces external to the field, in which case the convection current appears merely as an "impressed current".

Except for the above restrictions, equations (10) form a complete set; but for mathematical convenience two other equations are usually adjoined. These are

$$\begin{aligned} \iint \epsilon\, E_n\, dS &= q, \\ \iint \mu\, H_n\, dS &= 0, \end{aligned} \tag{11}$$

where the double integration is extended over a closed surface. The first of these equations states that the total electric displacement through a closed surface is equal to the net enclosed electric charge; the second denies the physical existence of magnetic charge. These equations can be derived from (10) and for this reason are not quite on the same footing with them.

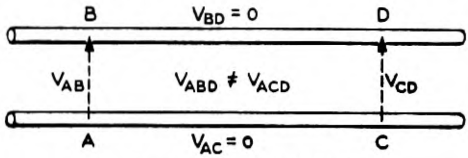

Fig. 8—A pair of parallel wires.

Equation (10) tells us that, except when the field is static, we cannot speak of the electromotive force or the voltage between two points without specifying the path along which we add up the elementary voltages. In fact, equation (10) gives us the difference between the voltages along two different paths connecting the same pair of points. To illustrate, consider a wave along a pair of perfectly conducting wires, Fig. 8. Voltages V_{AC} and V_{BD} along the wires are equal to zero; transverse voltages V_{AB} and V_{CD} are usually unequal; hence $V_{ABD} \neq V_{ACD}$.

If two points are infinitely close, then we can define the voltage unambiguously as the product $E_s ds$ of the electric intensity and the distance between the points. The difference between this voltage and the voltage along any other infinitesimal path is an infinitesimal of the second order, being dependent on the area enclosed by the two paths. In practice two points are sufficiently close if the distance between them is small compared with one quarter wavelength.

Since, except in electrostatics, we cannot speak of the voltage between two points without specifying the path, we cannot speak of the *potential* *difference.* In mathematical terms we should say that the differential voltage in a varying electromagnetic field is not an exact differential. To illustrate: $2x\,dx + 2y\,dy$ is an exact differential equal to $d(x^2 + y^2)$ and for this reason its integral depends only on the difference between the values of $(x^2 + y^2)$ at the end points of the path of integration; but $2x\,dx + 2x\,dy$ is not an exact differential and cannot be integrated except when y is given in terms of x so that the path of integration is prescribed.

If equations (10) are applied to infinitesimal closed curves, the following differential equations are obtained:

$$\text{curl } E = -\mu \frac{\partial H}{\partial t}, \qquad \text{curl } H = gE + \epsilon \frac{\partial E}{\partial t}. \tag{12}$$

The expressions curl E and curl H are merely the symbols for the maximum emf's and mmf's per unit area. These equations are not as general as (10) because they assume that E and H are continuous and at least once differentiable. The equations do not hold across the boundary between different media, where they have to be supplemented by the so-called *boundary conditions* which are obtained from (10). Equations (12) do not hold at a wavefront where E and H are discontinuous; there also we have to supplement them by appropriate boundary conditions, which connect the solutions on the two sides of the wavefront.

Analytic Functions

An advance of fundamental importance is made when the field intensities are represented by complex quantities $E\,e^{j\omega t}$ and $H\,e^{j\omega t}$ where ω is the frequency in radians. The equations become

$$\text{curl } E = -\,j\omega\mu\,H, \qquad \text{curl } H = (g + j\omega\epsilon)E, \tag{13}$$

and are thus freed from one independent variable, the time t. This does not mean that we have restricted our analysis to steady state fields; Fourier analysis supplies a general rule for passing from steady states to any state whatsoever. Computational difficulties are great but no greater than they would be in any other method.

A still more important advance is made when the field intensities are represented by $E\,e^{pt}$, $H\,e^{pt}$, where the *oscillation constant* $p = \xi + j\omega$ is a complex number. The equations become

$$\text{curl } E = -\,p\mu H, \qquad \text{curl } H = (g + p\,\epsilon)E. \tag{14}$$

The solutions of these equations are analytic functions of the complex variable p and a way is open for application of the theory of functions of a complex variable.

Thus if we write

$$E = \sum_{n=0}^{\infty} e_n p^n, \qquad H = \sum_{n=0}^{\infty} h_n p^n, \tag{15}$$

and substitute in (14), we obtain

$$\begin{aligned} \operatorname{curl} e_0 &= 0, & \operatorname{curl} e_{n+1} &= -\mu h_n, \\ \operatorname{curl} h_0 &= g e_0, & \operatorname{curl} h_{n+1} &= g e_{n+1} + \epsilon e_n. \end{aligned} \tag{16}$$

If these equations are solved subject to the prescribed boundary conditions, E and H will be expressed as power series in the oscillation constant p.

The function theory has already been used successfully in the *restricted circuit theory*; that is, in the theory of finite networks composed of ideal (independent of the frequency) resistances, inductances and capacitances. Likewise, some very general theorems have been established concerning any *physical* input impedance. Whereas the poles and zeros of a function can be anywhere in the complex p-plane, the poles and zeros of the input impedance of a passive system never lie to the right of the imaginary axis. This leads to a theorem to the effect that all poles and zeros on the imaginary axis are simple. The resistance components of the input impedance on the imaginary axis determine the reactance component and hence the complete impedance function except for a purely reactive impedance. The zeros and poles of an impedance occur always in conjugate pairs. These are some of the general theorems of impedance analysis. Not very long ago I came across an expression for the input impedance of a spherical antenna which was obtained by what appeared superficially as a straightforward conventional method; but as soon as I observed that some poles were situated to the right of the imaginary axis, I knew that the expression had to be false. The existence of poles in this region meant a possibility of oscillations which would increase indefinitely of their own accord.

The difference between finite and infinite networks consists in that the former possess a finite number of zeros and poles. All physical structures always possess an infinite number of such singularities; but a finite number of them may form a cluster in the vicinity of the origin, far removed from all other zeros and poles. When this happens we have a physical finite network. In a reactive network all zeros and poles lie on the imaginary axis. In a slightly dissipative system these zeros and poles move a little to the left of the imaginary axis. This happens, for instance, in the case of a thin antenna. The field in the vicinity of a thin wire is large and the radiated power is only a small fraction of the stored energy. The distribution of poles (the solid circles) and zeros (the hollow circles) is illustrated in Fig. 9. The zero frequency is always a pole for an open type antenna and a zero for a perfectly conducting loop antenna. As the frequency passes through a zero, the antenna impedance passes through a minimum. As the frequency goes through a pole, the antenna impedance passes through a maximum. The disposition of zeros and poles gives us a qualitative idea of the behavior of the impedance as the frequency varies.

As the radius of the antenna increases, the zeros and poles move farther to the left of the imaginary axis. At the same time some zeros and poles, which for a thin antenna are so far to the left that they have very little effect on the impedance, move nearer the origin. For spherical antennas the number of zeros and poles around the origin is considerably larger than for thin doublets.

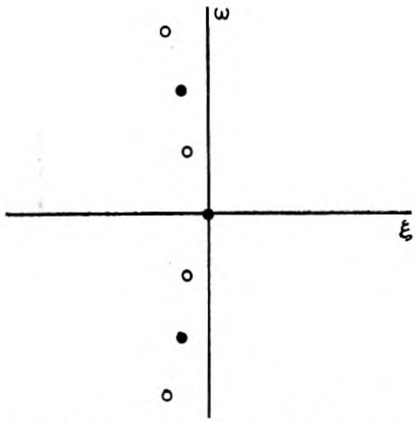

Fig. 9—Distribution of zeros and poles in a dipole antenna: solid circles represent poles; hollow circles zeros.

Circuit and Field Equations

In conclusion I should like to make a few remarks on the relationship between Kirchhoff's circuit equations and Maxwell's field equations. Are the former approximations; and, if so, in what sense? The answer depends on what is meant by Kirchhoff's equations, for their meaning has changed with passing years. It was exactly a hundred years ago that Kirchhoff stated his equations in a kind of postscript to his paper in Poggendorf Annalen; but he contemplated only the d-c networks. Yet nowadays we interpret these equations in such a way that they are applicable to a-c circuits. Some thirty years went by before Maxwell thus generalized the original Kirchhoff equations with the aid of Lagrange's concepts. Maxwell wrote his circuit equations (not the field equations) in a form applicable only to networks with a finite number of degrees of freedom; but nowadays we interpret these equations in such a way that we can apply them to one-dimensional transmission lines. In so doing we refrain from making approximations which we normally make when applying Kirchhoff laws to networks of lumped elements. In the latter case it is usual to ignore the inductance of the connecting leads or rather the inductance associated with the loop formed by the leads; but in the case of two-wire transmission lines the "connecting leads" constitute the entire network and the loop inductance is no longer ignored. In the case of lumped networks the capacitance between the connecting leads is normally neglected; but this capacitance is scrupulously included in the analysis of two-wire lines since in this case the "lead capacitance" is all the capacitance there is. And I have already referred to a recent contribution of Kron's who presented a three-dimensional network such that if we apply Kirchhoff's laws to it, we shall obtain Maxwell's field equations. The merger between the two points of view is now complete. In its growth, each theory has developed concepts peculiar to itself. The net result is that we are now in a position to understand electromagnetic phenomena better than ever.

The Evolution of the Quartz Crystal Clock*

By WARREN A. MARRISON

SOME of the earliest documents in human history relate to man's interest in timekeeping. This interest arose partly because of his curiosity about the visible world around him, and partly because the art of time measurement became an increasingly important part of living as the need for cooperation between the members of expanding groups increased. There are still in existence devices believed to have been made by the Egyptians six thousand years ago for the purpose of telling time from the stars, and there is good reason to believe that they were in quite general use by the better educated people of that period.[1] Since that period there has been a continuous use and improvement of timekeeping methods and devices, following sometimes quite independent lines, but developing through a long series of new ideas and refinements into the very precise means at our disposal today.

The art of timekeeping and time measurement is of very great value, both from its direct social use in permitting time tables and schedules to be made, and in its relation to other arts and the sciences in which the measurement of rate and duration assume ever increasing importance. The early history of timekeeping was concerned almost entirely with the first of these and for many centuries the chief purpose of timekeeping devices was to provide means for the approximate subdivision of the day, particularly of the daylight hours.

The most obvious events marking the passage of time were the rising and setting of the sun and its continuous apparent motion from east to west through the sky. The first practical measure of the position of the sun of which any record is known was the position or the length of shadows of fixed objects, resulting through a long period of development in the well-known sundial in its many forms. But the sundial was in no sense an instrument of precision and in no sense could be considered as a time *keeping* device. Even after the development which resulted in mounting the gnomon parallel with the axis of the earth, the largest, most elaborate, and most carefully made instruments could at best indicate local solar time. Furthermore, the sundial has value only in daylight hours and then only on

* The subject matter of this paper was given before the British Horological Institute in London on the occasion of the presentation of the Horological Institute's Gold Metal for 1947 to Mr. Marrison in consideration of his contribution toward the development of the quartz crystal clock. The present text is substantially as published in the *Horological Journal*.

days when the sun shines clearly enough to cast a shadow. These shortcomings became more and more important with advances in society and, for measuring duration, man soon began inventing timekeeping means that would work without benefit of the sun.

The evolution of timekeeping devices may be divided into three main periods, each employing a specific type of method, although overlapping to some degree in their applications, and characterized by increasing orders of accuracy.

A graphical representation of this evolution, indicating these three periods of development, and showing the relation between some of the major contributions to time keeping and the resulting accuracy of time measurement, is shown in Fig. 1. The methods employed chiefly during these three periods may be classified broadly as CONTINUOUS FLOW from the beginning up until about 1000 A.D., as APERIODIC CONTROL from then until about 1675 A.D. and as RESONANCE CONTROL from that time up to the present. Keeping in mind the logarithmic nature of the time and accuracy scales used in this graph, it can be seen readily that most of the advancement has been made in a very small part of the total time, corresponding to the resonance control epoch.

The Epoch of Continuous Flow

Perhaps due to a feeling that the passage of time was like the flow of some medium, the first time *measuring* devices were those depending on the flow of water into or out of suitable basins. It was recognized that, with an orifice properly chosen, the time required to fill or empty a given basin should be about the same on repetition, and hence was born the first reliable means for measuring time at night or on overcast days. A great variety of devices operating on this principle were constructed and used, some of the earliest having been made by the Babylonians and the Egyptians 3500 years ago.

Some of these water clocks, or clepsydra as they were called, had floats or other indicators which were intended to subdivide a unit of time into substantially uniform divisions. Others were constructed so that successive fillings of the basin would be counted or would operate a stepping device, associated with a dial or other indicator. Through the centuries great numbers of such devices were constructed, with some of the later ones having elaborate mechanisms for striking the hours or for animating figures of people or animals.

For use in places where water was not readily available and where sand was plentiful, clepsydra were developed that would operate with the flow of sand in much the same way as with the flow of water. The basic ideas were not greatly different, the substitution being merely one of expedience.

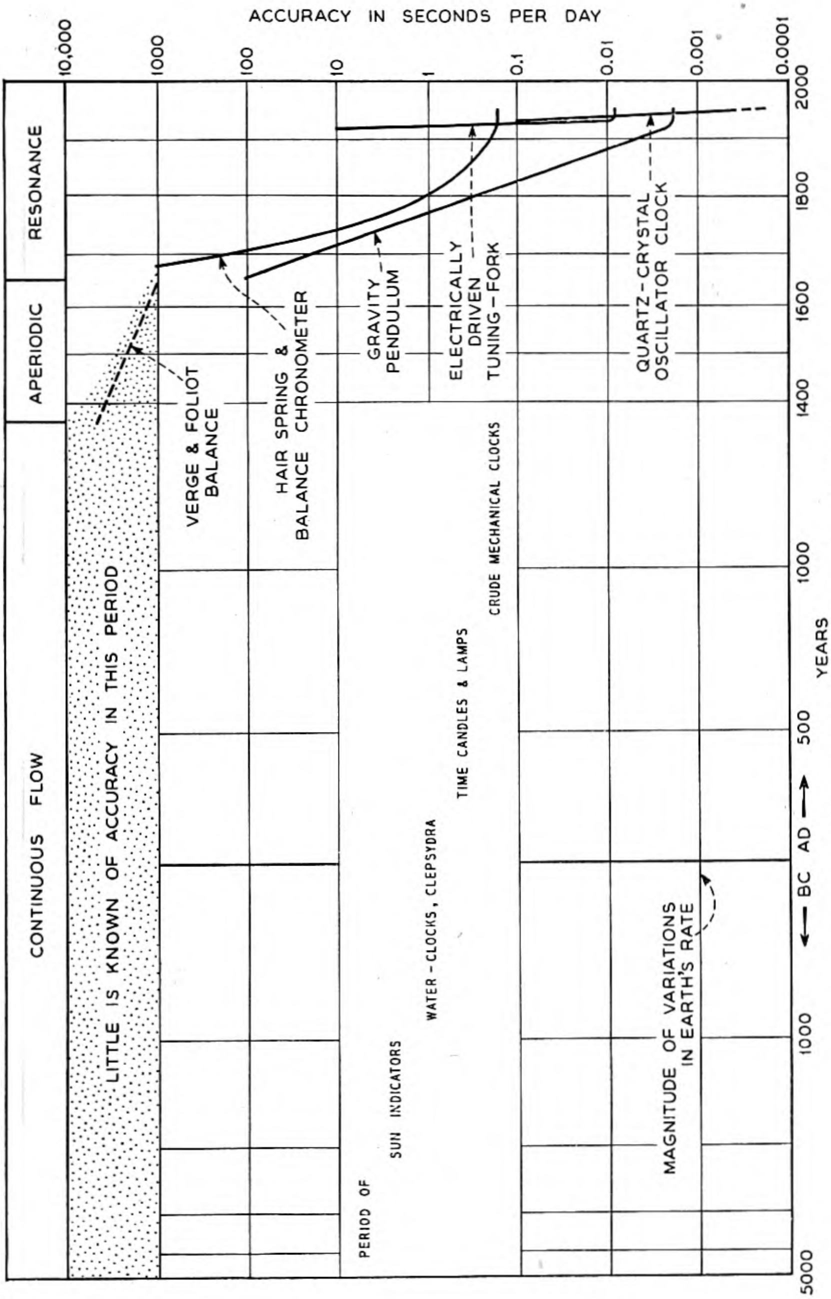

Fig. 1—The accuracy of timekeeping through history.

The hour glass, and its smaller counterparts, is one of the most convenient forms of this device and until quite recent times served a useful purpose where accuracy was of no great importance. The hour glass shown in Fig. 2 was used by a pastor in the early eighteen hundreds to determine the length of his sermons. The average variation among a set of ten one-hour determinations made recently with this glass was 3 minutes, or about 5 per cent.

The clepsydra that were designed to repeat and totalize an endless succession of cycles were especially adaptable to the measurement of extended intervals of time, although with very poor accuracy as we now think of it.

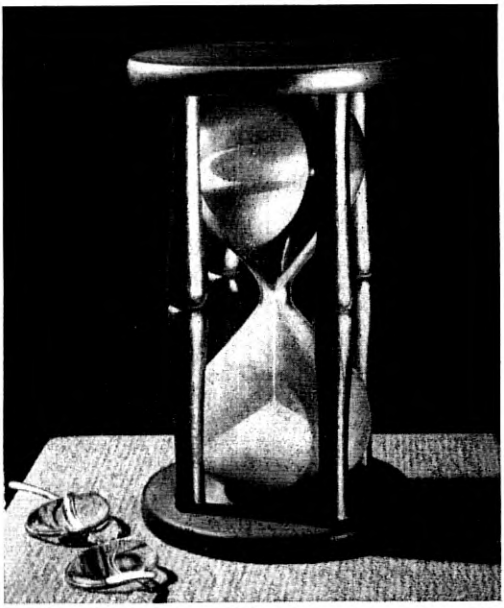

Fig. 2—Hour glass.

By suitable design any desired number of cycles could be made equal to the natural large unit, the day, so that any fraction of a day within the accuracy of a given instrument could be determined simply by counting off the number of cycles from a particular starting point such as sunrise, sunset, or high noon. It was possible with these devices to operate without calibration over periods of several days, although the cumulative error inevitably was very large.

An error of a few hours was of small importance in the days when the speed of communication and travel alike depended on pack animals or the caprices of the wind. And so, in spite of the inaccuracies of the water clocks and sand clocks, they served their purpose well through many centuries.

In fact, it was not until the tenth century A.D. that any really novel effort was made to improve upon them as timekeepers. The first efforts to improve upon them, making use of falling weights for motive power and various frictional devices to control the rate of fall, were not very successful because no satisfactory means were known to keep a friction-controlled device sufficiently constant for the job. Clocks so constructed were no better timekeepers on the whole than the traditional clepsydra. They had, however, the hope of compactness, and much ingenuity was exercised in their design over several centuries.

Also in the category of continuous flow devices should be mentioned the methods depending on the rate of burning, such as in time candles, time lamps and their numerous variations. Such timekeepers are not very accurate but are thoroughly reliable in dry, quiet places, even providing their own illumination at night. Such timekeepers are known to have been used before the tenth century A.D. and certain variations still are used by a few isolated tribes, especially in the tropics.

The Epoch of Aperiodic Control

In or about the year 1360 the invention of an escapement mechanism for controlling an alternating motion from a steady motive power, such as a suspended weight, was the first really important step in the history of precision clock development, and marks the beginning of the second major epoch in timekeeping evolution. The escapement in one form or another was soon applied in practically all timekeepers, the most outstanding example of an early application being a clock constructed by Henry De Vick for Charles V of France in or about the year 1360 A.D. and still in use—with extensive modifications—in the Palais de Justice in Paris.

This invention was important, not because De Vick's clock, or any of its immediate successors, were good timekeepers, but because this was the first time that vibratory motion in a mechanism was used deliberately to control the rate of a time-measuring device. All precision clocks depend in one way or another on using energy to produce vibratory motion, and on using the rate of that motion to regulate suitable dials and other mechanisms.

No simple improvement on De Vick's clock could ever have produced a precision clock in the modern sense, however, because the essential rate-controlling feature was still lacking. His invention consisted of the use of a verge escapement which produced oscillatory motion in a dynamically balanced member, known as a foliot balance, having essentially only moment of inertia and friction. The rate of oscillation, therefore, depended to a large extent on the applied force exerted by the falling weight through a train of wheels, and upon the friction of the escapement parts and of the oscillating member itself.

This sort of operation is known sometimes as relaxation oscillation and appears in many forms. In the clock, the rate-controlling feature depends upon the length of time it takes a member having a given moment of inertia to move from one angular position to another under a given applied torque. Thus, the rate depends to first order on the applied torque.

Although De Vick's clock was one of the most famous in all history, it was not because of its good record of timekeeping. In its original form, it is said that it often varied as much as two hours a day from true time. Outwardly, this clock on the Palais de Justice appears about the same as it did originally, but the "works" have been modernized and it keeps much better time now.

The history of timekeeping during the next three hundred years consisted mainly in improvements and in a great variety of applications of the principles contained in De Vick's clock. During this period great numbers of clocks of all sizes, from tower clocks to portable table clocks were made, controlled by various forms of the crown wheel, verge and foliot balance. All of these timekeepers belong to the class that we have just called aperiodic. Their accuracy, in general, was still poor and the indicator on their dials consisted of but one hand—the hour hand. It was not until the invention and application of the pendulum that the next major improvement was born in timekeeping.

The Epoch of Resonant Control

All that has been said so far is a prelude to the shortest but by far the most productive epoch in timekeeping, that of resonant control. The heart of every precision clock is an oscillatory device which depends upon *resonance* for its constancy of rate. The history of precision clock development consists largely of the choice and design of stable resonant elements and of devising means for using them so that as far as possible their inherent properties alone control their rates of oscillation. Once in stable oscillation, it is only necessary to control the indicating of dials and other suitable mechanisms in order to constitute a complete clock.* Presumably this can always be done, but in some cases it is more convenient to do than in others, as will appear.

The resonant element may be any of a wide variety of forms, mechanical or electrical, all characterized by the single property that, if deformed from a rest condition and released, the stored energy is transformed back and forth from potential to kinetic at a rate depending chiefly on the effective mass and the effective stiffness, or other like properties, a small proportion

* Encycl. Brit. 14th Ed. "A clock consists of a train of wheels, actuated by a spring or weight or other means, and provided with an oscillating governing device which so regulates the speed as to render it uniform."

of the energy being lost in internal friction at each oscillation. Some resonant elements which have been used in timekeepers are illustrated in Fig. 3.

The simplest appearing of all these is that of a mass, M, supported by a spring with stiffness, S. From the equation of motion

$$Sx = M \frac{d^2x}{dt^2}$$

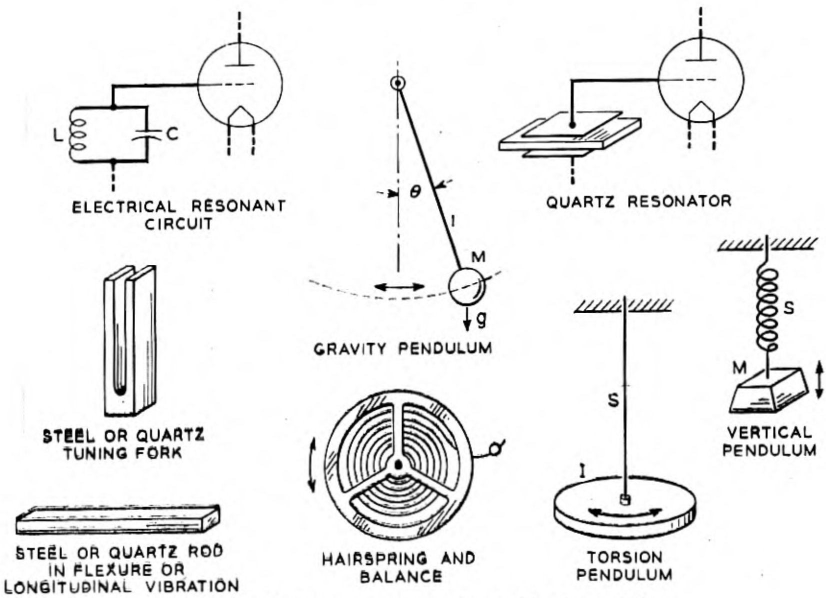

Fig. 3—Typical resonant elements used in timekeeping.

the period of oscillation may be derived simply and is found to be

$$T = 2\pi \sqrt{\frac{M}{S}}$$

Similarly for the simple electrical resonant circuit where current flowing in an inductance, L, behaves like a mass, and current flowing in a condenser, C, behaves like the reciprocal of a stiffness, the period may be written.

$$T = 2\pi\sqrt{LC}$$

Similar expressions are derivable for the periods of oscillation of all simple oscillating systems, including the pendulum for which the period (for small amplitudes) is given by

$$T = 2\pi \sqrt{\frac{\ell}{g}}$$

where ℓ and g are respectively the length and gravity expressed in the same system of units, for example, the c.g.s. system.

When any such resonant element is strained from its rest condition, and released, it will oscillate with gradually decreasing amplitude until all of the stored energy has been dissipated in internal friction or resistance, and in the friction or resistance of the coupling with the supports. In general, the resulting amplitude of free oscillation may be given as

$$A = A_0 e^{-kt} \sin pt$$

the graph of which is a damped sine wave. The rate of free oscillation, p, is dependent chiefly on the effective mass and stiffness and to a small degree on the effective resistance of the element, while the rate of loss of amplitude, that is, the logarithmic decrement, k, is dependent on the ratio of effective resistance to effective mass.

If the resistance could be made exactly zero, such a motion once started would continue forever and its rate would be controlled wholly by the effective mass and stiffness of the resonant element. Actually, of course, such a condition cannot be realized in practice but, by the selection of suitable materials and environment, and by special control means, it is possible to approach very closely to the ideal condition by causing the oscillation to be maintained *almost* as though there were no damping.

The evolution of precision timekeeping, whether consciously or not, has centered around the study and development of these two ideas: to discover resonant elements whose rate-determining properties are inherently stable, and to discover means for sustaining them in oscillation as though they had no effective resistance; or in employing means to circumvent or to compensate for any such resistance. The high precision of rate control that can now be obtained has been the result largely of developments in these two categories.

The Pendulum

The gravity pendulum was the first truly resonant element to be used to regulate the rate of a clock and for nearly three centuries maintained the supremacy for precision measurements of time. The pendulum was more a discovery than an invention, the popular story of its origin being that, while still a youth of seventeen years, Gallileo Galilei chanced to notice that a hanging lamp in the Cathedral of Pisa seemed to swing at the same rate regardless of amplitude. This he confirmed approximately by comparison with his pulse, and later made an extensive study of the isochronism of swinging bodies. These studies were in progress as early as 1583. Nearly sixty years later Gallileo described to his son Vincenzio how a pendulum could be used to control a clock, but no concrete result of this advice is known to have been made at that time. A working model of this clock, made subsequently from the original drawings, is on exhibition in the South Kensington Science Museum, London. The first authentic record of the actual use of a pendulum in a clock is attributed to the great Dutch scientist, Christian Huygens, who produced his first pendulum clock in 1657. This was described by him in the *Horologium* in 1658.[2]

The performance of pendulum clocks was so good that almost immediately clocks of all other types were modified to include a pendulum. So complete was this transformation that very few unmodified clocks are now in existence which antedate the first application of the pendulum to timekeeping. This, as a matter of fact, is one of the major reasons that so little is known about the actual mechanisms used in mechanical clocks that were made before the introduction of the pendulum.

The subsequent history of pendulum clock development is well described in numerous books and papers and covers a wide field. Only those factors that relate the pendulum to other means of rate control will be discussed in the following.

The properties of a pendulum which make it such a good timekeeper are easily seen from a study of the forces on the bob as illustrated in Fig. 3. Since these forces must be in equilibrium at all times we may write (assuming no friction)

$$Mg \sin \theta = M\ell \frac{d^2\theta}{dt^2}$$

The nearly isochronous property of the pendulum is contained in this relationship since the period, on solution, is

$$T = 2\pi \sqrt{\frac{\ell}{g}} \left(1 + \frac{1}{4} \sin^2 \frac{\theta}{2} + \frac{9}{64} \sin^4 \frac{\theta}{2} + \cdots \right)$$

where θ is the maximum semi-amplitude of swing expressed in radians. When this arc is small the period approaches a minimum. For small angles the natural period depends almost wholly on the ratio of ℓ to g and the stability of T depends chiefly upon the constancy of ℓ and g. Figure 4 shows the relation between period and the arc of swing, expressed as seconds per day departure from the theoretical rate for zero arc.

The sum of all the terms that depend upon powers of $\sin \theta/2$ is known as the circular error, relating to the fact that the bob is constrained to move on the arc of a circle. It was shown theoretically by Christian Huygens[3] that if the bob could be constrained to move on the arc of an epicycloid it would be truly isochronous, that is, the period would be completely independent of its amplitude of motion. It is of interest to note at this point that in no other resonator used for precision timekeeping is there the direct counterpart of circular error, for in all other cases the restoring force varies linearly with displacement in the region of operation and not as a sine function of it.

In the early stages of pendulum clock development it was not necessary to consider the arc error because other errors were of greater magnitude. But it is by no means a negligible factor, and in all precision timing by pendulums it must be accounted for, either by allowing for an arc correction, as is done commonly in geodetic survey work, or by keeping the arc small and precisely controlling it. According to F. Hope-Jones[4], referring to the master pendulum in the famous Synchronome free-pendulum clock: "A variation of only 0.01 mm. in the excursion of the bob or 2 secs. of arc will by circular error alter the rate by 0.00145 sec. per day,—and if it arose un-

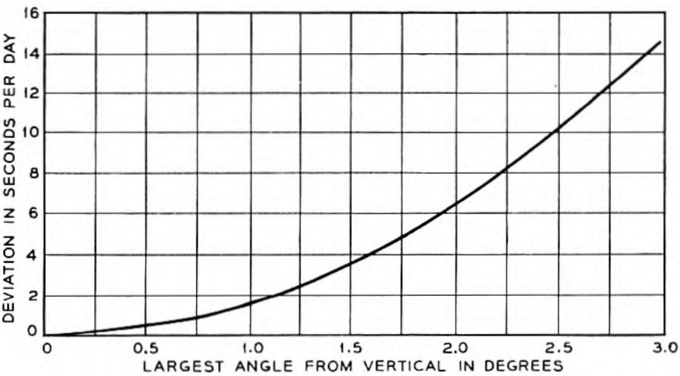

Fig. 4—Relation between arc and rate of pendulum.

perceived and was steadily maintained, it would produce an accumulated error of half a second in a year, so the necessity for this close observation is obvious."

The control of arc has almost invariably been accomplished by keeping constant the amount of energy applied per swing so that the actual amplitude obtained is that value for which all of the applied energy is dissipated in the pendulum system. In a sense this method of control of arc puts a penalty on improvements in design that would reduce the friction, because the better a pendulum becomes in this respect the less stable becomes the arc control. Since even the best pendulums develop unexplainable small changes in arc, it has been common practice in some observatories to record the arc frequently and to make allowance for changes in it when making the most precise time determinations.

The inherent constancy of rate of a pendulum, with small or constant amplitude of swing, depends to the one-half power on the stability of ℓ/g.

The changes in ℓ and g are quite independent of each other and so can be treated separately. Other factors that will be described also affect the rate, and it is the object in every precision clock design to reduce such variable effects to the absolute minimum.

Some control can be exercised over every factor except g, which remains a property of space and is dependent only on the proximity of matter and on acceleration. As is well known, the value of g varies over the surface of the earth due chiefly to its deviation from spherical shape, and because of the uneven distribution of matter. It also varies with vertical displacement or tides at any location to such an extent that a gravity clock that keeps accurate time at ground level will lose a second a day or more in a tall building. Actually, it is now possible to chart variations in g with high precision through measurement of the rate of a pendulum clock against a standard whose rate does not depend upon gravity.

Most of the factors that can affect ℓ have been studied critically and means have been found to reduce them to very small effects. The chief source of variation was at first the temperature coefficient of the pendulum rod. With ordinary metals the rod expands from 10 to 16 parts in a million per degree C, causing a proportionate change in rate of half this amount, corresponding to from one-half to two-thirds of a second per day. Many ingenious means were developed to reduce this effect, starting with George Graham's mercury-filled bob in 1721, followed by John Harrison's grid-iron pendulum in 1726, and a great number of variations on these ideas, all depending on the differential coefficient of expansion of dissimilar materials.

About the year 1895, Charles Edouard Guillaume of Paris developed an alloy, consisting chiefly of nickel and iron, which he called Invar, because it had a very small temperature coefficient of expansion, from which pendulum rods could be made. The use of this material made it unnecessary to resort to complex compensated pendulums with their own inherent instabilities, and the accuracy of timekeeping was increased another step. The residual temperature effects could be measured readily, and compensated if desired, by the use of a small bar of aluminum attached to the bob.

Some other important factors that affect the working length of a pendulum are the aging of the supporting rod, the "knife edge" or spring used for the suspension, the nature of the main supporting column or frame, and some atmospheric effects caused by changing temperature and pressure. In the most accurate pendulum clocks, the atmospheric effects are greatly reduced by mounting the pendulum in partially evacuated, hermetically sealed enclosures which can be temperature controlled. All of these factors and many others are discussed in every good treatise on accurate pendulum clocks. They are mentioned here chiefly for the purpose of comparison with like factors in the quartz crystal clock and to show how in many cases the difficulties introduced by such factors may be more easily and more positively controlled.

In every primary clock mechanism the resonant governing device must be sustained in oscillation, and the manner in which this is done has a strong bearing on its rate regardless of the quality of the governing element. The basic requirements are the same for any kind of oscillator, whether a pendulum, an electrically resonant circuit comprising inductance and capacitance, a steel tuning fork, or a quartz crystal resonator. The requirements were first stated for the case of the pendulum by Sir George Airy in 1827 and it has always been the aim in the design of every good pendulum driving means to satisfy Airy's condition.

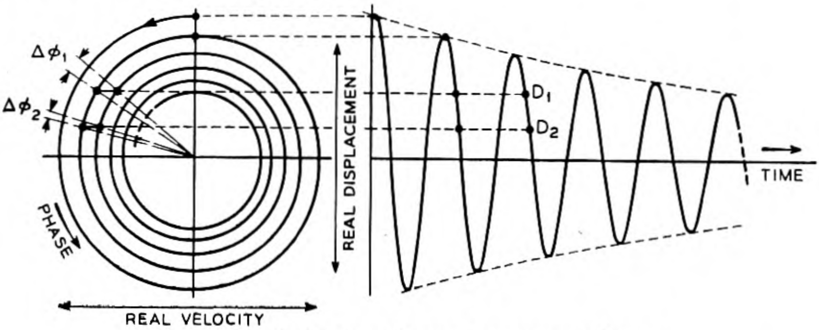

Fig. 5—Amplitude-phase diagram for resonant element.

This condition is conveniently illustrated by the diagram of Fig. 5 which shows the two most familiar representations of damped sinusoidal motion. In order to provide a convenient scale in the drawing an impractically large damping is represented, corresponding to a Q of 20. The Q of a resonant circuit is related to the logarithmic decrement, δ, by the relation $Q\delta = \pi$. The factor δ is the logarithm, to base $e = 2.718 \cdots$, of the ratio of the amplitudes at any two successive periods. It should be noted that the Q of a good electrically resonant circuit is in the order of 200, that of a good pendulum from 10,000 to 100,000 and that of a good quartz resonator from 100,000 to 5,000,000. The significance of these higher values of Q will be evident from the following discussion.

In Fig. 5 the damped sine wave shown corresponds, point by point, to the phase diagram, which is simply a logarithmic spiral. By suitable choice of scale the spiral can be interpreted to represent either the amplitude or the velocity—in which case the real amplitude is vertical and the real velocity horizontal. In this representation the velocity is shown maximum when the amplitude is zero, which is a very close approximation to fact for all practicable values of Q. The discussion will center on the velocity spiral.

Let us assume that the pendulum is sustained in oscillation by a succession of short impulses, one for each swing applied at some phase angle φ_1. If the impulse is really short, the *velocity* will be increased to the value that the pendulum had when it occupied the same *position* during the last swing. This change of condition is represented by the short horizontal path on the velocity-phase diagram and, as indicated, is accompanied by an advance in phase $\Delta\varphi_1$. This can be interpreted as meaning that the period of a pendulum sustained in oscillation in this way is reduced from its natural period in the ratio of $\frac{2\pi - \Delta\varphi_1}{2\pi}$. It is obvious from the diagram that $\Delta\phi_1$ becomes smaller and that this ratio approaches unity as the phase of the applied impulse approaches that of the maximum velocity—that is, when the pendulum is in the center of its swing; and this is Airy's condition. It is clear also that if the impulse is applied after (instead of before) the instant of maximum velocity, the period will be correspondingly increased. From the geometry of the figure, it can be seen that, in the neighborhood of the optimum condition, the deviation from natural period is very closely proportional to the amount of the phase departure.

The closeness of spacing of the turns of the spiral depends directly on the Q of the resonant element. For a Q of 200, the turns will be packed ten times closer than shown, and the corresponding $\Delta\varphi$ will be only one tenth as great, other conditions being comparable. For a Q of a million or more, $\Delta\varphi$ becomes very small indeed, especially when φ is properly chosen—and the *variation* in $\Delta\varphi$, which is a measure of the variation in rate due to the driving means, may be made vanishingly small.

The importance of the above properties to timekeeping depends upon how well conditions can be set up to realize them. At first wholly mechanical means were employed and, with the advent of the dead-beat and detached escapements and by careful design and operation, quite remarkable performance was obtained.

A new approach in timekeeping methods was introduced by Alexander Bain[5] in 1840 when he first used electrical means for sustaining a pendulum in oscillation. The importance of Bain's invention of the electric clock is indicated by a long controversy over the priority of the invention with Charles Wheatstone, who was working along similar lines at the same time as a by-product of his extensive researches on the electrical telegraph. A brief story of this controversy entitled "The First Electric Clock" was written for the one-hundredth anniversary of Bain's invention[6]. The first electric pendulum clocks could not compare in accuracy with the best mechanically driven pendulums of the period but, in spite of a great deal of initial skepticism on the part of those brought up in the mechanical tradition, electrical maintenance and control has been applied in the most accurate pendulums in the world.

The free-pendulum clock makes use of the idea, first proposed by Rudd, of allowing a master pendulum to swing free of all sustaining or other mechanism for a considerable number of periods and of imparting to it, after each group of free swings, a single impulse large enough to maintain the next equal number. The advantage is that no friction effects of driving mechanism are coupled to the pendulum except during that minimum time required to impart energy to it. Actually, in theory, the phase error introduced by one large impulse after n free swings is exactly the same as the sum of the phase errors for n small impulses. That can be deduced from the phase diagram of Fig. 5. But experience has shown that a pendulum is actually more stable when the sustaining mechanism is detached from it the greater part of the time.

The Synchronome free-pendulum clock includes also the basic idea of the gravity remontoir first applied by Lord Grimthorp (then Sir Edmund Beckett Denison) in the design of the mechanism of Big Ben, London, constructed in 1854—and still in continuous operation. The ingenious application of these principles and the electrical means devised by F. Hope-Jones and W. H. Shortt for its accomplishment have resulted in the construction of the most accurate pendulum clocks in the world by the Synchronome Clock Company of London. The history and development of the free-pendulum clock is elegantly described by F. Hope-Jones in his book on Electric Clocks[7].

The predominant characteristics of a pendulum resonator, as used in a clock, have just been discussed in order to show the parallel between them and the properties of other resonant systems. It will be shown how some of the factors that have been troublesome in the development of pendulums have been rather easily taken account of in other types of control devices and in particular in the quartz crystal clock.

The Evolution of Electric Oscillator Clocks

It almost never happens that a result of any considerable value is obtained at a single stroke or comes through the efforts of a single person. More often even the most important advances come as the climax of a long series of ideas which have accumulated over a period of years until the next step becomes almost self-evident and is accomplished either through the necessity for a new result or as a logical next step.

This was preeminently the case in the crystal clock development and involved the putting together of a considerable number of ideas that had been accumulating through a century or more of related activity. The chain of events which led eventually to the crystal clock followed a course quite independent of pendulum clock development, although parallel with it, and meeting it from time to time on the way. From the start, it involved the use of resonant elements whose frequencies do not depend upon gravity for controlling the frequency of oscillations in a positive feedback amplifier. From a rather simple beginning, taking advantage of a series of discoveries and inventions through about a century of progress, there has evolved a clock whose stability is comparable with that of astronomical time itself, as heretofore defined in terms of the earth's rotation, and having a versatility far exceeding all other existing means for the precision measurement of time.

Electric Oscillators

The first recorded experiments that relate directly to this development were those of Jules Lissajous[8] who, in 1857, showed that a tuning fork can be sustained in vibration indefinitely by electrical means, using an electromagnet and an interrupter supported by one of the prongs. The idea of using an interrupter to sustain vibration was not new with Lissajous, but had been invented by C. G. Page[9] and described by him as early as April 1837, to obtain a regularly interrupted electric current. Credit for this important invention is often given to Golding Bird[10] or Neeff[11] who evidently were working along similar lines concurrently although quite independently of each other. Page, Golding Bird and Neeff were all medical doctors and evidently were interested in their devices more for their therapeutic interest than for the general scientific value, since "galvanic" electricity was attributed at that time with marvelous healing powers.

Lissajous was probably the first to make use of the idea for accurate measurements of rate, being a prolific experimenter in mechanics and acoustics, and the originator of the famous method bearing his name for the study of periodic motions. Indeed, the electrically operated fork was developed especially for use as a standard to be used in studying the rates of other vibrators. In principle, the electrically operated fork is like the pendulum drive of Alexander Bain, except that the rate of vibration in this case is not a function of gravity but for the most part is controlled by the effective mass and elastic stiffness of the vibrating member.

The tuning fork itself was invented in 1711 by John Shore, a trumpeter in Handel's orchestra[12], and was developed to a high state of perfection by the great instrument maker and physicist of Paris, Rudolph König. To establish an accurate standard of pitch for calibrating these forks König developed what he termed an "absolute" method for the determination of frequency. This consisted of a tuning fork having a frequency of 64 vibrations per second, with delicate mechanical means, similar to a clock escapement, for sustaining the fork in vibration and for counting the number of vibrations over any desired interval of time. For this purpose, the escapement mechanism was geared to the hands of a clock, so that when the fork had its nominal frequency the clock would keep correct time. Dr. König credits the invention of the fork-clock to N. Niaudet[13] in these words:

"Cette disposition avait été realisée pour la première fois dans l'horloge à diapason que N. Niaudet fit présenter à l'Académie des Sciences le 10 décembre 1866, et que à figuré aux expositions universelles de Paris 1867 et de Vienne 1873."*

Thus, as early as 1866, the essential elements had been developed separately from which a clock of the electric oscillator type could have been constructed. But it was not until more than half a century later, when there was more apparent need for such a clock, that it was actually realized. It was chiefly for the purpose of studying temperature coefficients and like properties of tuning forks that König constructed and used his famous mechanical fork-clock. There is no evidence that there was at that time any idea of using a fork-clock as a timekeeper.

It was for the purpose of making still more precise studies of the properties of tuning forks that H. M. Dadourian[14] in 1919 made use of the phonic wheel motor for the first time for counting the number of cycles executed by a fork over an extended period of time to measure its rate. By means of a chronograph the time interval corresponding to the total of a very large number of periods could be measured precisely in terms of a standard clock, thus providing a direct "absolute" measure of fork rate. For this he found already invented for him all of the essential component parts, including the fork with electromagnetic drive, and the phonic wheel motor.

The phonic wheel motor, which in some modified form is an essential part of nearly all oscillator clocks, was invented by two investigators, apparently quite independently and for entirely different purposes. The first published reports of each appeared in 1878.

The first of these is an American patent that was granted on May 7, 1878 to Poul La Cour[15], a Danish telegraph engineer. The application was filed in Washington on April 9 of the same year, and described a fork-controlled impulse motor similar to those still used in many modern synchronous clocks. The other publication was a report in Nature for May 23 of the March 30 Physical Society Meeting. In this, Lord Rayleigh described a motor which he developed to measure the frequency of sound by a stroboscopic method.[16] Both of these original disclosures indicated a

* "This apparatus was realized for the first time in the fork-clock which N. Niaudet described at the Academy of Sciences on December 10, 1866, and which was shown at the expositions of the University of Paris in 1867 and the University of Vienna in 1873."

considerable amount of previous study, even including the fluid-filled flywheel to reduce hunting. It may be impossible at this time to know who actually put in motion the first phonic wheel motor.

Difficulties inherent to contact-controlled devices prevented the development of highly accurate fork standards of this type, and there is no evidence so far that any thought had been given to the use of a tuning fork as a timekeeper.

The method of using a microphone instead of a contact was proposed by A. and V. Guillet [17], in 1900 and has been used considerably in frequency standards of moderate accuracy, but that too had limitations which made it impossible to utilize fully the inherent stability of a good tuning fork.

The Use of Vacuum Tubes

The first opportunity for really precise control of the frequency of a mechanical vibrating system, and the next step in the oscillator clock evolution, came with the invention of the thermionic vacuum tube at the turn of the century. The development of the vacuum tube has been a more or less continuous process[18] starting with the studies of electrical conduction in the neighborhood of hot bodies by Elster and Geitel, Edison, and Fleming, and later developed into the first practical devices by Fleming[19] and DeForest[20] in England and America respectively. The first patent for such a device, a two-element tube, was issued to J. A. Fleming in 1904.[21] The first patent on a tube containing three elements and suitable for use as an amplifier was issued to Lee DeForest in 1907.[22]

The vacuum tube as an amplifier found almost immediate and widespread application in telephony and, next to the basic telephone elements, was the most important single factor contributing to long distance communication. For this purpose large amounts of amplification were required. Very often in the operation of early amplifiers, enough signal from the output would somehow get coupled into the input circuit to make the entire circuit break into oscillation on its own account at some frequency for which the amplifier and feedback circuit were particularly efficient.

Although this was very annoying in an amplifier, it led naturally in 1912 to the invention of the vacuum tube oscillator, consisting essentially of an amplifier with coupling between the output and the input and some definite means for regulating the frequency of oscillation. The first to seek patent protection in vacuum tube oscillators were Siegmund Strauss[23] in Austria, Marconi Company in England[24], A. Meissner in Germany, and Irving Langmuir, E. H. Armstrong and Lee DeForest[25] in America. Many specific forms have since been invented and widely used, some of the more familiar types being associated with the names of Colpitts, Hartley and Meissner.

With the vacuum tube oscillator controlled by electric circuit elements, it would have been possible immediately to operate a clock by means of a phonic wheel motor. Even if this had been done, however, the accuracy would not have compared very favorably with that of good mechanical clocks of the period. This is because the rate-controlling element of such oscillators was subject to large changes due to temperature and aging, and because means were not yet known for avoiding the effects of tube and other variables on the resulting frequency.

The next important step in our evolution was the use of the vacuum tube to sustain the vibration of a tuning fork. This may be considered either as an improvement on the contact-driven fork by the substitution of a vacuum tube relay device instead of the contact, or as an improvement on the vacuum tube oscillator by the substitution of a mechanical resonator for the electrical resonant element. This achievement was first announced by Professor W. H. Eccles[26] in April or May, 1919, and was followed on June 20 by a note by Eccles and Jordan[27] in the London Electrician. Meanwhile, on June 16 of the same year, a similar announcement appeared in Comptes Rendus by Henri Abraham and Eugene Block[28], showing that parallel developments were in progress in both England and France. However, Eccles and Jordan in discussing their work at the National Physical Laboratory stated: "Several instruments of this kind have been set up and used during the past 18 months." From this, we may imply that they had vacuum tube driven forks in operation early in 1918.

One of the chief advantages of the use of the vacuum tube to sustain oscillations in a mechanical system is that the variable friction of the contact mechanism is avoided. Previously this had been one of the main causes of instability. With the new method it became possible to operate in a wide frequency range, continuously, and at small amplitude, and to deliver alternating currents of approximately sine wave form and having more constant frequency than heretofore had been possible. The judicious use of a vacuum tube in delivering power to sustain the vibration of a resonator is analogous to the ideal of the so-called free pendulum but may be utilized more effectively in freeing the resonator from disturbing influences associated with the driving means.

Another important advantage, which, however, was not realized immediately, is the ease with which the phase of the driving force applied to a mechanical vibrator can be adjusted for greatest frequency stability. In a manner analogous to the pendulum, in which it was shown that the rate is least affected when the driving impulse is applied at the instant of maximum velocity, the current delivered to the driving electromagnet and hence the force applied to the vibrating element, should be in phase with the *velocity* of that element. In the vacuum tube oscillator, it is a relatively simple matter to design the feedback circuits to meet this condition very accurately.

In 1921 and 1922 Eckhardt, Karcher and Keiser[29, 30] described the development of a precise fork and vacuum tube driving means, pointing out the following uses: "As a sound source; as a small scale time standard; as a current interrupter; as a synchronizer." The chief emphasis seems to have been on the second item because in the same year Eckhardt described a high-speed oscillograph camera using the same fork as a precise timing device. The study and improvement of the tuning fork oscillator were carried on continuously and soon such oscillators were used in several national physical laboratories and commercial research institutions as standards of frequency and time interval.

The next two reports of progress appeared in 1923, one by D. W. Dye of the National Physical Laboratory in Teddington, and the other by J. W. Horton, N. H. Ricker and W. A. Marrison of Bell Telephone Laboratories, New York City. Both of these papers disclosed work done over a period of two or three years and described apparatus that had been in operation for a considerable period. Dr. Dye employed a 1000-cycle steel tuning fork and a phonic wheel motor operating synchronously from it with a gear reduction and cam to produce periodic electrical signals which he compared with a clock by means of a chronograph[31]. Horton, Ricker, and Marrison used a 100-cycle steel fork, a synchronous motor with a gear reduction to produce electrical impulses at one-second intervals, and a clock mechanism operating directly from these signals[32]. This appears to be the first time that a vacuum tube-controlled oscillator was ever used to operate a complete clock mechanism. Shortly thereafter, a clock was built in which the 100-cycle motor was geared directly to the clock mechanism instead of operating through a stepping device. A contacting device was retained, however, for the purpose of making precise time measurements.

For precise measurements of rate over long time intervals, means were provided to compare the seconds pulses controlled by the synchronous motor directly with time signals received by radio from the Naval Observatory. To facilitate these comparisons, a two-pen siphon recorder was built by means of which the time marks were laid down side by side on a moving strip of paper in such a way that accurate subdivisions of a second could be made on any part of the record.

This same two-pen recorder and 100-cycle fork time standard was used during the total solar eclipse of January 24, 1925 to time the progress of the moon's shadow as observed at a number of stations in the path which were all connected by a round-robin telegraph circuit, through the Bell

Telephone Laboratories' headquarters in New York City [33,34]. A photograph of the original records is reproduced in Fig. 6. This is believed to be the first time that a vacuum tube oscillator type of time standard was ever used in the service of astronomy.

During the following ten years a great number of improvements were made in tuning fork oscillators and they became widely used as precise frequency standards. The Bell Laboratories' 100-cycle fork standard was mounted in a container which could be sealed at constant pressure or vacuum. It was carefully temperature controlled and provision was made to keep the amplitude within prescribed limits. In describing this improved standard[35], comprising a synchronous motor geared directly to a clock mechanism, the authors Horton and Marrison made the following statement:

"During tests on this frequency standard, it was found that it constituted a far more reliable timekeeper than the electrically maintained pendulum clock which was used to obtain the data already published. The pendulum clock was, therefore, dispensed with and all measurements of the rate of the fork are now made by direct comparison with the mean solar day as defined by the radio time signals sent out by the U. S. Naval Observatory."

In all fairness to the pendulum clock in question, it should be stated that the laboratory was situated on the seventh floor of a building adjoining a busy street and so was continually subject to vibration from traffic, wind, and other changing conditions. Disturbances of this sort have little or no effect on standards of the electric oscillator type but seriously impair the performance of most high precision pendulum clocks. The relative immunity of the oscillator standard to change of position and shock has an important bearing on its value in many applications.

Probably the most precise tuning fork controlled time and frequency standards ever constructed were those developed in the National Physical Laboratory at Teddington, as a continuation of the work begun there by Professor Eccles and carried forward by Dr. Dye and his staff. A report by D. W. Dye and L. Essen in the Royal Society Proceedings in 1934[36] described a number of refinements in the fork and method of use some of which had been suggested by Dr. Dye as a result of his studies ten years earlier. Among these was the use of *elinvar* in the construction of the forks in order to reduce the effect of variable temperature on the frequency. Elinvar is a nickel steel containing about twelve per cent of chromium, which on proper treatment has a small or zero temperature coefficient of elasticity. It was invented by Charles Edouard Guillaume[37,38] and was further studied by P. Chévenard[39,40]. The excellence of the N.P.L. fork standard can be appreciated readily from the conclusion of the 1934 report which states in part:

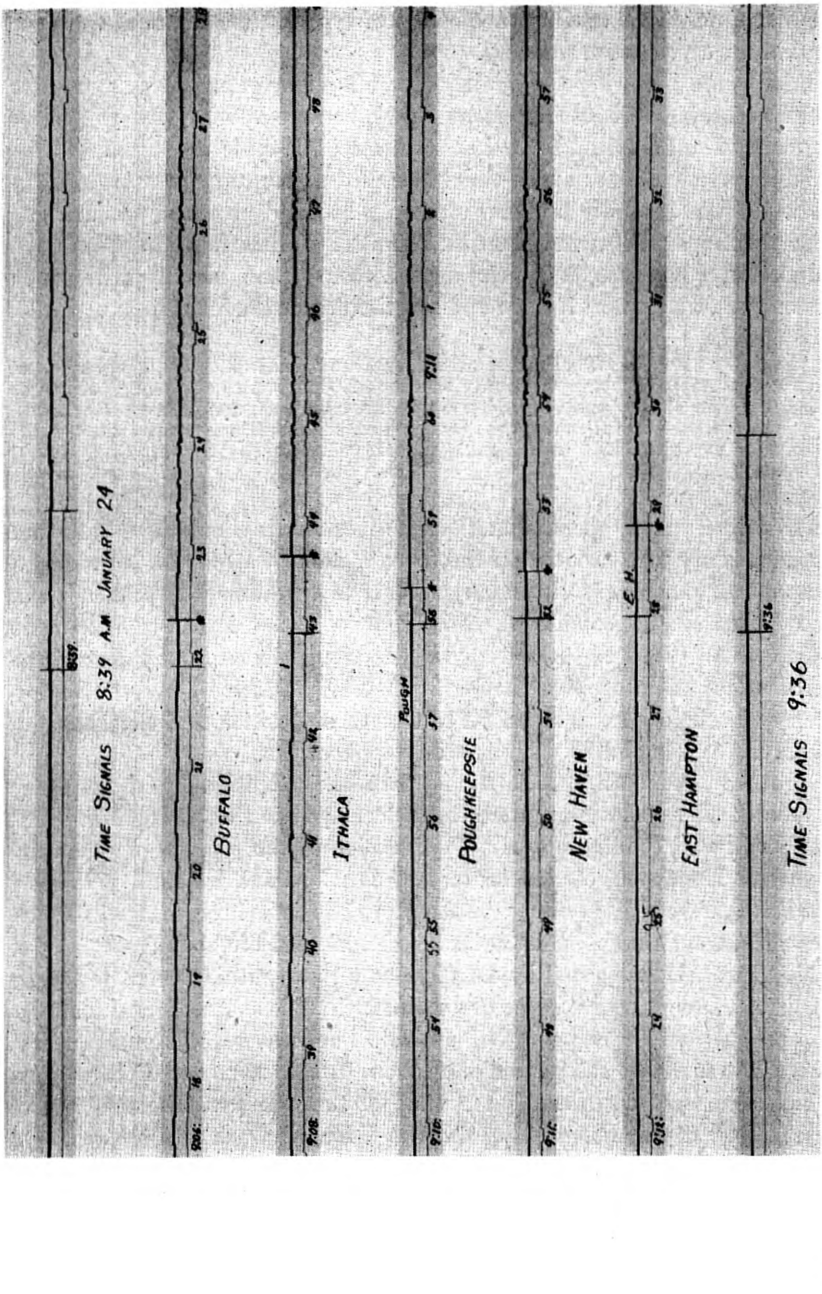

Fig. 6—Timing records of total solar eclipse of January 24, 1925

"The frequency of the fork in comparison with the N.P.L. Shortt clock can be measured at any time with an accuracy of 5 parts in 10^8. It is necessary to apply a correction for the rate of the Shortt clock, and the ultimate accuracy with which the absolute value of frequency is known depends on the accuracy of the time signals which are used to determine the rate of the clock. The final frequency can, however, usually be ascertained with an accuracy of ± 1.5 parts in 10^7. In its present condition the tuning fork maintains a frequency stability of the order of 3 parts in 10^7 over periods of a week or more."

A considerable amount of effort has been devoted to the improvement of tuning forks, directed mostly toward stabilizing the fork itself. Patents issued to H. H. Hagland[41], August Karolus[42] and Bert Eisenhour[43] have been concerned with the reduction of temperature coefficient by various methods of compensation in the alloy or in the mechanical structure of the fork. In recent years, alloys have been produced from which forks with a zero coefficient of frequency can be machined. These alloys have neither a zero expansion coefficient nor a zero elastic coefficient, but the two coefficients are so balanced that their effects cancel as they concern the frequency of a tuning fork.

One of the largest residual sources of error in a good fork is that caused by the coupling through the mounting. A fork which is efficient as a producer of sound by coupling through the base would be quite useless as a precise standard of rate due to the losses introduced in this manner. It has been shown by S. E. Michaels[44] that the tines of a well-balanced fork can be so shaped that practically no energy at fundamental frequency is transmitted through the base.

By making use of all that is known about materials, shapes and mountings for tuning forks, and all that is known about stabilized vacuum tube circuits for driving them, it is quite possible that considerable further improvement could now be obtained in such a standard. But another line of development has shown greater promise in this field and the ultimate accuracy of tuning fork oscillators has never been pursued.

The Quartz Resonator

During the same ten years that the greatest advances were being made in the tuning fork art, the striking properties of the quartz crystal resonator were reviewed and first applied in the construction of frequency and time standards. Its use in primary standards for the most exacting measurements of frequency and time is now almost universal in national and industrial laboratories throughout the world.

Quartz crystal is the most abundant crystalline form of silicon dioxide, occurring, in some parts of the world, in large single crystals from which mechanical resonators of useful dimensions can readily be formed. The physical properties that make it eminently suitable for use in a standard of rate or time are its great mechanical and chemical stability. Having a hardness nearly equal to that of ruby and sapphire, and a rigidity of structure such that it cannot be deformed beyond its elastic limit without fracture, it might be expected to remain in a given shape indefinitely under ordinary conditions of use. Because of its great chemical stability, its composition is not easily modified by any ordinary environment.

In addition to its inherent physical and chemical stability, the elastic hysteresis in quartz is extremely small. For this reason, it requires only a very small amount of energy to sustain oscillation and the period is only very slightly affected by variable external conditions in the means for driving it.

A striking illustration of the importance of this property is indicated by the number of periods that a resonant element will execute freely, that is, without any sustaining forces whatever, during the time required for the amplitude to decrease to one-half of some prescribed value. For a good electrical circuit consisting of an air core inductance and an air condenser, this number is about 100; for a good tuning fork in vacuum, it is about 2000. For a good cavity resonator under standard conditions of temperature and pressure, the number may be as high as 10,000. The best gravity pendulums will swing freely from 2,000 to 20,000 times before they reach half amplitude. The effect is most striking of all in quartz crystal, in which the internal losses are extremely low. Professor Van Dyke has measured the rate of decay of oscillations under a wide range of conditions[45] and has found that, as ordinarily mounted, nearly all of the losses are in the mounting or in the surrounding atmosphere, if any, or in surface effects. Extremely small amounts of surface contamination will more than double the decrement. Recently[46] Maynard Waltz and K. S. Van Dyke have measured the decrement of one out of the first set of four zero coefficient ring crystals ever made[47] and found that, vibrating freely in vacuum and favorably mounted, it would execute more than a million vibrations before falling to half amplitude.

The advantage of this property is immediately obvious because of the relatively small amount of energy that must be supplied at each oscillation to keep the resonator in motion. As already discussed in relation to the pendulum, the amount that the rate of oscillation may be disturbed in a given structure is proportional to this energy and, to first order, on the departure from the ideal phase condition of the applied driving force.

The properties just enumerated are sufficient to assure the superiority of quartz crystal for the control element in a rate standard; no other vibrating system known at the present time is so sharply resonant or so stable. However, one more property, its piezoelectric activity, has added greatly to the convenience of its use in vacuum tube devices.

The piezoelectric effect was discovered by the Curie brothers in 1880,[48] and in the years following was studied extensively by them[49, 50]. They found that when quartz and certain other crystals are stressed, an electric potential is induced in nearby conductors and, conversely, that when such crystals are placed in an electric field, they are deformed a small amount proportional to the strength and polarity of that field. The first of these effects is known as the *direct* piezoelectric effect and the latter as the *inverse* effect. The amount of such deformation in quartz is extremely minute, a static potential gradient of 1 esu (300 volts) per centimeter causing a maximum extension or contraction, depending on the polarity, of only 6.8×10^{-8} cm per cm. If a crystal resonator is subjected to an alternating electric field having the frequency for which the crystal is resonant, the amplitude of motion will, of course, be multiplied many times. In practice, however, the actual amplitudes of motion are kept so small, by limiting the applied electric field, that even with the largest crystals used they can

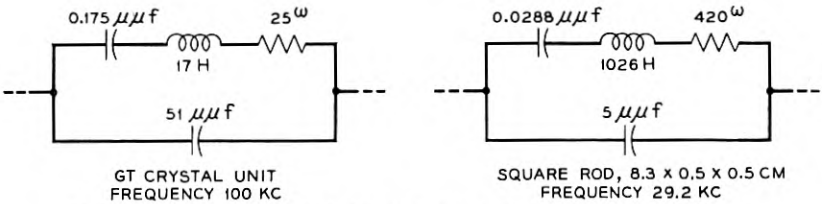

Fig. 7—Equivalent electrical circuits for typical quartz crystal resonators.

be observed only under a high powered microscope. This, in conjunction with means for precise amplitude control, is one of the reasons for the remarkable frequency stability of quartz crystal oscillators.

In practice, a quartz resonator is mounted between conducting electrodes which now most often consist of thin metallic coatings deposited on the surface of the crystal by evaporation, chemical deposition or other suitable means. Electrical connection is made to these coatings through leads which also support the crystal mechanically. The resonators with which we are chiefly concerned in this discussion have only two electrodes.

If such a two-terminal resonator is connected into any circuit, it will behave there *as though* it consisted of wholly electrical circuit elements, usually of such low loss as can not be realized by other means. The equivalent electric circuit for a quartz crystal resonator was first described[51] by K. S. Van Dyke in 1925 and, for some significant cases, is illustrated in Fig. 7. The part of such an equivalent circuit which in many cases cannot be duplicated by any ordinary means is the inductance element containing so little resistance. It is as though an electric resonator could be made and utilized constructed of some supra-conducting material.

Among the first serious efforts to utilize the piezoelectric effect in electrical circuits were those of Alexander McLean Nicolson who used rochelle salt crystal in the construction of devices for the conversion of electrical energy into sound and vice versa. He constructed loudspeakers and microphones during several years of study prior to the publication of his work[52] in 1919—ideas now being used extensively in phonograph pickups, microphones and sound producers. Nicolson also was the first to use a piezoactive crystal to control the frequency of an oscillator. His patent[53], applied for in 1918, shows a circuit which he operated successfully in 1917. The first actual use of resonators of quartz is attributed to P. Langevin[54, 55], who drove large crystals in resonance in order to generate high-frequency sound waves in water for submarine signaling and depth sounding.

The Quartz Crystal Controlled Oscillator

The first comprehensive study of the use of quartz crystal resonators to control the frequency of vacuum tube oscillators was made by Walter G. Cady in 1921 and published by him in April, 1922[56]. This was the step which initiated a most extensive and intensive research of the properties of quartz crystal and into methods for its use in numerous fields requiring a stable frequency characteristic.

The extent and importance of this research are well indicated by the number of investigators and published contributions to the art. Among these, a paper by A. Scheibe[57] in 1926 lists 28 articles on the subject, along with a description of his own extensive studies. Two years later Cady published a bibliography[58] on the subject, including 229 separate references to papers and books and 84 patents in various countries. R. Bechmann in 1936 published a review of the quartz oscillator[59] including 26 references to other original contributions in that field alone. More recently there comes at the end of Cady's 1946 book[60] on "Piezoelectricity", a bibliography of 57 books and 602 separate published articles on this subject. By any measure this represents a great amount of detailed effort for a single subject in so short a time—just about a quarter of a century. Of this great amount of material, it is feasible to review only a small number of the outstanding ideas relative to the evolution of the quartz crystal clock.

The first published quartz-controlled oscillator circuit is reproduced in Fig. 8A from Cady's 1922 article. In this oscillator the "direct" and "inverse" piezoelectric effects were employed separately, making use of two separate pairs of electrodes. The output of a three-stage amplifier was used to drive a rod-shaped crystal at its natural frequency through one pair of electrodes making use of the "inverse" effect, while the input to the amplifier was provided through the "direct" effect from the other pair.

The feedback to sustain oscillations in the electrical circuit could be obtained only through the vibration of the quartz rod and hence was precisely controlled by it. Cady's results were received with widespread interest and were duplicated and continued in many laboratories, which soon resulted in many new discoveries and inventions.

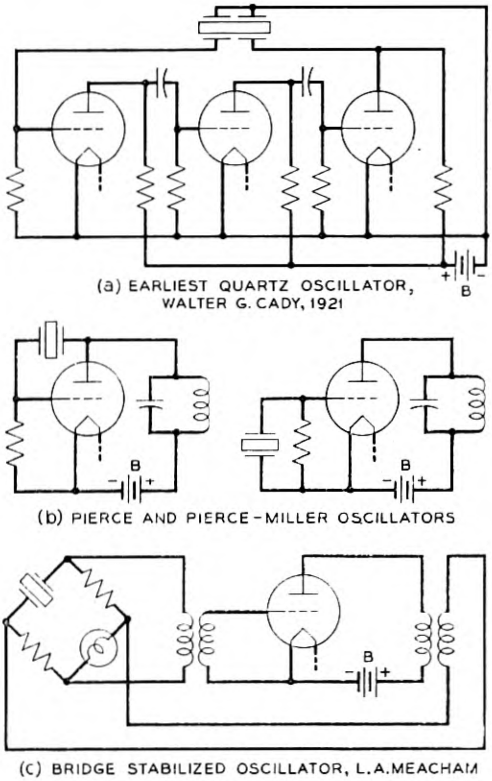

Fig. 8—Typical quartz oscillator circuits.

Important contributions were made by G. W. Pierce, who, showed in the following year that plates of quartz cut in a certain way could be made to vibrate so as to control frequencies proportional to their thickness[61]. He also proposed somewhat simplified circuits for their use which soon found very general application in the construction of wavemeter standards and later for oscillators used to control the frequency of broadcasting stations and for many other purposes. In 1924, the General Radio Company of Cambridge, Massachusetts, produced a commercial instrument based on these studies.

The significance of the unusually stable properties of quartz crystal—which at times were viewed with a sort of awe and a tendency at first to expect too much[62*]—was soon recognized in relation to precise standards of frequency and time, and many laboratories made experiments directed toward these applications.

For some years these efforts usually took one of two forms: either that of a quartz-controlled oscillator used as a comparison standard by various means[63], or that of using the quartz resonator itself as a portable standard, the high-frequency counterpart of an isolated tuning fork. Probably the most convenient standards of the latter sort were the luminous resonators first described in 1925 by Giebe and Scheibe[64]. The following year they proposed the use of such luminous resonators as frequency standards[65] and, shortly following, portable frequency indicators of this sort were made available for general use. The use of such a luminous resonator for the international comparison of frequency standards was reported by S. Jimbo in 1930.[66] The first international comparison of frequency standards making use of piezo resonators as isolated standards was carried out by Walter G. Cady in 1923, who by means of a set of early type resonators compared the existing standards at Rome, Livorno, Paris, Teddington, Farnborough, Washington, and Cruft Laboratory at Harvard University[67]. In the following year the U. S. Bureau of Standards carried out a similar international frequency comparison, but of greater accuracy, employing portable quartz crystal oscillators. This comparison and other important related studies were described by J. H. Dellinger in 1928—"The Status of Frequency Standardization"[68].

It was soon recognized that quartz oscillators could be built with a stability far greater than that of any other known type and that they possess qualities very desirable for a combined time and frequency standard. However, all early quartz oscillators had frequencies far too high to operate any synchronous motor and it was not immediately obvious how a clock could be operated thereby.

The Frequency Divider

The illustration in Fig. 9 from the author's notebook for November, 1924 is believed to be among the earliest means proposed to accomplish this. In brief, the proposal was to control the speed of a motor driving a high-frequency generator so that a harmonic of the generator output, say the

* In 1929, M. G. Siadbei wrote "Nous pensons que le quartz piécoélectrique peut trouver un nouvel emploi dans la chronometrie, étant donnée la conservation rigoureusement constant de ses oscillations."

"La seul cause de variation de la période d'oscillation résulte en effect du changement de la temperature. . . .'

tenth, would have a frequency of the same order as that of the crystal but differing from it by a relatively low frequency, f_1. This low frequency, derived from the modulator was to be used to *drive* the synchronous motor.

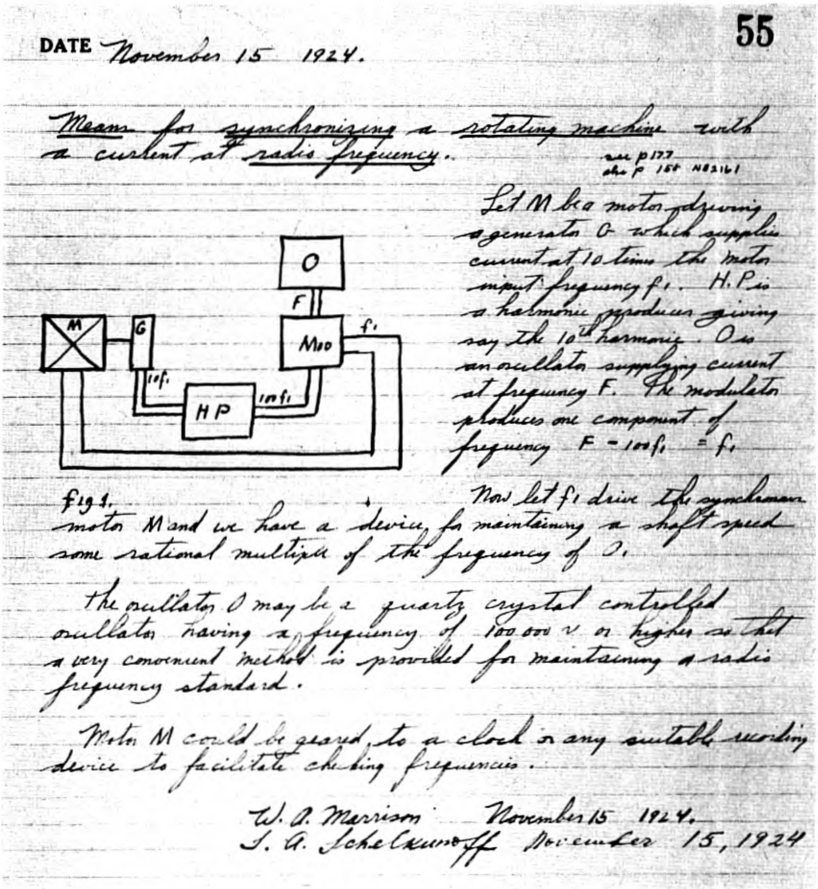

Fig. 9—Early suggestion of means to control a rotating device such as a clock from a high frequency.

The shaft speed of the motor-generator would, therefore, be integrally related to the crystal frequency and hence any mechanism geared to the shaft, such as a clock, would indicate time as dictated by the crystal. This method could have been carried through readily by a combination of means already developed for other purposes, and the construction of an apparatus based on this suggestion was soon begun. However, a simpler method[69], not involving a rotating machine in the control system, was suggested and the first quartz crystal clock was constructed using the simpler means. This apparatus was described by Horton and Marrison[70] before the International Union of Scientific Radio Telegraphy in October, 1927. The resonator in its mounting that was used in this first model is shown in Fig. 10. It consisted of a rectangular block of crystal, cut in the manner usually called X-cut, and of such size as to oscillate at a frequency of 50,000 cycles per second in the direction of its length. The temperature coefficient of this resonator was approximately 4 parts in a million per degree C at the

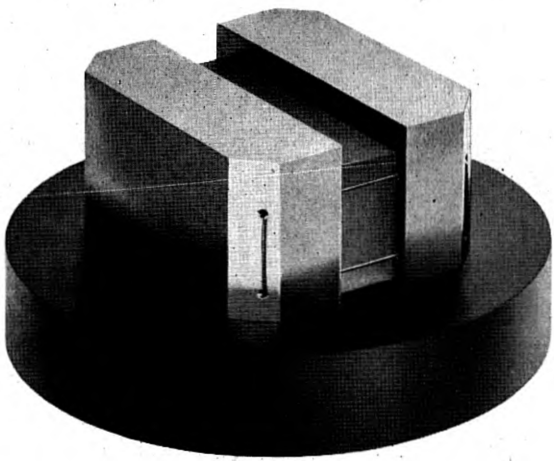

Fig. 10—50,000-Cycle quartz resonator, in original mounting, used in first quartz clock—1927.

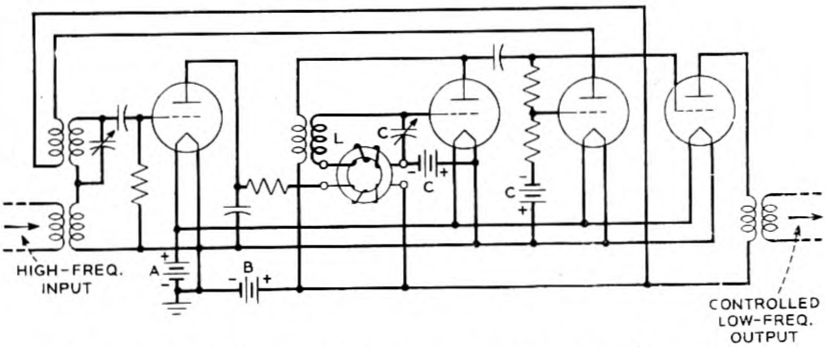

Fig. 11—Submultiple controlled frequency generator used in first quartz clock.

temperature of operation, which was controlled at a value in the neighborhood of 40 degrees C.

The method for frequency subdivision used in this first quartz crystal clock is illustrated in Fig. 11. The inductance element of an electric circuit oscillator, designed to operate at the desired low frequency, has a core of variable permeability so that the frequency can be adjusted over a narrow range through the control of direct current in an auxiliary winding. A harmonic of this low frequency, generated in the tube following the oscillator, is compared with the incoming high frequency in the vacuum tube modulator. The harmonic chosen has nominally the same frequency as that of the control, or crystal oscillator, so that one output of the modulator is a direct current whose magnitude and sign vary with the phase relation between the inputs to the modulator. The use of this method to regulate the low-frequency oscillator insures that the low frequency is some exact simple fraction of the high frequency. If, therefore, a synchronous motor is operated from the low frequency thus produced, its rate represents accurately that of the high-frequency source as though it had been possible to use that source directly.

Several other electrical circuits were proposed around 1927 for the subdivision of high frequencies. The method in most general use at present is an adaptation of the "multivibrator" first used by Henri Abraham and Eugene Block in 1919 for the measurement of high frequencies[71]. They used their circuit to produce a wave rich in harmonics and having a fundamental that could be compared directly with that of a tuning fork standard. By various means now well known the high frequency could be compared with one of the harmonics of this special oscillator.

This procedure was reversed by Hull and Clapp[72], who discovered that the fundamental frequency could be *controlled* by coupling the high-frequency source directly into the circuit of the multivibrator. This, in fact, is a general property of any oscillator in which the operating cycle involves a non-linear current-voltage characteristic, being most pronounced in those of the relaxation type. Van der Pol and Van der Mark in 1927 reported on some experiments on "frequency demultiplication" using gas tube relaxation oscillators[73]. The multivibrator is, in effect, a relatively stable relaxation oscillator[74], and with slight modification has been used extensively as the frequency-reducing element in quartz-controlled time and frequency standards throughout the world.

One serious difficulty with the multivibrator type of submultiple generator has been that, if the input fails or falls below a critical level, it will continue to deliver an output which, of course, will not then have the expected frequency. Certain variables in the circuit, such as tube aging, may cause a similar result. With this in view, a general method for frequency conversion has been developed by R. L. Miller[75], in which the existence of an output depends directly on the presence of the control input. The basic idea involved in this, now known as regenerative modulation, was anticipated by J. W. Horton in 1919[76] but had not been developed prior to Miller's investigations. The circuit of a regenerative modulator in its simplest form as a frequency divider of ratio "two" is shown in Fig. 12.

Soon after the announcement in 1927 of the first quartz crystal controlled clock,[70] the idea was studied and applied in many places notably in America and Germany, and at the present time it forms the basis for precise measurements of time and frequency in many government physical laboratories as well as in many astronomical observatories and industrial and university laboratories throughout the world.

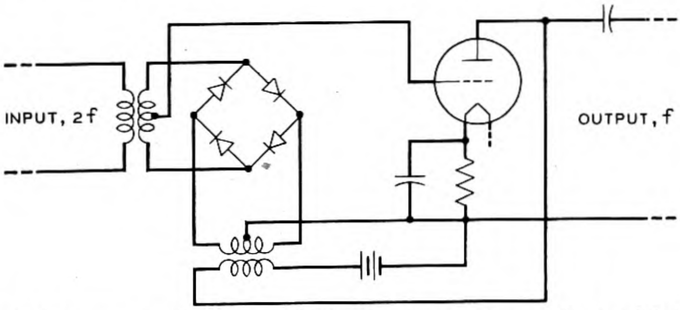

Fig. 12—Frequency divider for ratio TWO employing regenerative modulation.

Although the first results were quite satisfying, it was the immediate interest of all concerned to find out what improvements could be made, and these were not long in coming. As in the case of the pendulum already discussed, or with any other oscillator, the constancy of rate obtainable depends on two kinds of properties: those which concern the inherent stability of the governing device itself, and those concerned with the means for sustaining it in oscillation. Some of the factors in the two groups are interrelated and must be considered together.

The improvements in quartz oscillator stability therefore have been concerned with two main endeavors, namely that of cutting and mounting the resonator so as to realize effectively the unusually stable properties of quartz crystal itself, and that of coupling it to the electrical circuit in such a way as to avoid the effects of such variables as power voltage variation, aging of vacuum tubes, and the like, on the controlled frequency. The latter effects were not obvious at first because the temperature coefficient and the effects of friction and change of position in the mounting caused variations of considerably larger magnitude. It was natural, then, to see what could be done about these effects.

Zero Temperature Coefficient of Frequency

With the knowledge that X-cut resonators had negative coefficients, frequently as large as thirty parts in a million per degree C, and that Y-cut resonators in general had positive coefficients, often in excess of a hundred parts in a million per degree, the author undertook to make resonators of such shape that the oscillations would occur in both modes simultaneously, and so combine the coefficients, in the hope that the resultant could be made zero.[77]

The first experiments, made on two series of resonators both yielded encouraging results. The first was a series of rectangular X-cut plates of varying thickness shown in Fig. 13. The second was a series of three circular discs of different diameters, all being cut with the large surfaces in the plane of the Y and Z axes. The three discs were made from the *same* material, each smaller one being trepanned from the previous one after complete measurements had been made upon it. The set of circular crystals remaining after these tests were completed is shown in Fig. 14 and the slab from which they were cut is shown assembled with the original large crystal in Fig. 15.

Subsequent tests showed that the annular pieces could be designed for a low or zero coefficient and such a shape shown in Fig. 16 was employed for a number of years in the Bell System Frequency Standard in New York City[78]. As described in this reference, the reason for using the ring in preference to the solid disc or rectangular plate was in the convenience of mounting. The rings were formed with a ridge in the central plane of the hole so that they could be supported on a horizontal pin thus providing a one-point support at a position where the vibration is very small. The rings used in this first application of zero coefficient quartz resonators have been called "doughnut" crystals for obvious reasons. In Fig. 17, George Hecht is shown making a final adjustment, by "lapping" with fine abrasive, on one of the four original zero-coefficient ring crystals. Mr. Hecht made all four of these resonators, as well as many others of various shapes and sizes used in the early experiments in this work.

Supported as described, the rings hang in a vertical plane and, as first used, they were supported freely between solid electrodes rather closely spaced to the flat surfaces. The small amount of free motion relative to the electrodes, inherent in this sort of mounting, caused occasional changes in frequency if the support were disturbed, which at times would be as large as one part in ten million. To avoid this difficulty, other ring crystals were constructed with a sort of narrow shelf at the central plane that could be mounted in a horizontal plane on pin supports. The two methods of supporting the ring resonators are illustrated in Fig. 18. Such resonators were

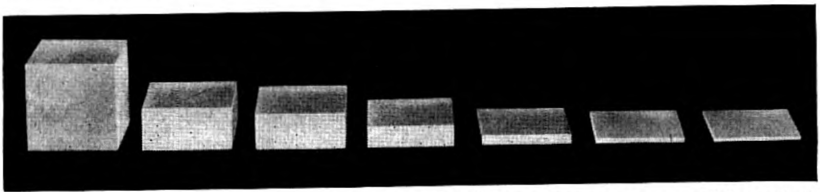

Fig. 13—Set of rectangular quartz resonators made for zero temperature coefficient study.

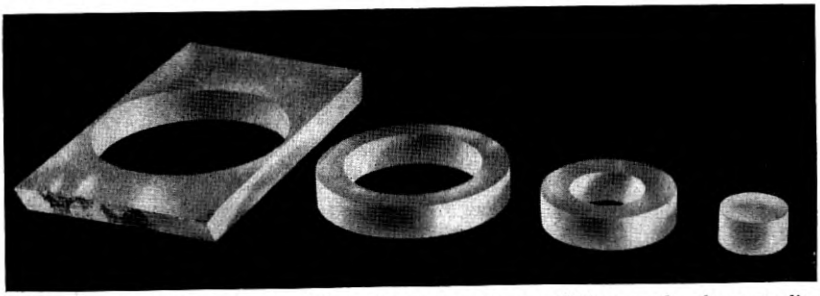

Fig. 14—Circular pieces remaining after temperature coefficient study of quartz discs and rings.

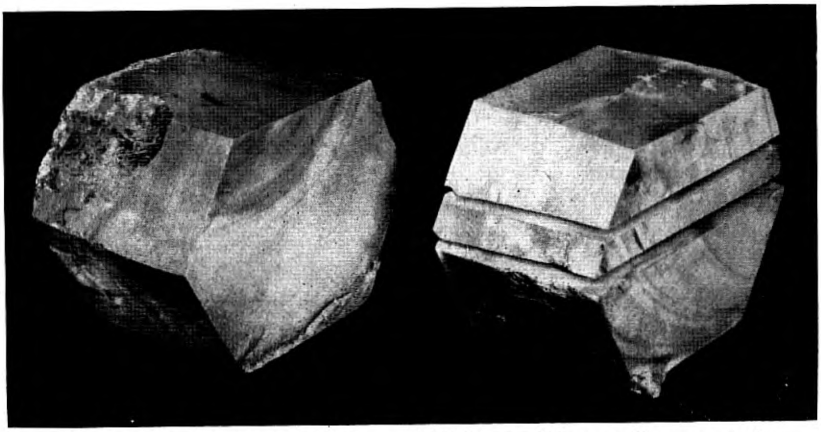

Fig. 15—Large crystal and slab from which low coefficient studies were made.

used in the Bell System Frequency Standard until 1937 when they were replaced by an entirely different type that will be described later.

The rings were adjusted to oscillate at 100,000 vibrations per second, the frequency which has been adopted in nearly all oscillators of extremely

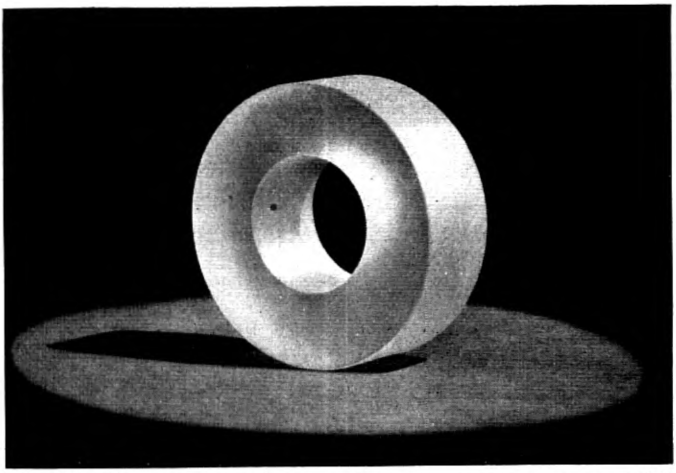

Fig. 16—100-Kilocycle quartz ring resonator with zero temperature coefficient.

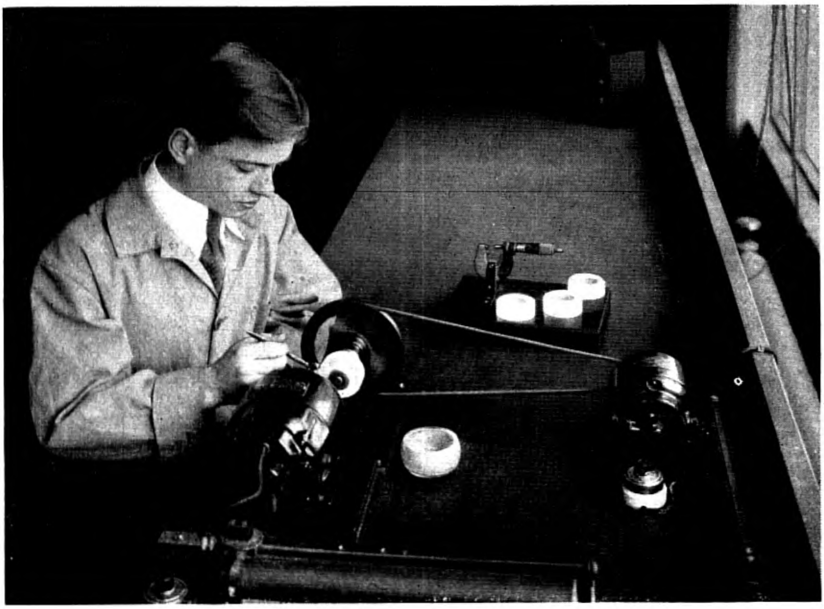

Fig. 17—George Hecht finishing the first set of zero-coefficient quartz rings.

constant rate. All of these rings were constructed to have a zero frequency-temperature coefficient at a temperature in the neighborhood of 40 degrees C, the frequency being a maximum at that point on an approximately parabolic characteristic. The zero temperature coefficient makes it possible to practically eliminate frequency changes caused by ambient temperature changes since, by relatively simple means, it is possible to control the resonator within ± 0.01 degree C, at the temperature for which the effect is substantially nil. The reduction of the effect of temperature, and the stabilization of the mounting, increased the stability of frequency control and oscillator-clock rate beyond anything that had ever been obtained before. Subsequent improvements that will be described later produced even greater stability.

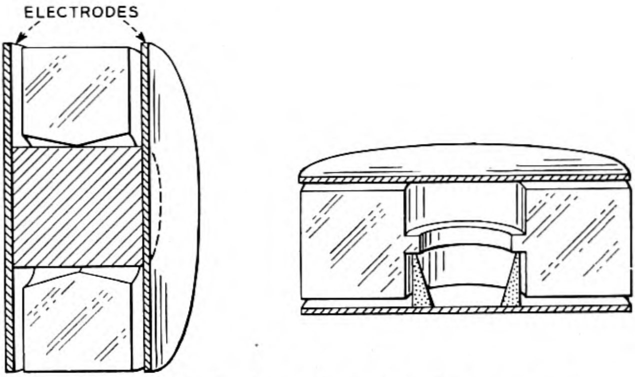

Fig. 18—Methods of mounting quartz ring resonators.

The Crystal Clock

The striking stability of the crystal oscillator clock led the author to propose the general use of this type of clock for precision timekeeping, the chief emphasis having been previously on the derivation of constant frequency. A paper entitled "The Crystal Clock,"[79] presented before the National Academy of Sciences in April, 1930, described such a clock and pointed out some of its properties and likely uses.

Chief among these properties, of course, is its inherent stability and relative freedom from extraneous effects. The quartz crystal clock is not dependent on gravity and, without any compensating adjustment, will operate at the same rate in any latitude and at any altitude. This property already has been useful in the *measurement* of gravity and gravity gradient by measuring the rates of pendulums on land and at sea.[80, 81]

The crystal clock is practically immune to variations in level and shock and can be used as an instrument of precision under conditions entirely unsuitable to pendulum clocks. For this reason it performs satisfactorily in practically any location, including earthquake zones, and may be used in transit as in a submarine, in an airplane or on the railroad.

Some of the outstanding properties of the quartz oscillator clock were discussed in 1932 by A. L. Loomis and W. A. Marrison[82], in relation to a series of experiments comparing the performance of quartz clocks at Bell Telephone Laboratories in New York and a set of synchronome free-pendulum clocks operating in The Loomis Laboratory in Tuxedo Park, about fifty miles away. The comparison was effected through a circuit maintained between the two laboratories over which a 1,000-cycle current controlled by a crystal in New York was used to drive the Loomis Chronograph[83] in Tuxedo Park. During part of the time, signals from the clocks were sent back over the same circuit and recorded on the Bell Laboratories' Spark Chronograph[84].

The quartz oscillator assembly at the Bell Telephone Laboratories at the time of these experiments is shown in Fig. 19. The four ring crystals in their individual temperature-controlled 'ovens' are mounted under hermetically sealed bell jars to avoid the effects of ambient temperature and atmospheric pressure changes. The vacuum tube oscillator circuits are immediately below the bell jars; and the control, monitoring and power supply equipment in the remainder of the space.

One of the most interesting results of these cooperative experiments was the measurement of a periodic variation in the rate of the pendulum clocks in phase with the lunar daily cycle. The amount of this daily variation is very small, being only a few tenths of a millisecond, but readily observable in comparison with a stable rate standard that does not vary with gravity.

Further Refinements in Quartz Clocks

The spectacular results from the use of the quartz crystal clock up to this time, about 1932, were due in part to its novelty and in part to the fact that it is quite independent of some of the variable factors that affect conventional precision clocks, including gravity itself upon which the rate of all pendulum clocks depends. The remarkable stability of present day quartz oscillators and clocks is the result of a series of developments and refinements extending over a number of years.

As mentioned previously, the factors that cause departure from constant rate in the completed operating device fall into two distinct classes, namely those which concern the inherent or natural frequency of the resonator itself, and those which concern the means for driving it at that inherent rate.

The first class comprises all those properties of the mounted resonator which tend to relate its inherent rate to ambient conditions such as temperature, atmospheric pressure, change of position and vibration, and to the passage of time—that is, aging. Since the final stability cannot exceed the inherent stability of the mounted resonator itself, its study is of prime importance.

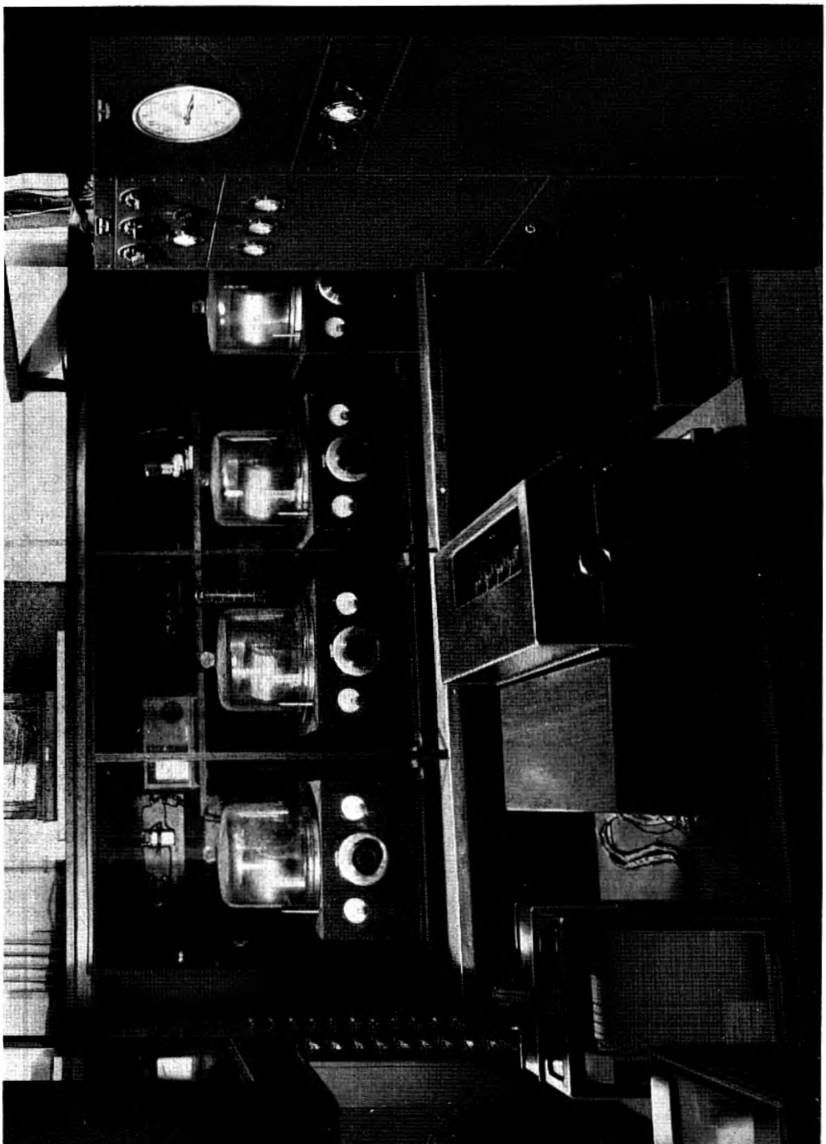

Fig. 19—Bell System Frequency Standard, 1930.

The second class comprises properties of the means for sustaining oscillations in such a resonator which relate the resulting actual rate to variations in the electrical circuits, in the power voltages, in vacuum tubes and other like effects. In the limit, it is the hope that the net result of all such effects can be eliminated so that the stability of the quartz crystal alone will remain the sole governing factor. This is the goal, and the inherent stability of the substance, quartz crystal, is the limit toward which the stability of the quartz crystal clock will approach but cannot exceed.

The development of the quartz resonator and its mounting for numerous applications is described in some detail by Raymond A. Heising and his collaborators[85] in their recent book, "Quartz Crystals for Electrical Circuits". Of all the types of resonator described in this work the one having the most extensive use at the present time, for quartz clock installations and for other applications of comparable accuracy, is the GT crystal resonator developed by W. P. Mason[86]. This resonator is cut from quartz crystal in such a way that the positive and negative coefficients are effectively neutralized over a range of about 100 degrees C, so that in any part of this range the resulting temperature coefficient of frequency is not more than one part in a million per degree C. With suitable precautions in manufacture, the tangent at the point of inflection in the frequency-temperature curve may be made horizontal, which means that the temperature coefficient may be made substantially zero over a considerable range of temperature.

The GT crystal resonator therefore introduces two significant advantages in timekeeping, namely that greater accuracy of rate may be obtained with a given accuracy of temperature control and that the value at which the temperature is controlled may be chosen in a considerable range. In fact, without any temperature control at all, the rate of a clock regulated by such a crystal may be accurate to a tenth of a second a day over an ambient range of 100 degrees C. Among the many quartz clock installations now using the GT resonator, all or in part, are the Royal Observatory at Greenwich, the British Post Office, the U. S. Naval Observatory and the U. S. Bureau of Standards.

One of the chief sources of variation in rate of quartz oscillators, in the early stages of their development, was in the means for mounting and in the electrical circuit connections. As mentioned previously, any variation in the effective resistance or in the effective mass or stiffness of a resonator has a direct effect upon its rate of oscillation. The problem reduces to that of supporting the resonator so that the frictional losses are small and constant and so that the coupling to the electrical circuit is as nearly as possible invariable.

The mounting of quartz crystal units is discussed at length by R. M. C.

Greenidge in Chapter XIII of Mr. Heising's book referred to above.[85] The most satisfactory means by far that has been found for mounting crystals of the GT type is that of actually soldering them to thin supporting wires by means of small discs of silver deposited on the crystal at its nodes. This method serves the double rôle of supporting the crystal and of providing electrical connection to metal electrodes plated on the crystal. Resonators so supported may be made almost immune to mechanical shock and will continue in satisfactory operation through accelerations of several times g. Nearly all crystals which vibrate in a long dimension are now mounted in this way. One manufacturer produced about 10,000,000 crystals of a single

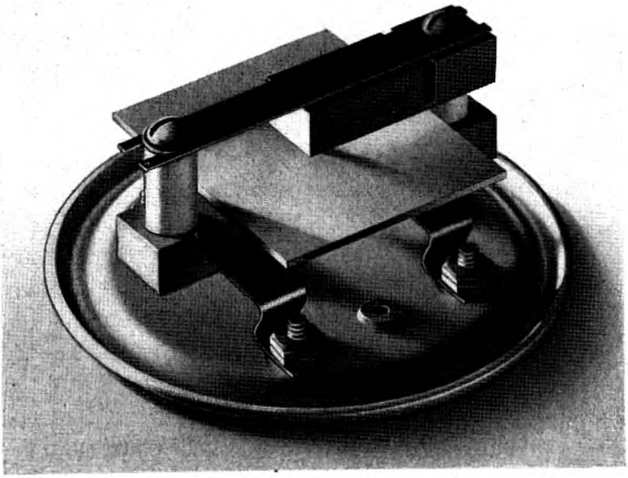

Fig. 20—Pressure-mounted GT crystal for sealing in a metal envelope.

type so mounted in a three-year period during World War II. Prior to the use of wire supports, such crystals were "pressure mounted" by means of small metal jaws which clamped from opposite sides at the nodes. A GT crystal mounted in this way is shown in Fig. 20. Crystals so mounted are still in use in the Bell System Frequency Standard, being the first of the GT crystals to go into actual service. This type of mounting is not quite so stable as the wire mounting and is somewhat more difficult to manufacture. One of the wire-mounted crystals such as developed for LORAN and other oscillators of comparable accuracy is shown in Fig. 21.

The plating of electrodes on the crystal surface has led to increased stability of frequency control, chiefly because the coupling to the electrical circuit may be kept more nearly constant thereby. When separate electrodes were employed, the variation in spacing was always found to be a source of instability, as mentioned previously in relation to the use of the first ring crystals. Plating of crystals is not a new idea but the application to quartz resonators of high Q requires a great amount of technical skill in order to obtain coatings which are mechanically and chemically stable and which utilize the minimum of added material. The use of too much metal

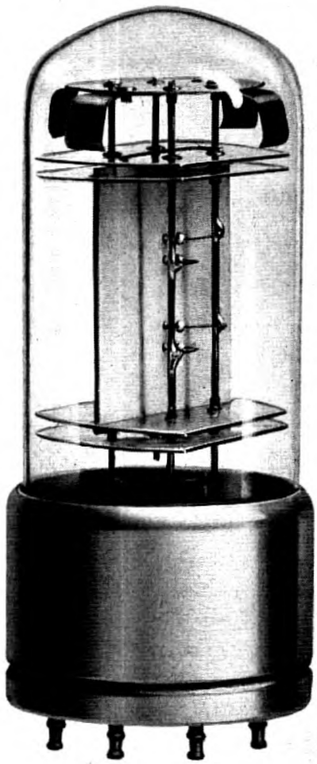

Fig. 21—Wire-supported GT crystal sealed in a glass envelope.

will, of course, impair the resonator by increasing its rate of energy dissipation and probably its aging rate. The metal most often used for electrodes is silver, although gold and aluminum have been used in special cases. Evaporation in vacuum has been found to be the most satisfactory method for the actual plating, giving very adherent coatings and being subject to precise manufacturing control. The art of plating quartz resonators is discussed in detail by H. W. Weinhart and H. G. Wehe in Mr. Heising's book.

Several other factors have had an important bearing on the final stability of quartz resonators. One of the most important of these is the care that must be exercised during fabrication in order to avoid setting up stresses in the material that subsequently can be relieved only slowly. By slow grinding with adequately fine abrasive such effects can be kept very small. Etching with hydrofluoric acid has resulted in much further improvement through the removal of stressed surface material and all potentially loose material which, formerly, often caused anomalous aging effects. Artificial aging by heating, and thorough cleaning before and after plating, have also contributed greatly to the final stability of the crystal unit. The resonator finally is mounted in high vacuum in a glass envelope in order to eliminate losses due to sound radiation and friction, and to protect it from surface contamination and chemical action.

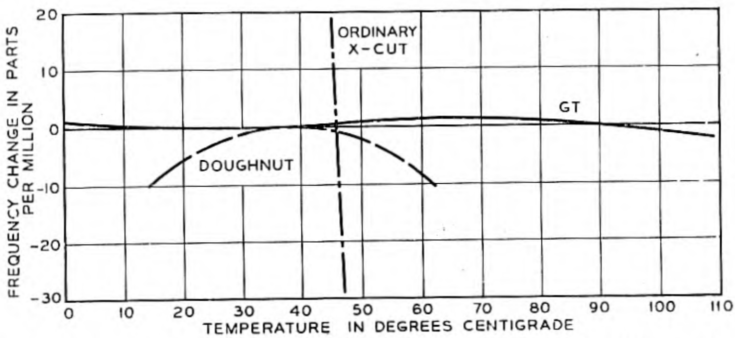

Fig. 22—Frequency-temperature characteristics for three types of quartz resonators.

Even the most perfect quartz resonator, in an ideal mounting, is unable to keep time unless it is maintained in oscillation; and, like a pendulum, its rate will depend in large part on the manner in which it is driven. The same general principles apply to both cases, except that usually a pendulum is driven by impulses which should be applied when the velocity is maximum, while a quartz resonator is usually driven by a sinusoidal force arising through the piezoelectric coupling, and so phased that the maximum force occurs when the velocity is maximum. This, in fact, is a required condition for maximum rate stability. The graphical analysis of Fig. 5 applies equally for the case of sine wave drive, since the sine wave can be considered as the summation of an impulse at its peak and of sets of pairs of impulses symmetrically disposed with respect to it. Obviously, the phase errors for each such pair of impulses cancel, bringing us back to Airy's condition, but with the broader view that, for the feedback or driving wave to have minimum effect on the rate of an oscillator, the force wave must be in phase with the velocity of the resonator.

Numerous vacuum tube circuits have been proposed and used for maintaining quartz resonators in oscillation, some of which are illustrated in Fig. 8. The one among these which at present most nearly approaches the ideal is that developed by L. A. Meacham, known as the Bridge Stabilized Oscillator.[87] This oscillator, in its original form or with slight modifications, is now used almost universally in England and America where the maximum stability of rate control is required.

In the bridge stabilized oscillator, the feedback path is through a Wheatstone bridge with the crystal in one arm and with resistances in the other three. The frequency of oscillation becomes that for which the reactance of the crystal approaches zero; the bridge can only be balanced when the crystal behaves electrically like a resistance. The unbalance voltage from the bridge is fed back into the amplifier, which should provide a relatively high gain, as will appear. The great frequency stability of this oscillator depends upon the fact that, in the neighborhood of balance, a small phase shift in the resonant elements causes an enormously larger phase shift in the unbalance voltage. But the actual amount of this unbalance phase shift is limited by the fact that it must be equal and opposite to that in the amplifier in order for oscillations to be sustained. This insures that at all times the phase shift in the crystal is much smaller than that occurring in the amplifier which itself can be made small by suitable design. The ratio of the phase shift of the bridge output to that of its input increases as balance is approached, making it possible to practically eliminate the effect of phase shift in the amplifier simply by increasing the amplifier gain. Most of the variable factors in the amplifier of an oscillator circuit affect the controlled frequency through the phase shifts caused by them. It is evident, then, that the bridge circuit, which permits only a small fraction of such phase shifts to become effective at the resonant element, will substantially free the resonator from variable effects in the amplifier and allow it to control a rate determined almost wholly by its own properties.

When the above condition is attained and the crystal resonator, when oscillating, acts in the circuit like an electrical resistance, it acts that way *because* the velocity is in phase with the applied mechanical force, which, as has been stated, is the condition for most stable rate control. In the crystal oscillator, this ideal condition is obtained simply by the automatic balancing of a bridge circuit, accomplishing in a most elegant manner the equivalent, in the case of a pendulum, of applying driving pulses at the exact center of swing.

The bridge-stabilized oscillator includes also an automatic control of amplitude. The variation of frequency with amplitude is very small and in no way comparable with the "circular error" of an ordinary pendulum, but in the quest for the highest attainable stability it must be taken into account.

The control of amplitude is obtained by the use of a resistance with positive temperature coefficient in the bridge arm conjugate to the crystal, chosen so as to have exactly the right value to balance the resistance of the crystal when a specified current is flowing in the bridge. If larger than normal current flows momentarily the resistance is increased, which decreases the feedback, thus stabilizing the amplitude at some predetermined value. For the highest stability it has been found advantageous to operate the crystal at a very small fraction of the amplitude that normally would be used in a power oscillator. In power oscillators the crystal sometimes is subjected to strains near the fracture point, which is not a favorable condition for precision control. The actual amplitude of motion of the crystal is of course extremely small. In the GT crystal, as currently used, the maximum change of dimensions during oscillation amounts to only about ±0.0006 per cent.

The improvements in quartz resonators, and in their driving circuits, have resulted in the construction of quartz crystal clocks that will keep time with an accuracy better than 0.001 second a day, so that measurements of time of great interest and value to astronomers and geophysicists can now be made with an accuracy hitherto unattainable.

Facility of Precise Time Measurement

In making such precise measurements of time it is of importance, second only to the inherent accuracy of the standards themselves, to have available means whereby they can be carried out with facility and within a reasonable time interval. The ease with which precise time measurements, and precise rate comparisons, can be made is an outstanding feature of the quartz crystal clock and already has an important bearing on the use of this type of clock in astronomical observatories. This facility depends chiefly on two properties of the oscillator clock: first, that *continuous* rotation of controlling and measuring devices can be produced having the stability of the primary control element; and, second, that the period of the control element, and therefore of alternating current controlled by it, is of very short duration.

The first of these, through simple devices controlled directly from the electrical output of the crystal oscillator, with suitable frequency reducing equipment, permits of ready comparison between any time phenomena in the form of electric or light signals, and of the derivation of precisely controlled time signals for radio transmission and for laboratory experiments.

Of prime importance among these comes the means for rating crystal clocks in terms of stellar observations using meridian transits or the photographic zenith tube[88]. It is possible to control a mechanism in the time-star observing equipment so that the difference between a star position *predicted* from the clock rate, and the *actual* star position, can be observed directly or recorded photographically with great accuracy. The difference thus observed, after allowing as well as possible for known systematic errors, is the best known single check on the time indication of a clock. A series of such observations constitutes the best known measure of the *rate* of a clock. The great value of the method is that the comparisons are made directly without the need of any intermediate mechanism thus eliminating a large part of the "personal error" of observation. The probable error of observation as derived from a number of such measurements on a good night may be as small as one or two milliseconds[89]. The average rate of a clock thus determined depends on the number of days over which the rate is computed and in a two-week period may be compared with the rate of the earth, that is, with astronomical time, with an accuracy of one part in one hundred

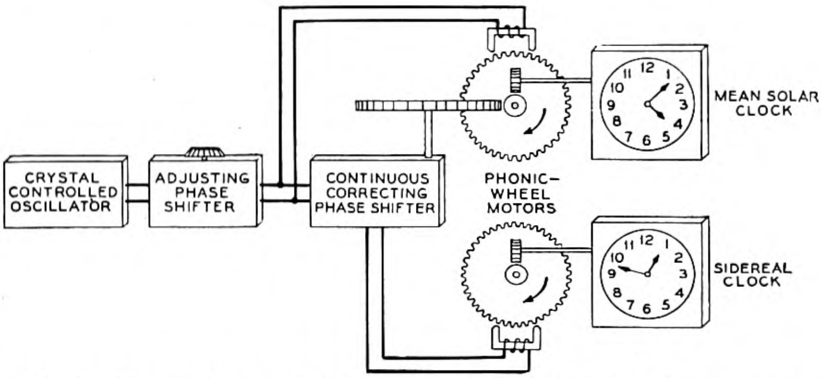

Fig. 23—The use of an electrical phase shifter to adjust the timing of a signal. (From 'The Crystal Clock", 1930)

million or about a third of a second a year. All this, of course, is contingent on the stability of the quartz clock, which, except for long-time effects, may be demonstrated independently.

A rotating mechanism controlled directly from a crystal clock is admirably adaptable to the transmission of precise time signals. Rhythmic signals of any desired structure can be produced readily by means of cams, special generators, or interrupted light beams, and the timing of those signals can be adjusted as precisely as the clock time is known by simply advancing or retarding the signal generators. Such adjustment is attained readily by means of differential gearing in the mechanical system, or by means of continuous phase shifters in the electrical driving circuit. The use of electrical phase shifters for this purpose was first proposed in "The Crystal Clock" paper[79] previously mentioned. Figure 23, taken from that paper, illustrates the manner of using the phase shifter with one type of time signal generator. Extremely fine control of timing is possible by means of the electrical phase shifter since it can be included in the circuit at any stage of frequency subdivision. If, for example, it is used at the lowest frequency, assumed to be 1,000 cycles, one complete turn of the phase shifter dial will cause a progressive time adjustment of one millisecond. When used at a higher frequency, the precision of adjustment is increased correspondingly. Continuous phase shifters suitable for such purposes were proposed as early as 1925.[90] The idea of utilizing continuous phase shifters for the purpose of making controllable changes in the frequency or indicated time in a standard time and frequency system[91] was first disclosed in a comprehensive patent filed in 1934 and issued to Warren A. Marrison in 1937. The most elegant type of phase shifting element suitable for such purposes was developed by Larned A. Meacham.[92] This has been used in many transmission systems requiring continuous variation of phase such as in variable direction radio beam systems [93] and LORAN.

The conversion between mean solar time and sidereal time, or for that matter between any time systems, may be accomplished very easily with the quartz clock. Having a rotating device, such as a dial or commutator, whose rate corresponds to mean solar time, it is only necessary to apply a gearing or the equivalent to obtain another rate corresponding to sidereal time. It has been shown by F. Hope-Jones[94], Ernest Esclangon[95] and others how any desired ratio, such as the ratio of the rates of mean solar and sidereal clocks, can be obtained with any required accuracy by gearing. A combined mechanical and electrical method was proposed in the "Crystal Clock" paper by means of which this ratio can be realized with an accuracy of one part in 10^{11} using simple gearing and a continuous phase shifter.

The potential value of the factors just discussed in precision time studies was realized early in the crystal clock development. This was indicated in the "Crystal Clock" paper written in 1930 which closed with the following paragraph:

> "It would thus be possible to combine, in a single system mean solar and sidereal time-indicating mechanisms, means for rating the clocks in terms of time star observations and means for transmitting time and frequency signals with the absolute accuracy of the time determinations."

It is of some interest to compare this prediction with the present trend of development. In describing the quartz clock installation at the Royal Observatory in Greenwich, Sir Harold Spencer Jones stated[89] in 1945:

> "The quartz clocks being installed at the Royal Observatory are all adjusted to give a frequency of approximately 100,000 per mean time second. By suitable gearing, the synchronous motor can give impulses every sidereal second and tenths of seconds. Thus, the same clock can be made to serve both as a mean time and as a sidereal time standard. All time signals are, of course, sent out according to mean time; the sidereal time is required only for the actual time determination so that it is not necessary for all the clocks to have the gearing to give sidereal seconds."

The importance of the convenient methods for measuring time and time interval inherent to the crystal clock is emphasized by the fact that some observatories employed crystal clock mechanisms in connection with stellar observations and in the transmission of time signals before they were used in the actual time *keeping* department[88].

The second property contributing greatly to the convenience of precise time measurements is the relatively very short period of the quartz clock control element. The chief advantage lies in the extreme accuracy with which the rates and indicated times can be compared by electrical methods. An example will suffice to illustrate this point.

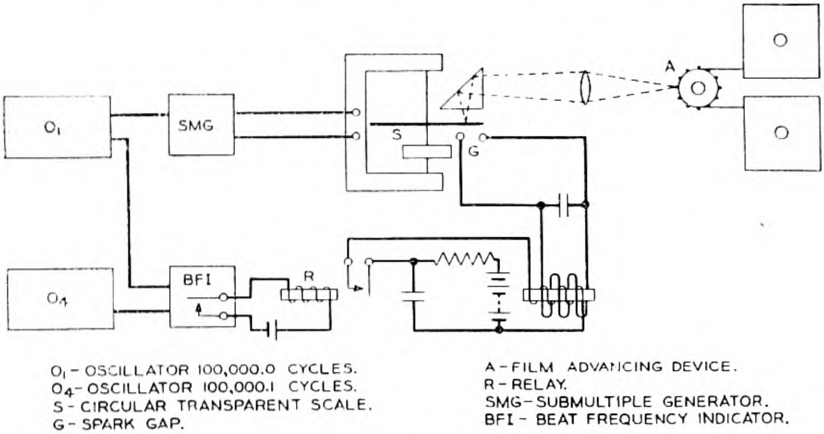

Fig. 24—Device for comparison of oscillator rates accurate to 1 part in 10,000,000,000. (From "High Precision Standard of Frequency", 1929.)

Since the *rate* of a crystal clock is the rate of oscillation of the crystal or of the current driving it, it is only necessary, in comparing clock rates, to measure the relative frequencies of the oscillators concerned. This can be done by any of the standard methods for frequency comparison[96] but, in the case of quartz clocks, since in general the primary frequencies are high and are nominally the same, special methods of extreme accuracy can be employed. The apparatus first designed for the ultra-precise comparison of quartz oscillators and capable of an accuracy of one part in 10^{10} was described by Marrison in 1929.[78, 97, 98] The principle of its operation is shown in Fig. 24, reproduced from the paper "High Precision Standard of Frequency".

Two oscillators to be compared were adjusted so as to differ by about one cycle in ten seconds. The problem reduces to that of measuring the beat frequency, nominally 0.1 cycle per second, with as great accuracy as possible.

This was done by measuring the duration of each beat by a photographic method. By means of a modulator, a relay, and induction coil, a spark was produced at the spark gap at a definite phase of each beat period. The spark illuminated the edge of a transparent scale rotating 10 revolutions per second under control of one of the oscillators, O_1. The transparent scale contained 100 numbered divisions, which therefore represented milliseconds in any time interval so measured. Each time a spark occurred, the portion of scale illuminated was registered on photographic film. Thus, the duration of each beat was registered photographically with an accuracy of one part in ten thousand. Since the beat frequency is one millionth of the high frequency, the resulting *comparison of high frequencies* is precise to one part in ten thousand million, or 1 in 10^{10}. Actually, it was possible to estimate fractions of a scale division which gave greater precision of measurement than was required in the study of oscillators of that date.

L. A. Meacham in 1940 improved upon this method of frequency comparison by substituting an electronic relay for the mechanical relay, and by using a discharge lamp instead of a spark for illumination. He used the improved apparatus[99] for studying the behavior of the then new and highly stable bridge stabilized oscillators.

Still further improvements in the general method have been reported by H. B. Law using a "phase discriminator" to trigger off a special chronometer, consisting of a decimal scaling counter, and thus avoiding the photographic process[100]. The scaling counter as used here counts the number of cycles of a 100,000-cycle input timing wave that occur during any one beat between the two frequencies being compared, and registers that number, in scale of ten, on a system of dials that can be read directly. In comparing frequencies that are free from interference, the accuracy of comparison by this means is limited chiefly by the precision with which the "phase discriminator" can mark the beginning of successive beats. An accuracy of one part in 10^{11} is claimed. This is one of the rate comparison means employed in the frequency and time standards of the British Post Office and in measurements involving the quartz clocks of Greenwich Observatory and the National Physical Laboratory.

The scaling counter is a particularly useful device for the precise measurement of any time or rate phenomena that can be reduced to the measurement of short time intervals. The counter idea originated some years ago as a means for counting alpha-particles and other phenomena associated with radioactivity studies, one of the original devices being the well known Geiger-Muller counter. The basic scaling circuit, used in many counters, was proposed in 1919 by W. H. Eccles and F. W. Jordan. An interesting history of counting circuits as applied primarily to the counting of electron and nuclear particles has been written by Serge A. Korff in his book on that subject published in 1946.[101] The early scaling circuits operated on the binary system, but recently various circuits have been developed that give the count in scale-of-ten notation with certain advantages, chiefly that of convenience, associated with the common decimal system of notation. A discussion of some modern binary and decade electronic counters[102] was published by I. E. Grosdoff in September, 1946.

Methods of measurement such as this, and the stable properties of the quartz clock which make them desirable, are of importance in the precise measurement of time because the *nature* of variations in rate, so small that, if continued unchanged they would accumulate to only one second in a thousand years, may be studied under controlled conditions in the laboratory, and with such facility that a comparison with this precision can be made every ten seconds.

In a simpler manner, the short period of one oscillation of the quartz oscillator is of direct interest to the astronomer in connection with means for the intercomparison of his clocks in time. This reduces simply to counting the number of cycles gained or lost by one oscillator, referred to another, and may be accomplished in a great number of ways, yielding, on the basis of whole numbers of cycles, an absolute accuracy of time comparison of 0.00001 second.

An elegant method for accomplishing this[103], which also indicates automatically which clock is fast or slow, employs a special vacuum tube circuit to produce a polyphase current having the frequency *however small* of the difference between any two oscillators nominally the same. This polyphase current is used to operate a special synchronous motor whose angular position corresponds at all times to the phase angle of the vector representing the polyphase current. This relation holds all the way to zero frequency difference, in which condition the angular position of the motor, now at rest, indicates the phase relation between the two high frequencies. If the beat frequency goes through zero, the motor reverses. By this means, it is possible with very simple equipment to set up dial indicators showing continuously the time comparisons between any group of quartz clocks, taken in pairs, with an absolute accuracy of 0.00001 second. Of course, to operate other indicators, contacts, etc. from this device is a simple mechanical problem.

The principle of operation of the polyphase modulator is illustrated in Fig. 25, which shows one of the many possible forms of this device. Other modulator elements than vacuum tubes are used in some applications. In the form shown here it is necessary only to assume that the vacuum tubes produce second-order modulation, the lowest-frequency component of which is employed. If inputs at the two frequencies f_1 and f_2, which are nearly the same, are delivered into the two balanced modulators A and B in such a

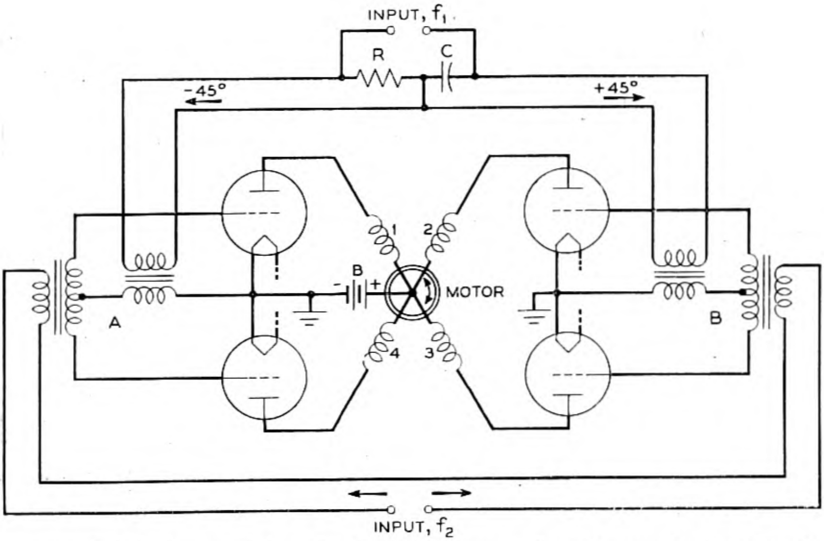

Fig. 25—Polyphase modulator for the absolute comparison of two oscillators of nearly the same frequency.

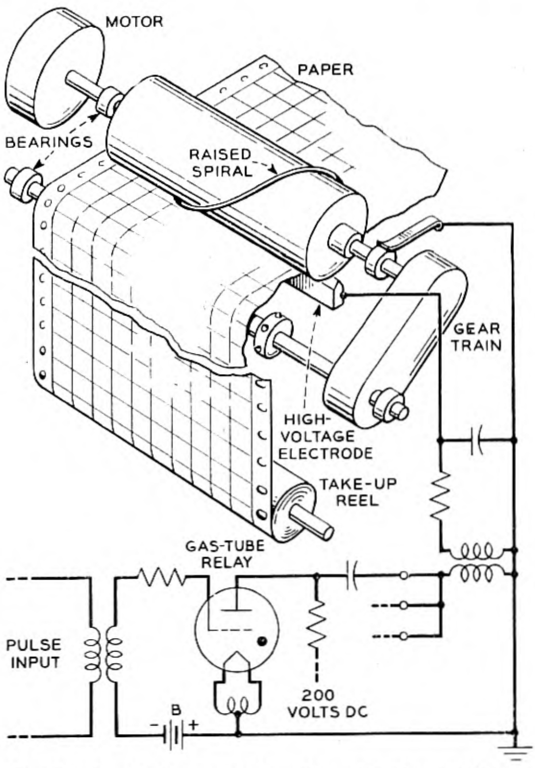

Fig. 26—Spark chronograph—schematic of operation.

way that there is a 90-degree phase shift between the two input voltages for one of the frequencies, the lowest-frequency component appears as a sinusoidal current in the output circuits 1, 2, 3 and 4 separated in phase by 90 electrical degrees in cyclic rotation. The principal output, therefore, is a 4-phase current having the frequency of the difference between the two inputs. If the magnetic circuits are arranged geometrically as shown, the resulting

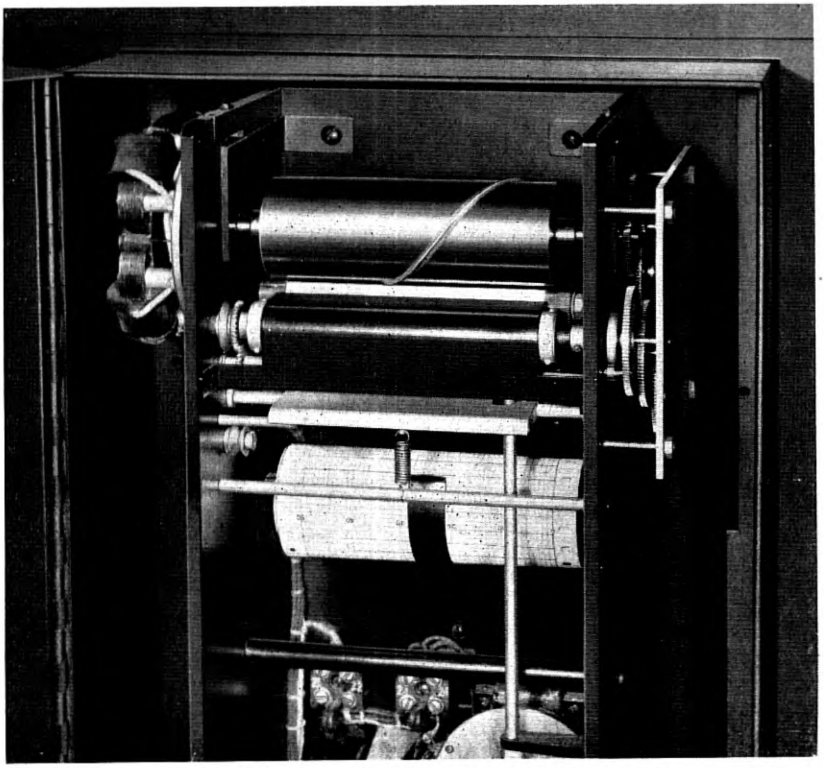

Fig. 27—Spark chronograph—close view of mechanism.

magnetic vector will rotate clockwise or counterclockwise depending on which frequency is high, or will remain stationary, indicating the phase relation, if the two frequencies are exactly equal.

Motors have been designed and are commercially available suitable for operating synchronously from such polyphase modulators, and form an excellent basis for the intercomparison of quartz oscillators and clocks with ultra-high precision.

For making records of time comparisons the spark chronograph[84] shown in Figs. 26 and 27 has served a very useful purpose, combining in a single convenient instrument the means for comparing recurrent time phenomena with an accuracy of a millisecond or two on a continuous chart which shows the records for an entire week. Electrical impulses, related to the time phenomena to be recorded, operate trigger tubes which discharge condensers through the primary of an induction coil and cause sparks to jump from a rotating spiral through a special chart paper having a dark colored backing

Fig. 28—Photomicrograph of single spark record showing nature of recording on wax-coated chart paper. × 100

and coated with a very thin layer of white wax. As the chart paper moves slowly under the spiral, corresponding to the time abscissa, the succession of sparks produces readily visible traces consisting of rows of tiny holes with small areas around them where the wax is melted revealing the dark background. The holes are so small as to be scarcely visible, the darkened areas constituting the visible trace. Figure 28 shows an enlargement of the record of a single spark illustrating the nature of the marking. A recorder[104] very much like the Bell Laboratories' spark chronograph is used currently as part of the standard frequency and time broadcast equipment of the U. S. Bureau of Standards.

Applications of Quartz Clocks

The many useful properties of the quartz crystal clock have been the reason for its wide and expanding application for the precise measurement of time and rate.

First in historical order was the application to the measurement and control of frequency in communication. In this, the clock, through comparisons with astronomical time, served as the means for determining the frequency controlling it, the stability from the outset being great enough over intervals of a day or more so that the average rate, as determined by daily checks with time signals, was a very close approximation to the instantaneous rate at any time intervening. The first of these clocks, already referred to[70], was constructed in 1927 at the Bell Telephone Laboratories, in New York City, primarily for use as an accurate standard of frequency. Since that first experiment, three subsequent installations have been built in replacement with progressively improved performance. The standard now in operation (1947) was installed in 1937, using the first laboratory model GT crystals and the first set of four bridge-stabilized oscillators, and has been in operation continuously since that time. Two of the four oscillators, mounted in a temperature controlled booth, are shown in Fig. 29, and part of the auxiliary equipment, including a clock dial, a spark chronograph and some monitoring equipment, is shown in Fig. 30. This apparatus serves as the standard for precise measurements of frequency and time throughout the Bell System and is used to regulate the telephone Time of Day Service in New York City. It is the standard of reference for the electric light and power services in Metropolitan New York[105], and is used for a number of other similar services, distributed through the medium of a submaster installation[106] maintained by the Long Lines Department of the American Telephone and Telegraph Company. The original oscillators in this submaster installation were controlled by electrostatically-coupled 4000-cycle steel tuning forks *in vacuo* but recently have been replaced by improved oscillators controlled by 4000-cycle bi-morph quartz resonators.

A clock shown in Fig. 31, which is on display in a window of the American Telephone and Telegraph Company at 195 Broadway, is controlled from this source. It is sometimes called "The World's Most Accurate Public Clock".

The facility with which standard frequency and time services can be provided and distributed is an outstanding feature of the quartz clock development. Such services, having the accuracy of the primary controlling standard, may be provided anywhere that can be reached through a suitable communication channel. As an example of this, a new primary standard equipment is being constructed for installation at the Murray Hill, New Jersey location of Bell Telephone Laboratories, the services of which will be

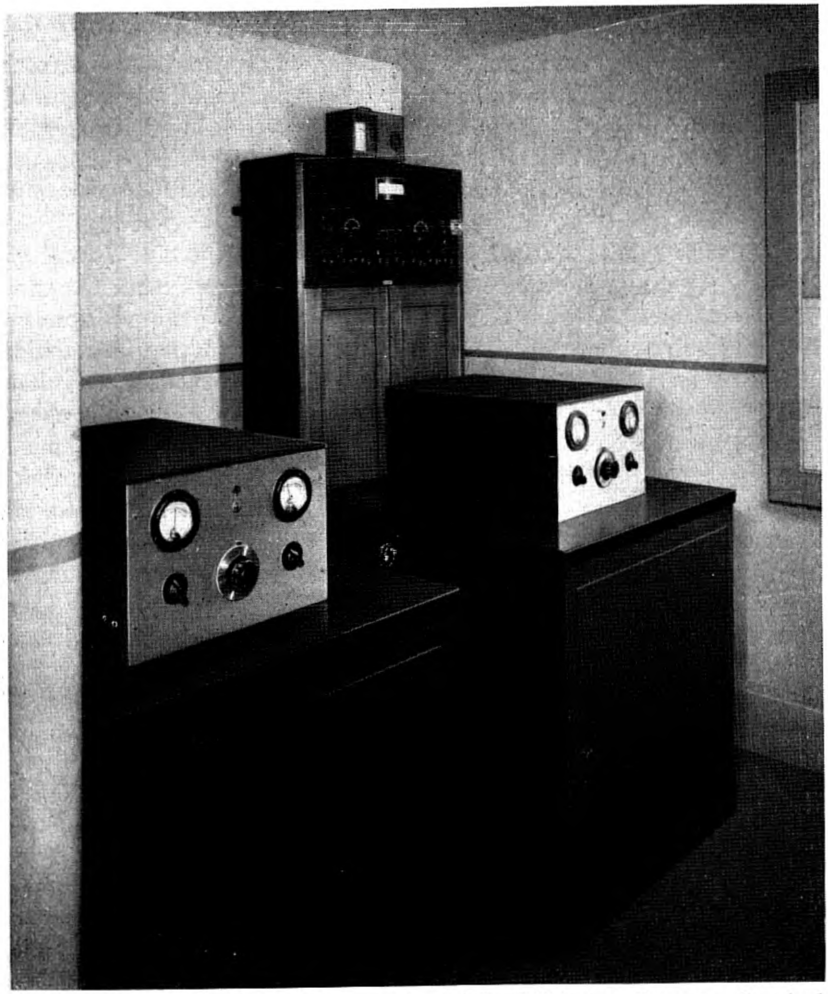

Fig. 29—Two of the four quartz oscillators of the Bell System Frequency Standard, 1937 to date.

made available through permananet wiring to all departments concerned. A number of frequencies in the range from 60 to 10,000,000 cycles, all controlled from the same crystal source, will be made available at some thirty locations at the Murray Hill Laboratories, as well as to other laboratories of the Bell System and, through the Long Lines Department, to outside agencies.

A considerable number of quartz clocks have been built and used in laboratories and observatories all over the world, some as standards of

Fig. 30—Clock dial and monitoring equipment associated with the Bell System Frequency Standard, 1937 to date.

frequency, some as precise clocks, and others for general use in all measurements of rate and time. It would be impossible to mention all of these, for already there are many of them. But certain installations are of especial interest and will be discussed briefly.

When the Crystal Clock was first described as such in April 1930, the idea was discussed quite widely in Europe and America, and it was not long before the work was duplicated and extended in other places. The first outstanding application of the quartz clock to astronomy was made in

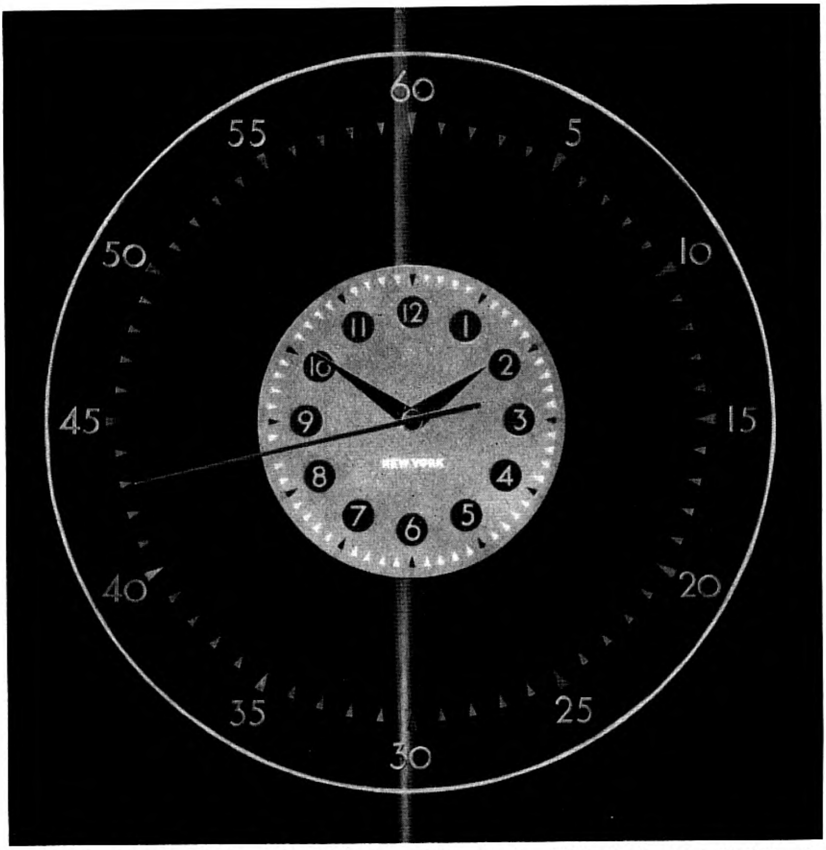

Fig 31—Display clock at 195 Broadway, New York. This clock, controlled by the Bell System Frequency Standard, shows the same time as that of the New York Telephone Time Service.

Germany with the installation at the Physikalisch-Technische Reichsanstalt This was described by Scheibe and Adelsberger in 1932[107] and 1934[108], and reports of its splendid performance continued periodically. It was with this installation that it was possible for the first time to observe and measure variations in the earth's rate occurring over intervals as short as a few weeks. Previous measurements of such variations, involving studies of motion of the moon, the planets, and Jupiter's satellites, had required years to obtain comparable information which, of course, by nature, could never reveal short-term factors.

Soon after the inauguration of the quartz clocks at the Physikalisch-Technische Reichsanstalt, somewhat similar installations were made at the Prussian Geodetic Institute at Potsdam[109], and at the Deutsche Seewarte in Hamburg[110]. The latter has been moved because of war conditions and is now the Deutsche Hydrographische Institut. The quartz resonators used in these installations are believed to be similar to those in Clocks III and IV in the Physikalisch-Technische Reichsanstalt installation except that some of them were made for 100 kilocycles instead of the original 60 kilocycles. They were made by the firm Rohde and Schwarz where also is maintained a quartz clock installation of extremely high precision [111].

For a number of years the U. S. Bureau of Standards at Washington, D. C. has maintained a quartz clock installation for their extensive constant frequency and time services. The early history of this installation was described in some detail by E. L. Hall, V. E. Heaton and E. G. Clapham in 1935.[112] As is now well known, the Bureau broadcasts a number of precisely controlled carrier frequencies at all times, all of which carry standard time and frequency modulations, including audible pitch standards and time signals. The audible pitch standards are 4000 cycles and 440 cycles, while the time signals consist of a succession of seconds pulses, continuous except for certain omissions for the purpose of identifying longer time intervals. All of these rates, including the carrier frequencies, are derived directly from crystal oscillators and are known so well that their accuracy as transmitted is estimated as one part in 50,000,000 at all times. The relative rates of the standard oscillators are compared and recorded continuously at the Bureau of Standards with an accuracy of one part in 10^9. The time signals involved in these transmissions are so precise, and so convenient to use, that they may be employed for the high-precision intercomparison of quartz clocks across the Atlantic and for studies in astronomical time, heretofore difficult or impossible to accomplish by any other means.

The present standard frequency and time service facilities at the U. S. Bureau of Standards, which have been instituted under the general direction of J. H. Dellinger, are described in recent separate articles[113, 114] by Vincent E. Heaton and W. D. George respectively of the Bureau, both of whom have made very substantial contributions to this development. The transmitting station for the standard frequency broadcasts, which comprises a complete set of quartz oscillators and control and measuring equipment, is shown in Fig. 32.

The absolute rates for the crystal oscillators at the Bureau of Standards are determined through cooperation with the U. S. Naval Observatory, also

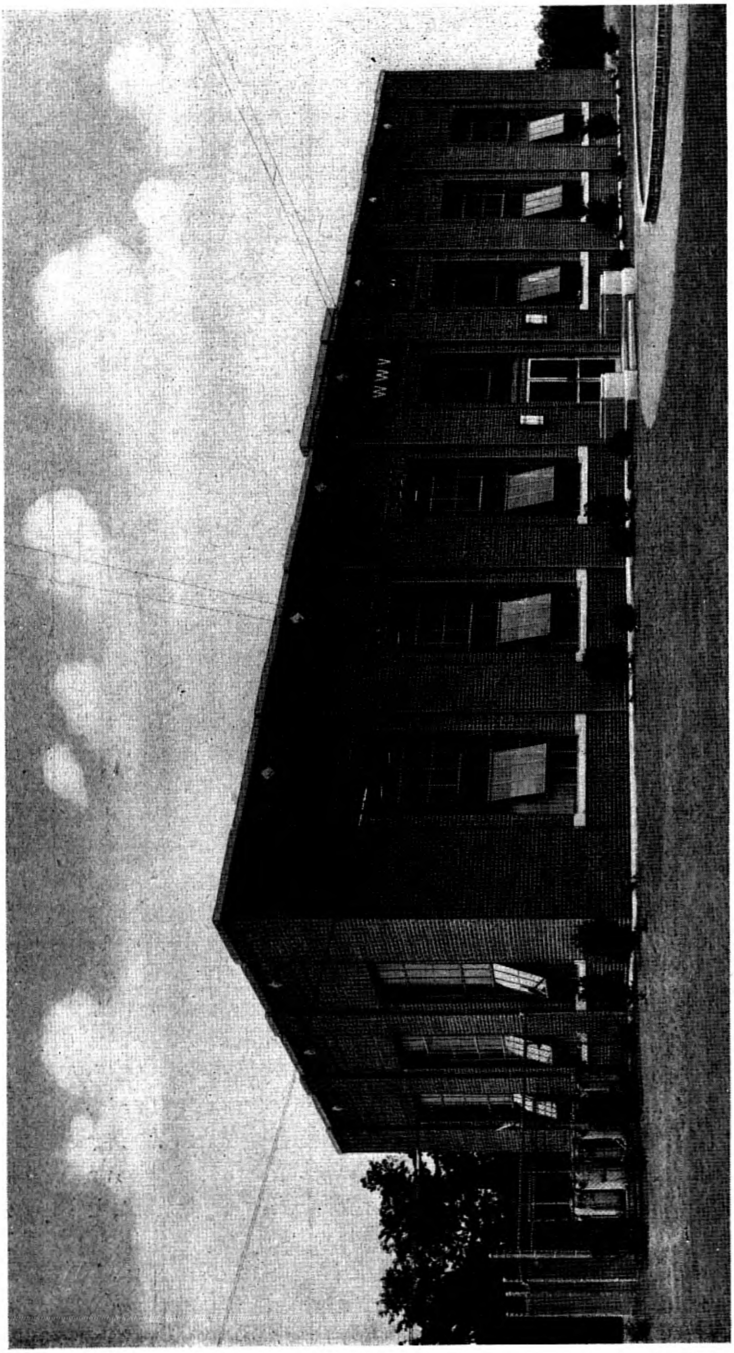

Fig. 32—WWV.—The Standard Frequency Broadcasting Station of the U. S. Bureau of Standards.

at Washington, where time determinations of great accuracy are made by means of a Photographic Zenith Tube and a set of quartz clocks. A continuous precise check is maintained between these organizations by radio communication so that the Naval Observatory time signals sent out from NSS at Annapolis and other Navy stations, and from WWV the Bureau of Standards radio transmitting station at Beltsville, Md., as well as all the carrier frequencies from Beltsville, are very accurately determined and maintained in agreement throughout.

The time studies of the U. S. Naval Observatory up to 1937 are described in two important articles by J. F. Hellweg, then Superintendent of the Observatory. The first of these[115] in 1932 describes the state of the art just before the quartz clock entered the scene, and the second[88] in 1937, already referred to, tells of some of the first improvements brought about by its use including the elegant method for making direct photographic time-star checks of the crystal clock rate by means of the Photographic Zenith Tube. Many of the advances involving the use of quartz clocks at the Naval Observatory have not as yet been published.

The British Post Office and the National Physical Laboratory with laboratories at Dollis Hill and Teddington respectively, in cooperation with the Royal Observatory at Greenwich, have done much the same sort of thing in England in relation to time and frequency measurements and broadcast services as has just been described. Considering the number of crystal units among these organizations and the precise nature of the intercomparisons maintained between them, this is probably the most extensive and elaborate quartz clock system in the world. In connection with Greenwich Observatory alone, the complete installation includes eighteen or more such clocks used in deriving the best possible mean rate from steller observations at Greenwich and from studies of other time observatories throughout the world.

An outline description of the quartz clocks of Greenwich Observatory, and of their function there, has been discussed by Humphry M. Smith in Electrical Times[116] (London) in March 1946. These clocks employ for the most part the GT cut crystal, first described by W. P. Mason, the bridge stabilized oscillator circuit developed by L. A. Meacham, and the regenerative modulator type of frequency dividers similar to those first developed by R. L. Miller.

The accuracy of the quartz clocks exceeds that of the best pendulum clocks with the result that quartz clocks are now used exclusively in the most precise measurements of time. Some of the considerations[117] leading up to the adoption of quartz clocks at Greenwich were discussed in 1937 by H. Spencer Jones, Astronomer Royal. Since then, reports have appeared from time to time by the Astronomer Royal[89, 118] and others[119] concerning the adoption and use of quartz clocks there. Some interesting sidelights on this "Precision Timekeeping Revolution" were written by F. Hope-Jones in two articles[120] for the Horological Journal during the same year. The quartz clock itself, as developed by the British Post Office for Greenwich Observatory, was described[121] in some detail by C. F. Booth in the P.O.E.E. Journal for July 1946. A more general treatment involving some of the same apparatus was presented[122] by C. F. Booth and F. J. M. Laver in the I. E. E. Journal of the same month.

Fig. 33—Crystal chronometer for geophysical studies, consisting of 100 KC. GT-cut crystal, bridge oscillator, and frequency converters to derive precision 500-Cycle output to operate timing devices.

An outstanding example of the versatility of the quartz clock has been its application to the measurement of gravity at sea. Knowing of its stable properties and its independence of gravity, Dr. Maurice Ewing in December 1935, asked the Bell Telephone Laboratories whether a portable quartz clock could be made available for use during a proposed gravity measuring expedition by submarine in the West Indies. Since this was in line with experimental work already in progress at the time, the first portable "crystal chronometer", shown in Fig. 33, was assembled for this occasion, and was taken by Ewing and his colleagues in the U. S. Submarine *Barracuda* on the trip[80, 81] which began at Coco Solo on November 30, 1936. This was the first application of the GT crystal and the bridge stabilized oscillator in portable equipment. This original crystal chronometer has been on several gravity-measuring expeditions and is still in active service, having been used again under Dr. Ewing's direction during the summer of 1947.

Gravity determinations at sea are made by measuring the rate of a special triple pendulum that was invented by F. A. Vening Meinesz especially for use in unsteady environments[123]. Previously, the standard of rate had been the usual ship's chronometers, but Ewing found the crystal chronometer to be an improvement for his purposes, saying in part: "This chronometer is not thermostatted, and temperatures in a submarine change greatly during a dive. No elaborate control over battery voltages was used. The cruise started in the tropics and ended in Philadelphia in mid-winter. It is highly significant that the interval between NAA-time and the chronometer-time never exceeded 0.6 second during the six-week's cruise and that the variation in this interval is very regular. The crystal chronometer has reduced errors in gravity-measurements at sea, due to the rate of the chronometer, to the point where they are negligible."

Some years previous to the construction of the crystal chronometer, a self-contained quartz clock was made to illustrate the possibility of a compact assembly, but it was not sufficiently portable for the submarine expedition. This earlier clock was regulated by a quartz sphere such as used by 'crystal gazers'. The frequency of the sphere was not adjusted, but its natural frequency, which happened to be 33212, was adopted to operate a mean-time dial by the choice of a suitable gear train. Since that time much more compact assemblies have been built using more suitable crystals for control.

The stable properties of the quartz clock have been useful in a number of cases requiring precise synchronization. Perhaps the most noteworthy among these is the application to Long Range Navigation known as LORAN. In this application, pairs of transmitting stations, usually on shore and separated by accurately known distances, send out distinctive signals in synchronism. The time interval between these signals, as received by a ship, identifies the locus of all the points corresponding to that time interval. The set of curves corresponding to all feasible time intervals defines one of the coordinates in a two-coordinate system. The other coordinate is provided in identical manner by another pair of shore transmitters (which may have one station in common with the first pair). The resulting coordinate system consists of two families of intersecting hyperbolas. From the geometry of these curves, and the constants of the signals, the complete figure bounded by the ship and the transmitters can be determined readily.

The need for stability is evident from the fact that the relation between time error and location error is roughly 5 microseconds per mile. In some cases, location within a mile is highly desirable even at considerable distances. Sometimes the two shore stations, operating as quartz clock time transmitters, must operate for hours without intersynchronization, which calls for very great constancy of rate. One microsecond per hour corresponds to one part in 3.6×10^9.

The precise synchronization of mechanical parts in remotely situated stations can be accomplished readily. For a number of years, the 5-band privacy system of the transatlantic radio telephone service has been thus synchronized, the apparatus at the American terminal being controlled by the Bell System Frequency Standard while that at the English terminal is controlled independently by similar equipment in the British Post Office. The accuracy requirement for this particular purpose is not very great. However, it has been found possible to maintain two or more rotating shafts at remote and independent stations so precisely controlled by independent quartz oscillators that they never depart, during hours of operation, by more than one fifth of one degree of arc.

A major project in which the quartz clock is destined to take an important part is that of making world-wide land and water surveys in order to locate more accurately boundaries and other features of the earth's surface. There would be applications to sea and air navigation and it would be of great value to geophysicists in studying the figure of, and changes in, the earth's surface. By the combination of a widely dispersed set of Photographic Zenith Tubes associated with quartz clocks and time signal means for communication, and with the powerful ranging techniques growing out of LORAN and RADAR, it should be possible to obtain a new order of accuracy in long distance surveying.

The new order of accuracy of time measurement has made it possible for the first time to study directly the variations in longitude caused by the irregular wandering of the poles. These are small effects and heretofore could only be determined by inference from observations of apparent latitude variations at remote stations. With the added new techniques it should be possible to learn a great deal about these and other phenomena related to real or apparent variations in longitude.

Two other possible applications, involving the precise control of angular movement so readily obtainable with synchronous motors operated from quartz crystal controlled alternating current, are of considerable interest. The first is that of operating the right ascension control of a telescope directly from the amplified output of a crystal-controlled low frequency. Vacuum tube amplifiers and synchronous motors are commercially available with which this could be accomplished by suitable gearing. In addition, of course, it would be necessary to include auxiliary controls to allow for atmospheric and other transient effects, and for obtaining rates of motion other than sidereal. For small and slowly changing effects this could be taken care of very simply by means of electrical circuits now well known for adding or subtracting small changes in the control frequency.

The other application refers to a suggestion made by the author a few years ago[124] for the measurement of gravity, and changes in gravity, by comparison of the forces Mg and $M\omega^2R$. The proposal was based on the idea that ω can be measured or produced with an accuracy two or more orders greater than required, and that the problem reduces to that of balancing two forces and of measuring a linear displacement. The physical set-up would be some form of conical pendulum driven at constant angular velocity about the vertical axis under control of a crystal. Some such arrangements are shown in the reference.

Future Possibilities

It is part of the nature of a scientist to extrapolate ahead of any current development and to wonder what lies beyond. That feeling is certainly justified in the field of time measurement, for the major advances have taken place in so short a period and so recently, as compared with the thousands of years during which Man has been time-conscious in some degree, that it is reasonable to expect continued advancement for many years to come. Such advancement may come as improvements and refinements in existing techniques, or radically new methods may be developed with inherently more stable potentialities.

Accuracy of Rate

In the first place, it is not reasonable to suppose that the final accuracy that can be attained with the quartz crystal clock has been reached; in view of the rapid current progress indicated in the chart of Fig. 1, it is much too soon to assume this, and there is considerable evidence that improvements could be made by making fuller use of some of the stable properties of quartz crystal and of refinements in the mounting and sustaining circuits. The quartz oscillator assemblies in most general use at the present time embody some compromises which it would not be necessary to make if an all-out effort were being made to construct a few clocks having the highest attainable stability under the most favorable conditions of operation.

The first of these concerns the shape and size of the resonator itself and is related to the frequency of oscillation. From the standpoint of stability of operation, the actual frequency that is used in the oscillator is of little concern because it is now a very simple matter to obtain low frequencies, suitable for the operation of mechanisms, starting with any frequency that can be controlled by a crystal resonator. The choice of 100,000 cycles for the first zero-coefficient resonator was made because, as a standard of frequency, that value was a good median for the range of frequencies then used in electrical communication. For use in a clock any other frequency would answer just as well, so the inherent stability of the resonator should be given first consideration.

One of the inhibitions imposed on the design of quartz resonators has grown out of the dwindling available supply of large pieces of perfect crystal quartz. Where large quantity production is involved this is an important consideration, but for the small numbers required in a few observatories and national laboratories it should not be a limiting factor.

Except for whatever added difficulties might be entailed in the mounting, it seems reasonable that a large resonator should be more stable than a very small one. The most fundamental reason for this is the proportionate change in effective size that would result from the transfer of any surface material including even the quartz itself.

Every substance is supposed to have some vapor pressure although in some cases it is very minute. However, we are concerned with very minute effects, and it is worthwhile to consider what would happen if there were any evaporation or condensation of material. The possibility of this being an important effect is evident when we realize that the removal of a single layer of molecules from the end of a resonator one centimeter long would increase its frequency by about five parts in a hundred million. The effect on frequency would vary about inversely as the effective length, which favors a large crystal. Such a transfer of material could be inhibited to some extent by operating at a low temperature and by seeking equilibrium between the quartz material of the resonator and other quartz material within the same envelope. Of ourse, other materials than quartz may be involved in similar surface phenomena and should be thoroughly studied and controlled. This has a strong bearing, of course, on the use of conductive materials deposited on a resonator for the purpose of electrical coupling to it.

The slightest trace of surface contamination has a deleterious effect on the damping coefficient. Professor K. S. Van Dyke in 1935 made a series of measurements on resonators of uniform shape and size but constructed with a considerable range of surface treatments[45]. In the construction of different resonators used in these tests he used different grades of abrasive and various amounts of etching with hydrofluoric acid. In these experiments he operated them under varying degrees of refinement with regard to contamination of the surfaces and found that the highest Q was obtainable only after the utmost care was exercised in keeping the surfaces free from foreign material. The effect is so striking, in fact, that it leads one to wonder whether there is *any* actual elastic hysteresis in the material of quartz crystal, or whether the minute energy losses observed are entirely surface and coupled effects. Since, for a given shape, the volume increases with linear dimension in greater porportion than the surface area, it can be inferred that surface phenomena would affect a large resonator less than a smaller one.

This is also a reason for employing a stubby shape, in order that the volume of crystal may bear as large a ratio as possible to its surface area. From this standpoint alone a sphere would be ideal but for other reasons, chiefly concerned with the temperature coefficient, it would be unsuitable. It is probable that a polished prolate spheroid, properly oriented with respect to the crystal axes, would satisfy both conditions. Such a resonator could be supported by a pair of wires, serving also as electrical leads from metal-plated electrodes, using techniques already well established.

Crystal resonators as now used in many of the most stable oscillators have been constructed to withstand severe mechanical shock while in operation. It is likely that a slight improvement in frequency stability might be obtained by relaxing a little on the mechanical stability of the present support. Where the greatest accuracy of rate is desired, such as in national standards laboratories and in astronomical observatories, it should be possible to provide suitable mountings for crystal resonators having more delicate supports than those required in mobile equipment. The GT crystal illustrated in Fig. 21 is mounted on eight supporting wires for applications requiring great mechanical stability, and at the same time remains one of the most stable frequency controlling resonators ever produced. It would be reasonable to expect a little improvement in frequency stability at the expense of some mechanical stability if four supports were used instead of eight.

There is a good possibility also that some improvement could be obtained by reducing the electrical coupling to the crystal. At present, the plates are usually provided with plated metal electrodes which cover the entire large surface areas. Some increased stability in frequency might be expected by the use of relatively smaller electrodes covering only the central part of the resonator where the amplitude of vibration is small. At least two advantages might be expected from such a modification. One is that the loading effect is least near the node for vibration, another is that any looseness of material, or elastic hysteresis, would be least troublesome where the motion is least. Of course, it is chiefly the *variations* in such effects that concern us. One would expect, however, that if such effects exist at all they might be minimized by the use of smaller electrodes.

These particular effects may be eliminated completely, of course, by the use of isolated electrodes spaced from the crystal—but at the expense of other possible variations related to changes in electrode spacing. There is considerable promise in such means, the end result depending upon how precisely the resonator may be held in a fixed position by means that will not change its resonance characteristics. Such means have, in fact, been used successfully in a number of German quartz clocks such as at the Physikalisch-Technische Reichsanstalt[108], and with the Dye ring resonator developed by D. W. Dye and L. Essen at the National Physical Laboratory[125, 126], England.

For any given resonator and circuit a careful study would probably reveal an optimum amplitude of oscillation that would yield a maximum stability against residual uncontrollable variables. With the GT crystal, as used currently, the maximum amplitude of motion is about 0.00006 mm. It would be possible to limit the motion to a tenth or a hundredth of this value if it should be found desirable.

Further studies of the factors contributing to aging of the quartz material also should produce valuable improvements. Since resonators, which appear to be alike in all other respects, often age at greatly different rates, some being very small or substantially zero, it would seem that some reason should be discoverable for such variations and some effective control established.

There are other relatively massive shapes that should be investigated further such as the ring crystal, mentioned earlier in this paper, and as developed and studied by Dye and Essen[125, 126]. The ring may be excited in various modes of vibration some of which are more favorable than others from the standpoint of mounting. By choice of orientation relative to the crystal axes, and of dimensions, certain of these can be designed to have zero temperature coefficients in a restricted temperature region.

Another shape that holds great promise because of its convenience of mounting, along with the other desirable properties, is the rectangular rod vibrating longitudinally in its second or higher overtone such as first described by Scheibe and Adelsberger[108]. Still another possible massive shape is a much thicker version of the GT crystal which would combine the very favorable temperature-frequency characteristic with that of reducing the ratio of surface area to volume.

In seeking the highest possible accuracy a precise temperature control is essential in all cases, even with the GT type of resonator with its wide region of low-temperature coefficient. The reason for this is that the frequency of oscillation depends not only on the mean temperature of the resonator but also upon the temperature gradient throughout its volume. Thus, even if a resonator has the same frequency exactly at different mean temperatures, its frequency will vary a little while the temperature is varying from one value to another. The effect of this can be reduced by enclosing the crystal unit in an envelope with thermal lagging so that such *variations* as do exist at the temperature control layer are prevented from reaching the crystal.

This is no longer a serious problem for there are various electronic means such as described by C. F. Booth and E. J. C. Dixon[127] for continuous temperature control, by means of which the variations may be kept very small, and very effective thermal lagging methods[128] are well known.

The bridge method for temperature control has been applied in many forms. One of the simplest and most effective procedures has been to utilize a bridge-stabilized oscillator of the type developed by L. A. Meacham for *frequency* control, and to use it instead for *temperature* control. For this purpose, all four arms of the bridge are noninductive resistances wound as heaters on the oven to be controlled. In the feedback circuit of the oscillator, a rough frequency control is included simply for the purpose of setting up an oscillation in the circuit which includes the bridge. The conjugate pairs of bridge arms are made of resistance wire with different temperature coefficients and so proportioned that the bridge balances at the desired temperature. The *amplitude* at which this bridge oscillator oscillates depends upon the temperature departure from the balance value. Since the alternating current output of the oscillator flows in the bridge arms, the amount of heating is proportional to the temperature error, and hence the control is automatic.

Continuity of Operation

An astronomical clock, in addition to having as nearly constant a rate as can be attained, should also be able to operate over long periods of time without change or interruption. The reason for this is that many of the phenomena that are of interest in time measurement occur in continuous succession and the greatest amount of information can be obtained only by the use of clocks with which measurements can be made in unbroken sequence. Quartz clocks that have been used for astronomical purposes to date have not had a very commendable record in this respect and already a good deal has been said in the clock literature about this aspect—as though it were an inherent property of the quartz clock.

However, it is only a matter of simple engineering, making use of techniques and apparatus already well known and available, to design a quartz clock which should operate continuously for many years. A chain is only as strong as its weakest link—and the clock comprises a chain of apparatus parts every link of which must function perfectly and continuously. This chain consists of (1) the crystal-controlled oscillator, (2) a frequency demultiplier to obtain a low frequency to operate a motor, (3) a power amplifier to obtain sufficient current to drive the motor and (4) the motor itself, associated with any of a wide assortment of time signal-producing or measuring equipment. In addition to the links in this chain, a power supply must be maintained, and the temperature of the crystal must be controlled, both continuously.

The crystal itself is no problem as far as continuity of operation is concerned. Its motion is so very small there is no likelihood at all of failure on that account. Mountings are very stable and in all likelihood will be improved. The oscillator circuit, the frequency demultiplier, the power amplifier and the temperature-control circuit are all vacuum-tube devices and deserve special consideration. In all of these circuits, vacuum tubes have been used in some installations which do not have a very long life, some even becoming defective within a year of operation. On the other hand, there are tubes which have been developed for use in continuous telephone circuits where failures would be troublesome and costly. Some of these tubes in current production have an expected life of more than ten years. There is good reason to believe that a quartz clock installation equipped with such vacuum tubes throughout, and engineered so as to make effective use of their special properties, would operate continuously for ten years or more.

The remaining "link" in the chain is the synchronous mechanism operated from the crystal-controlled circuits and used for totalizing continuously the oscillations of the crystal and for producing suitable time signals at specified intervals of time thus measured off in terms of the crystal rate. This mechanism usually consists of a small synchronous (phonic wheel) motor operated from a submultiple of the crystal frequency and geared to commutators or cams or other means for producing the electrical signals used in making time measurements. Many of the troubles in quartz clock installations have occurred in this 'link'. There is every reason to believe, however, that suitable synchronous motors geared to cam-controlled electrical contacts can be built that will operate continuously through many years. To insure long operation it would be desirable to employ motors with low rotation speed in order to reduce bearing wear. With the present knowledge of bearing materials and lubricants, it should be a simple matter to design such a motor that would operate without failure for ten years or more.

A relatively trouble-free electrical time signal producer, suitable for operating under the control of a quartz oscillator, with frequency demultipliers to 100 cycles, could be constructed as indicated schematically in Fig. 34. This is not intended to be an actual design, but is intended to indicate how an apparatus could be designed that would circumvent some of the troubles now experienced which prevent long continuous operation.

The basic apparatus consists of a crystal oscillator, presumably 100,000 cycles, with a frequency divider to obtain controlled 100-cycle current to drive the 100-pole phonic wheel motor at one revolution per second. Obviously, other crystal frequencies and step-down ratios could be used, the important thing being to obtain a rotation speed of 1 rps. This is a very low speed for a phonic wheel motor but has the obvious advantage of great simplicity since it permits of controlling seconds devices without the use of gearing. Only one shaft is involved and the bearing problem is reduced to the simplest possible terms. A hardened steel cam, integrally mounted with the phonic wheel rotor, is used to operate a single electrical contact, so connected into the circuits controlled by it that the *instant of break* is the sole time-determining operation. A *break* signal is preferable to a *make* signal chiefly because it is easier to avoid irregular effects, such as result from contact chatter, when a circuit is being opened than when it is being closed. If a pallet of sapphire or ruby is used for the mechanical contact on the cam,

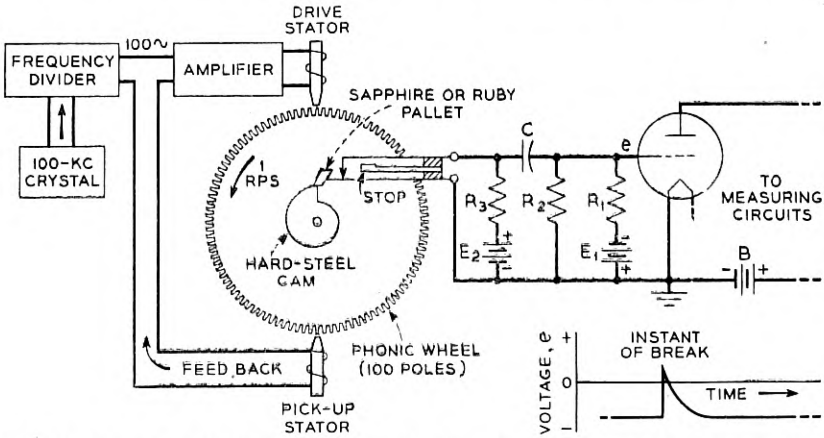

Fig. 34—Suggestion of elements for a quartz clock for long time continuous operation.

and if small currents are used through the contacts, made preferably of platinum-iridium or similar alloy, it would be reasonable to expect trouble-free performance through several hundred million operations.

Ordinarily, the "hunting" of a phonic wheel motor operating on a frequency as low as 100 cycles would cause time errors too large to neglect in a device such as just described. However, by the use of feedback in the motor amplifier circuit, such as indicated schematically in Fig. 34, the effective hunting can be reduced to the point where the time errors caused by it would become negligible for most purposes.

Various circuits could be suggested for making use of the break signal for timing purposes, the one shown in Fig. 34 being typical and suitable for various methods of precise measurement and control. It is capable of providing an electrical impulse with a steep wave front and of adjustable duration. The grid of the vacuum tube is normally biased to cutoff by the negative voltage, $\frac{E_1R_2}{R_1+R_2}$. While the contact is closed, the battery E_2, with resistance R_3 in series, is short-circuited. But at the instant of opening the contact, current flows momentarily in the circuit including E_2, R_3, C and R_2. By making E_2 positive, and equal to or larger than E_1 numerically, the plate circuit of the tube becomes conducting for a short interval, the duration of which is determined by the time-constant of the condenser circuit, each time the contact is opened. At all other times, the plate circuit is nonconducting. The sharply defined electrical signal thus produced in the plate circuit can be used by well-known means for direct time comparison with signals from other sources.

Making use of the *duration* of the impulse thus produced, it is possible to use it as a selecting means to isolate a single more precise signal from a continuous chain. For example, the 100-cycle wave controlled by the crystal can be modified by a simple vacuum tube circuit to consist of a continuous sequence of very sharply defined impulses. By using the pulse circuit just described as a bias control on an amplifier, it would be readily possible to select one out of every hundred of these impulses and thus provide an extremely precise seconds signal, the accuracy of which is determined wholly by electronic means.

It would be readily possible to vary the time relation of the seconds signal while in operation, by the use of electrical phase shifters in the driving circuits, or by rotating the stator of the phonic wheel motor, but for long continuous operation it would be desirable to keep the number of apparatus parts comprising the clock at a minimum.

It is not necessary, of course, to employ a complete frequency divider and phonic wheel apparatus for each quartz crystal oscillator. As mentioned previously, the relative time rates of quartz oscillators can be measured with very high precision and be very simple means through a direct comparison of the high frequencies.

Other Means for Precise Rate Control

In addition to making improvements on the quartz crystal resonator, and on methods for sustaining it in vibration, there are two other avenues of investigation which may yield comparable results, with possibly some additional advantages. Not much can be said about them at this time except to point out their possibilities because no appreciable work has been done so far to explore their merits as timekeepers.

The first is in the field of very low temperatures where some quite remarkable properties are obtained. Chief of these for our purpose is the supraconductivity of some metals, and the constancy of shape of most materials, at temperatures in the neighborhood of absolute zero. It seems reasonable to suppose that an electrically-resonant circuit maintained at a temperature in this region could be made to have a very high Q, and very stable dimensions, and so have the chief desirable properties for rate control that obtain in a quartz resonator. Resonant cavities used at high frequencies have many of the properties of other electrical resonant circuits, and in particular their energy dissipation for electric oscillations can be very substantially reduced when cooled to superconducting temperatures. In some experiments made recently at Massachusetts Institute of Technology[129] it has been shown that a cavity resonator made of lead, which for 3-cm. waves has a Q of about 2,000 at room temperatures, is so much improved at a temperature of 4 degrees absolute that the Q approaches a million. Such a resonator could be used as the stabilizing element in an oscillator and hence in a clock. The relative stability over long periods could, of course, be determined only by experiment.

Maintenance of the required low temperature would add considerably to the complexity of such a system, but if the advantages were such as to produce a new order of stability, and particularly if it should make possible a clock system with small or zero aging, it certainly should be justified for future time measurement studies.

The other avenue of approach is through the application of certain resonance phenomena in atoms and molecules that do not depend upon aggregates of matter as is the case with all mechanical systems used heretofore in time measuring means. The extreme fineness of structure and the constancy of atomic and molecular resonance phenomena have long been recognized through studies of line spectra, and in the field of spectroscopy these properties have been used as standards of wavelength ever since the early studies of Joseph von Fraunhofer, reported in 1815.[130] Wavelength, λ, and frequency, f, are associated by the simple relation $f = \frac{c}{\lambda}$ where c is equal to the velocity of light. For visible radiations f turns out to be extremely large, for the red light, 6500Å, it is 462 million million vibrations per second. So far, such high frequencies have not been observable or measurable directly but can only be deduced from wavelength measurements as just stated—which inevitably involve the use of man-made standards of length and the combined errors of two quite different sorts of physical measurements.

It has long been the dream of physicists to find some way to tie in directly with the natural frequencies of atoms and molecules and to derive from them a direct measure of rate, and, of course, of time interval. It has been thought, for example, that the red radiation from cadmium vapor, whose wavelength was measured by C. Fabry and A. Perot in terms of the standard meter as accurately as that standard could be defined, would also make a good standard for time measurements. A step in the right direction was made later by A. A. Michelson whose precise determination of the wavelength of this radiation made possible the redefinition of the International Meter as a definite number of such wavelengths, measured in vacuo. From this definition, it is now possible to duplicate the primary standard of length with great accuracy, and to check such secular changes as may occur in the original standard, the distance between two marks on a metal bar. The constancy of the standard, as defined by Michelson, depends upon properties of primary particles of matter, and upon properties of space, which, as far as human beings are concerned directly, appear to be quite independent of time or location. A similar definition of rate, or time interval, is very desirable.

A ray of hope came out of the important work of Nichols and Tear[131] who proved that electric waves which could be produced electrically were of the same stuff as radiation from hot bodies. They were able to detect radiation of either sort by the same receiving device and showed that they both had the same properties of refraction, polarization, etc. Later, Cleeton and Williams[132] were able to produce *continuous* electric waves at very high frequencies—corresponding to about 1 cm. wavelength—and to show that they also had the important properties of light waves. Now the range has been extended somewhat more and there are reports[133] of experimental generators that can produce continuous waves of a few millimeters wavelength. This is an active development and, of course, the end is not in sight. From continuous waves of any frequency it is believed possible by general techniques now well known to control lower frequencies, and from them eventually all sorts of time measuring and indicating devices as previously described.

Within the last few years, the missing link has been discovered which, with suitable instrumentation, may make it possible to construct a clock controlled by atomic- or molecular-resonance phenomena. There are a great number of resonance phenomena associated with the molecules in a gas, or in molecular beams, which are responsive to electric waves that can be produced continuously by modern vacuum tube means. In some cases, the sharpness of resonance is such that changes of frequency of one part in 10^8 or less can be detected, leading to the idea that such resonance phenomena may be utilized in some way to *control* the frequency of a suitable oscillator and hence, through frequency conversion circuits, to control frequencies low enough to operate clocks and other mechanisms. Some of the resonance phenomena in point are in the one-centimeter region, a field that is rapidly being exploited in radar and communication applications. It is to be expected, therefore, that techniques for dealing with such high frequencies will be developed in the near future thus facilitating a study of this new approach to timekeeping. The idea of utilizing such resonance phenomena for the measurement of time was suggested in January, 1945 by Professor I. I. Rabi of Columbia University at an address before the American Physical Society and the American Association of Physics Teachers.

These resonance phenomena, involving the interaction of microwave electromagnetic radiation with atoms or molecules of matter, have been discovered only quite recently and it is likely that a great deal more will be learned about them in the next few years. The results already obtained are very promising and investigations already under way may well lead to the means for creating an entirely new type of standard of time interval and rate—both of prime importance in Physics.

The studies of greatest significance for such purposes now in progress fall in two main branches involving quite different techniques. The actual means for regulating a clock would be quite different in the two methods, but would be possible in either. With what is known up to the present time, however, the construction of such a clock would be a considerable undertaking, especially to make one that would operate over long periods. The two chief phenomena involving atomic or molecular resonances are: (1) the absorption of high-frequency energy in certain materials, particularly in gases, exhibiting ultra-fine absorption spectra; and (2) the deflection of beams of atoms or molecules under special conditions of magnetic and electric fields. The earliest reported work on the absorption of microwaves in gases was done by C. E. Cleeton and N. H. Williams[134] in 1934. With the development of improved high-frequency generators and measuring techniques the work has been extended considerably during the last few years by C. H. Townes[135], W. E. Good[136] and others. It is believed that with modifications of methods, such as used by them, it would be possible to control the frequency of the short-wave generators such as used in making these studies; and, if this can be done, the adaptation for use in time-measuring devices would follow naturally as in the case of any other stable oscillator.

The general method using molecular beams has been a gradual development over some years, but the first published suggestion of the applications which relates closely to this work was made in 1938 when I. I. Rabi, J. R. Zacharias, S. Millman and P. Kusch first used the beam deflection method for measuring nuclear magnetic moments.[137] Two articles[138, 139] in *Reviews of Modern Physics* in July 1946 give a good description of the molecular beam method and the results of some studies of fine structure resonance phenomena. The resonance curve shown in Fig. 35 obtained recently by P. Kusch and H. Taub of Columbia University, and hitherto unpublished, illustrates the resolution obtainable by molecular beam methods. According to theory, the actual *width* of the resonance should be substantially independent of the applied frequency and they expect to be able, when employing frequencies corresponding to centimeter waves, to obtain a hundred or more times this resolution. If this should be realized, it suggests the possibility of a clock with an accuracy of better than one part in 10^8.

Perhaps the greatest advantage that might be expected from such a method lies in the possible long-time stability or freedom from aging. Every existing means for timekeeping involves in some manner the motion of large aggregates of matter which, when they rearrange themselves in any way, vary their rates of rotation, or of oscillation, as the case may be, in ways

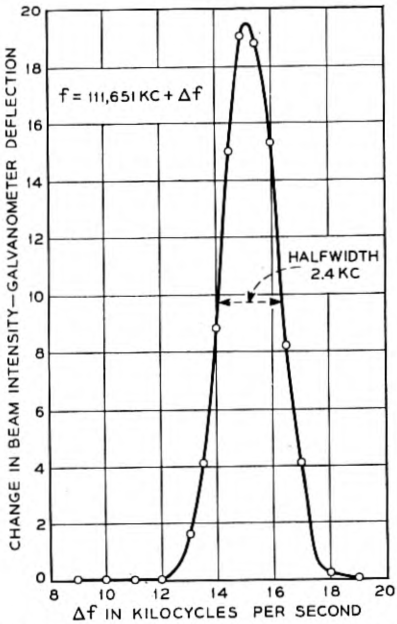

Fig. 35—Typical resonance curve for a line in the radio frequency spectrum of atomic K^{39} observed by the method of molecular beams. Experimental data supplied by P. Kusch and H. Taub, Columbia University Physics Department.

that are not wholly predictable. It may well develop that a method based on the behavior of single particles of matter will be ageless and, with proper instrumentation, that it will permit of setting up an *absolute* standard of rate and time interval. The actual value of this rate would be indeterminate by a small amount depending on the sharpness of resonance and the precision of control that could be effected from it, in addition to any uncontrollable effects of the actual resonance frequencies such as result from temperature, pressure, and electromagnetic and gravitational force fields. In the case of some of the resonance phenomena all the latter effects are believed to be vanishingly small. In any case, one would not expect to experience a progressive change in rate as in the case of the rotation of the earth which now is the measure and definition of astronomical time. On the average the earth is said to be slowing down at the rate of a thousandth of a second per day per century[140] and, according to the astronomers[89], the day will continue to lengthen until finally, at some time in the distant future, the earth will always face one side toward the moon and the length of the day will become about 47 times as long as it is at the present time.

Meanwhile, if an absolute standard could be established, such as now appears feasible through atomic- or molecular-resonance phenomena, it would be possible to record these changes through the centuries and to establish a relatively stable "second" that could be used for all time in physical measurements in place of the elastic second of the cgs system which, as now defined, must stretch with the inevitable variations in the mean solar day.

Whether or not such an "absolute" clock becomes a reality at some time in the future, the quartz crystal clock, because of its accuracy, compactness, great convenience and versatility is likely to continue to be a most useful instrument in all precision measurements of time.

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Abstracts of Technical Articles by Bell System Authors

Experimental Determination of Helical-Wave Properties.[1] C. C. CUTLER. The properties of the wave propagated along a helix used in the traveling-wave amplifier are discussed. A description is given of measurements of field strength on the axis, field distribution around the helix, and the velocity of propagation. It is concluded that the actual field in the helix described is slightly weaker than would be predicted from the relations presented by J. R. Pierce for a hypothetical helical surface.

Results of Microwave Propagation Tests on a 40-Mile Overland Path.[2] A. L. DURKEE. This paper gives the results of a series of microwave radio propagation tests over an unobstructed 40-mile overland path. The purpose of the tests was to investigate the transmission characteristics of such a path at centimeter wavelengths over a long period of time. Statistics on the transmission results at wavelengths ranging from 1.25 to 42 cm. are given. The tests extended over a period of about two years.

A Tunable Vacuum-Contained Triode Oscillator for Pulse Service.[3] C. E. FAY* and J. E. WOLFE. A tunable push-pull triode oscillator is described in which the vacuum-tube components and the entire r.f. portion of the oscillator circuit are contained in an evacuated metallic envelope. A terminal is provided for coaxial output into a 50-ohm transmission line. The oscillator was developed for the frequency range of 390 to 435 Mc. and is tunable by mechanical means continuously through this range. Pulse power of above $\frac{1}{2}$ megawatt is obtained with pulse voltages of 15 to 17 kilovolts applied.

A Proposed Loudness-Efficiency Rating for Loudspeakers and the Determination of System Power Requirements for Enclosures.[4] H. F. HOPKINS and N. R. STRYKER. Experimental and computed data relating to the loudness contribution of various ranges of the frequency spectra of speech and music are correlated with the corresponding energy distribution. A relatively simple measurement of sound pressure and a knowledge of certain acoustic radiation phenomena are applied to this correlation to form the basis of a

[1] *Proc. I. R. E.*, February 1948.
[2] *Proc. I. R. E.*, February 1948.
[3] *Proc. I. R. E.*, February 1948.
* Of Bell Tel. Labs.
[4] *Proc. I. R. E.*, March 1948.

method for predicting the loudness established by loudspeakers in enclosures. A loudness-efficiency rating for loudspeakers is suggested, and its application to sound-system engineering problems is described.

A Sheet of Air Bubbles as an Acoustic Screen for Underwater Noise.[5] DONALD P. LOYE* and WM. FRED ARNDT. In Pearl Harbor, where there often were eight hundred ships of all kinds, the underwater noise level was high. No place was found where noise measurements could be made satisfactorily, and therefore it was decided that the best arrangement would be to insulate Auxiliary Repair Docks and measure the noise of submarines while they were in the docks. This was done by the development of a suitable air bubble screen across the open end of the dock. Such an acoustic barrier was comparatively easy to install, did not interfere with submarines entering and leaving, kept ocean surface oil out of the dock, insulated against low- as well as high-frequency noises as was required and, after extensive experimentation, the noise of the screen was reduced to a level that did not interfere with the noise measurements. The insulation of the screen upon the noise of a nearby submarine charging batteries is illustrated by a phonograph recording.

A Method of Determining and Monitoring Power and Impedance at High Frequencies.[6] J. F. MORRISON and E. L. YOUNKER. A method and newly developed devices for determining and monitoring power and impedance levels in transmission lines at high frequencies are explained. Practical considerations influencing accurate determination of power and impedance levels are analyzed, and the previous and newly developed methods of monitoring these important quantities under changing conditions of load are compared.

Automatic Volume Control as a Feedback Problem.[7] B. M. OLIVER. Feedback amplifier theory is shown to be applicable to the usual a.v.c. system. Expressions are derived for the loop gain in terms of the design requirements and the gain-control characteristic of the controlled amplifier. Using these expressions, the design of an a.v.c. system is quite straightforward and its characteristics, such as regulation and effect on desired modulation, are readily predictable.

[5] *Jour. Acous. Soc. Amer.*, March 1948.
* Of Western Electric Co.
[6] *Proc. I. R. E.*, February 1948.
[7] *Proc. I. R. E.*, April 1948.

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