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Pulp Insulation for Telephone Cables *

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Pulp insulation is a new type of insulation that has been developed to replace the well-known spirally wrapped ribbon paper insulation in certain kinds of telephone cables. It consists of a continuous pulp sleeving formed directly on the wire by a modified paper making process. The raw material for this insulation is commercial Kraft pulp and its preparatory treatment in the beaters corresponds to that given in the regular paper making process.

The machine used to apply this pulp to the wire is a modified single cylinder paper machine equipped to insulate 60 wires simultaneously. The wires are taken from the supply spools by means of flyers so as to allow the brazing of the wire on a nearly empty spool to a conveniently located full one. This gives continuous operation. The wires are fed to the machine through an electrolytic cleaner for the removal of residual drawing compound. The surface of the mold or paper forming mechanism is divided into 60 narrow portions in such a way as to form that many narrow sheets continuously. The wires are brought into contact with the mold in such a way that, as it rotates and forms the sheets, a single wire is embedded in each sheet. These sheets and wires are transferred from the mold to a traveling wool blanket by the pressure of the couch roll. The traveling blanket carries the sheets and wires through the presses for dewatering and consolidating, and delivers them to the polishers where the sheet is turned down by a rapidly rotating mechanism into a cylindrical wet sleeve surrounding the wire. The moisture is driven from the wet insulation by passage through a box type electric furnace one end of which is maintained at a rather high temperature. The insulated wire is then taken up on spools ready for the twisting operation. The speed of the machine is about 130 feet per minute.

The major difficulties in the process have been overcome and the basic properties of the insulation have been determined. Equipment for the production of about 225 million conductor feet per week has been provided and the entire output of 24 and 26 A.W.G. cables is being made in pulp.

These cables are designed to the same size as the ribbon paper cables which they replace and compare favorably with them in their electrical characteristics except that the mutual capacitance is slightly higher. The impairment in transmission efficiency due to the higher capacitance is, however, more than offset by the lower cable first cost.

Standardized installation practices are followed except that a softer and more lubricating type of boiling-out compound than paraffin wax is required, particularly at low temperatures. A suitable compound has been found by adding paraffin oil to wax in varying proportions depending upon the temperature at the point of splicing.

The anticipated savings have been realized in the operation of the commercial units and the further expansion of the uses of this insulation is being studied.

In this paper a more complete and technical treatment of the pulp insulation development is presented than was given in previous papers.[1]

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[1] *Bell System Technical Journal*, Vol. X, pp. 432–471, "Developments in the Manufacture of Lead Covered Paper Insulated Telephone Cable"—J. R. Shea. *Bell Telephone Quarterly*, Vol. X, No. 4, pp. 211–215, "An Important New Insulating Process for Cable Conductors"—H. G. Walker. *Bell Laboratories Record*, Vol. X, No. 8, pp. 270–278, "Pulp—The New Cable Insulation"—L. S. Ford.

Introduction

OPEN wires were almost universally used for the transmission of speech in the days when telephony was young but gradually as the need arose the art of cable making was evolved. Today, except in the rural districts, the open-wire lines have been almost entirely replaced by aerial or underground cable. The conductors in the early metal covered cables were insulated with one or two servings of cotton but in the late eighties Bell System engineers developed a spirally wrapped paper insulation so much better electrically and lower in cost that it was shortly adopted as the standard insulation for telephone cables by the growing industry. Now after some forty years of service this type of insulation is being rapidly displaced for inter-office and subscriber loop cables by a pulp insulation applied directly to the conductor by a process which brings the paper mill into the cable plant and combines the paper making and insulating operations into one process with the elimination of a number of costly intermediate steps. In addition, this process makes possible the use of a less expensive material as an insulating medium.

In order to establish a background for the logical consideration of the pulp insulation development it is desirable to cover briefly the materials, equipment and methods that have gradually been developed for the rapid application and economic use of paper ribbon insulation and indicate the limitations involved.

For many years the standard paper in this country for insulating conductors for lead sheathed telephone cables was made from a stock composed of all old rope or old rope and a small admixture of cotton, the fibres of the rope being chiefly manila from the plant Musa Textilis or hemp from the plant Cannabis Sativa. Papers of such composition, slit into long narrow strips, were applied helically around the wire to form the insulated conductor. Experience had proved them to be highly suitable as an insulating medium, both as to structural permanency and electrical characteristics and to be sufficiently flexible and strong mechanically to admit of ready application to the conductor in manufacture and to withstand subsequent handling in service. With the mounting demand for insulating papers, however, came the urge for the finding of a suitable less expensive fibre and the year 1920 saw the adoption, for the larger sizes of paper only, of a formula composed of about 40 per cent chemical wood pulp and the remainder rope stock. This wood fibre is of the spruce or other coniferous tree species prepared by the sulphate or "Kraft" process. It is required to have a high cellulose content and to be as free from water soluble salts as the best manufacturing practice will permit. Extensive tests have demonstrated that it compares favorably in stability and permanence with the well established manila fibre. In the case of pulp insulated cable which is discussed in this paper the raw material used is 100 per cent of this wood fibre.

The present day ribbon paper insulating machine as developed by the Western Electric Company is essentially a rotatable hollow tapered spindle centrally mounted on and integral with a light weight disc about 15 inches in diameter. The wire to be insulated passes through

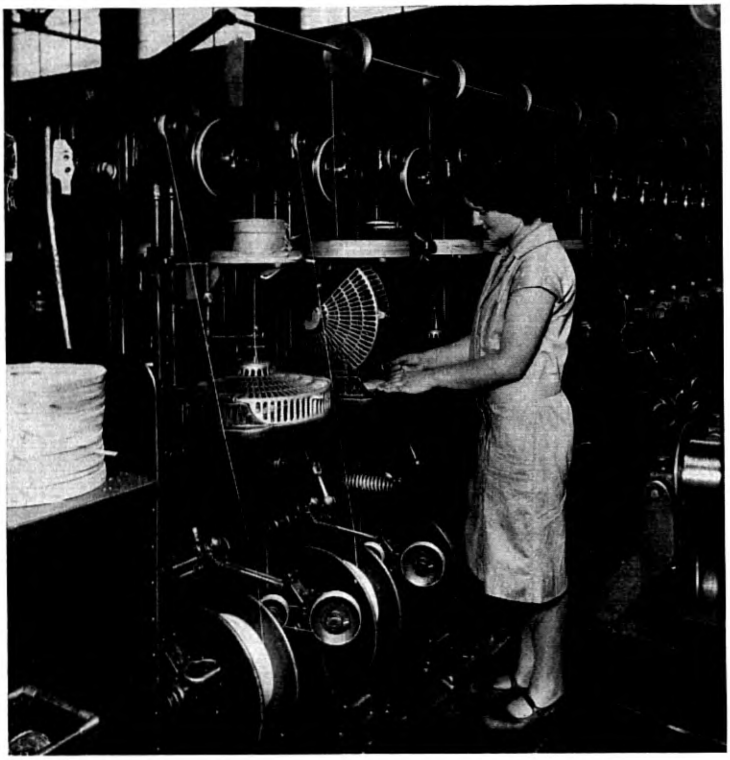


Fig. 1—Ribbon paper insulators.

the spindle over a capstan and to the take-up spool. The insulating paper wound into a pad or disc is slipped over the spindle and supported by the metal disc. This whole assembly is arranged to rotate rapidly around the wire and feed the paper ribbon from the periphery of the pad through guides so as to form a continuous spiral wrapping around the wire which is advanced at a definite speed by the capstan. The speed of rotation of the spindle is approximately 3300 R.P.M.

and the wire advances from 175 to 200 feet per minute depending on the length of wrap.

From a process standpoint manila paper was selected originally because of its strength and elasticity and in the development of equipment to serve it full advantage was taken of these two characteristics, particularly for the insulation of the finer gauges of wire. This fact tended to handicap the adaptation of cheaper papers to this purpose when the changing conditions in the paper industry made such a step desirable, since the readily available substitutes were somewhat inferior in these two respects. Studies were undertaken to modify the equipment for serving ribbon paper with the idea of adapting it for handling this paper, but with only indifferent success. The mixing of varying amounts of wood pulp with manila stock proved to be a successful solution in the case of heavier papers, but in the thinner ones the results were not satisfactory, and progress in this direction was at a standstill.

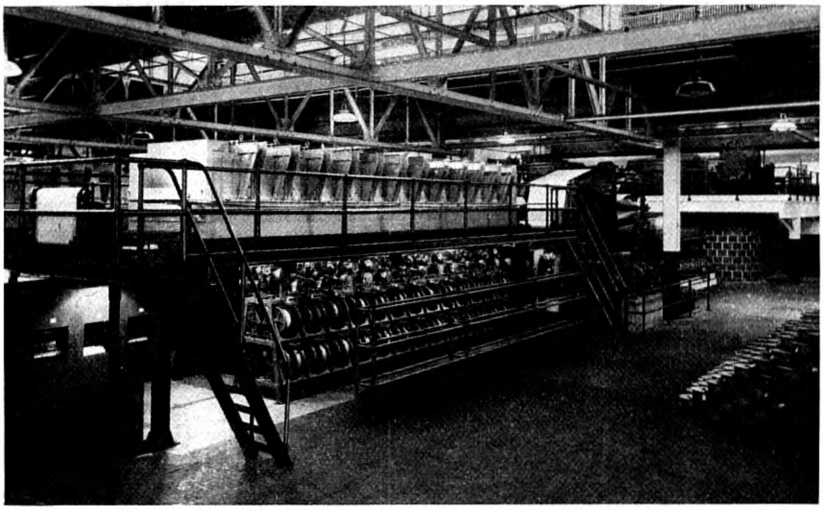


Fig. 2—General view of pulp insulating equipment. Take-up and dryer in left foreground, polishers and wet machine in right center and wire supply and pulp preparation equipment in right and background.

The Development of Pulp Insulated Wire

In line with the generally recognized need for a radical change in the insulating situation some work was initiated in 1921, with the idea of determining the possibilities of producing a continuous homogeneous paper covering directly on the wire and a scheme was worked out which after some preliminary experiments gave sufficient promise of success to suggest the desirability of going ahead with the development of the idea and the mechanism to carry it out.

A crude paper machine of the cylinder type was built and with this the feasibility of the basic idea was demonstrated. Dryers were next improvised and sufficient wire was insulated to give a few short test cables. These, of course, were made from carefully selected insulation for only a small part of the wire made was usable. The test results on these cables were sufficiently interesting to warrant proceeding further with the project. After considerable study and experiment it was decided to build a ten-wire machine adaptable to future expansion if

Fig. 3—Wire supply and pulp preparation equipment.

the anticipated results were realized. This ten-wire machine was started up with very indifferent success in January, 1924. During that year an operating technique was gradually developed and numerous improvements made in the equipment. In 1925, a great many test cables were made and several were installed for use in the telephone plant. Experience with the ten-wire operation and product finally became so satisfactory that it was decided to expand the machine to a fifty-wire capacity and put it on as near a commercial basis as possible, in order that its operation, product and economics might be studied to better advantage. Accordingly, the necessary auxiliary equipment was purchased and installed and the machine converted to a fifty-wire basis.

The installation was completed early in 1928 and the machine put in experimental operation about March of that year. As rapidly as possible crews were broken in and late that summer the machine was placed on a regular operating basis with three complete crews on a twenty-four-hour day and six-day week. It continued to operate on this basis until 1931, when ten more wires were added. This product was cabled into 26 and 24 A.W.G. cables on standard cabling equipment with no major difficulties and installed in commercial telephone plant by the operating companies. No serious operating trouble has developed in any of this cable.

The pulp insulated wire capacity now at the Hawthorne and Kearny plants is approximately 225 million conductor feet per week and all 24 and 26 A.W.G. exchange area cables are being manufactured from pulp insulated wire.

Process

Essentially the process consists in forming simultaneously on a modified cylinder paper machine 60 narrow continuous sheets of paper with a single strand of wire enclosed in each sheet, pressing the excess moisture from the sheets, turning them down so as to form a uniform cylindrical coating of wet pulp around the wire and then driving the water from this coating by drying at a high temperature.

The insulating material is given practically the same treatment in a beater as it would receive in paper making, but without the addition of sizing or loading. The beaten pulp is stored in a large tank from which it is pumped to a mix box for dilution with water before passing to the screen where coarse particles and lumps are removed.

For the next operation a modified paper machine of the cylinder type is used. The mixture of pulp and water is fed into the cylinder vat by gravity from the screen. The cylinder mold itself is divided into 60 narrow, uniform sections by dams or deckels on the surface of the wire cloth covering. The bare conductors coming to the machine are guided so that one conductor passes around the mold in each of the sections. As the mold is rotated in the water suspension of pulp in the vat, a narrow continuous sheet of paper with a conductor embedded in it is formed in each section by the simple paper making process of straining the fibres from the suspension as the water flows through the fine wire cloth covering the mold, under the slight head maintained outside the mold. These sheets are transferred from the mold to a woolen felt by the pressure of a couch roll and carried by it through two presses which take out a considerable part of the water and leave the material in shape to be turned down to the final form. This is done by passing the conductors embedded in the narrow sheets through individual polishers which turn the wet sheet down into a uniform covering of a size determined to a large extent by the amount of pulp deposited in the sheet. These polishers are simply rapidly rotating heads carrying three specially shaped blades so arranged that one blade deflects the traveling wire and sheet from a straight line against the other two with a pressure controlled by the tension on the wire. The wet cylindrical insulation is then dried to about a 9 per cent moisture content by a single passage through a horizontal electric furnace 26 feet long the wet end of which is maintained at a temperature of about 1500° F. and the dry or tempering end at something under 800° F. The wires are carried through the drier by a rotary pulling mechanism designed to minimize the crushing or flattening of the dried insulation. This device delivers the finished product to the take-ups for spooling. The machine is operated at about 130 feet per minute.

Considerable amounts of water are used in the process, for in this, as in all paper making processes, water acts not only as a carrier for the fibres, but it forms some sort of a loose chemical or mechanical combination with them in the beater which is one of the principal factors in determining the final characteristics of the material. The approximate fibre concentrations at the various steps of manufacture are as follows:

Beater	3.5–4%
Storage	1.3%
Screen	0.07%
Cylinder Vat	0.05%
Polishers	28%
Completed Insulation	91%
Finished Cable	100%

Some Problems Involved

In theory the whole process is remarkably simple, but from the practical standpoint, many intricate problems had to be solved before satisfactory operation was possible. In some cases it was rather difficult to segregate the problems for study as there were so many variables involved. Gradually, however, these details have been cleared up and today operation is quite satisfactory. A brief survey of some of the more important problems and their solutions may be of interest.

Continuous Operation

It is quite essential, from an economic standpoint, that the machine should operate continuously. The fact that the supply spools carry only a limited amount of wire necessitated the working out of a dependable means for shifting from an empty to a full spool without a shutdown or break in conductor or insulation. This is accomplished by taking the wire off over the head of a spool by means of a flyer and brazing the inner end of the wire on one spool to the outer end of the next. Again in spooling the finished material at the dry end, the wire must be transferred from a full spool to an empty without interfering with the operation of the machine. This has been taken care of very

Fig. 4—Changing spools at supply end.

simply by providing two spool positions for each wire with a simple manual means of shifting from one to the other.

Broken Wires

In spite of all the care that can be exercised, wires break at times and as a matter of economy, methods of restringing the broken wires with the machine in operation had to be worked out. Continuous six-day week operation is now possible without shutdowns except for the midweek clean-up.

Wire Cleaning

The supply wire comes to the machine on spools. It is spooled on the wire drawing machine and annealed on the spool. The surface of this wire, annealed with the drawing compound on it, seems to act somewhat as a repellant to wet pulp and causes a ragged, broken insulation. This is probably due to a surface tension effect. This action caused considerable trouble in the early stages of the work as the blame was placed on polishers, pulp, felts, or anything but the wire surface. Finally it became apparent that the surface condition of the wire was a large factor and the trouble was eliminated by passing all the bare wires through an A.C. electrolytic cleaner between the supply stand and the wet machine.

Tensions

Fine gauge copper wire is soft and easily stretched, pulp insulation in the wet form possesses very little strength, and in the dry form its elongation is much lower than spirally wrapped ribbon insulation; hence it is necessary at every step in the process to maintain minimum tensions in order that the wire may not be stretched and the insulation opened. Devices have been developed that are quite efficient in holding tensions within the safe range.

Pick-Up

In the early operating stages the pick-up from the mold was at times ragged and uneven and the sheet formation not all that could be desired. It was found that these conditions could be materially improved by the addition of a very small amount of soap to the pulp suspension immediately before it reaches the machine.

Polishing

In connection with the operation of polishing the sheet down to a circular insulation it has been found that a water content of approximately 72 per cent is preferable to a dryer or wetter sheet as it seems to felt down and form a more homogeneous insulation. The polisher itself has required a considerable amount of development work to insure a continuous uniform product and avoid stripping when a lump or break in the sheet occurs.

Drying

Several methods of drying pulp insulation were given a thorough trial but a completely satisfactory drier did not prove a simple thing to find. Finally, however, it was discovered that very rapid drying caused less shrinkage than slower drying, and so resulted in a less dense insulation. As a low density insulation is very desirable electrically, this was the deciding factor in adopting high temperature radiant heat drying and experience has amply justified the decision.

Operation

The development of operating technique and methods offered some difficulties as the process is neither wholly paper making nor wire handling. Preliminary methods were worked out by engineers on the machine. Then regular operators were recruited for the most part from the operating organization and broken into the work. Most of

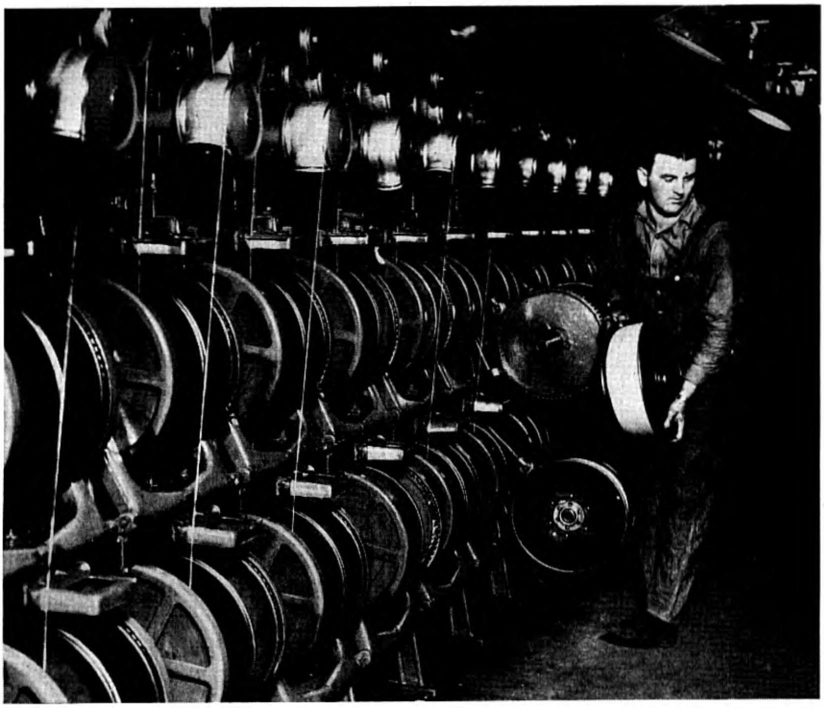


Fig. 5—Changing spools and take-up.

them had never seen a paper machine before but they became very efficient in a surprisingly short time and there have been no prejudices acquired on regular paper machines to overcome.

Making Narrow Ribbons

The question of making narrow uniform ribbons has given considerable trouble. The most satisfactory solution of this problem to date is the use of deckels or dams painted on the mold mechanically at spaced intervals. Apparently very good life can be expected from such a mold.

Defective Wire

It is necessary to mark defects in the completed wire by placing a white tag in the winding in order that repairs may be made by the twister operators, as there is no opportunity to make them at the pulp machine take-ups. Short breaks in the insulation were often passed

Fig. 6—Stringing in a wire at polishers.

unnoticed by the operators so a bare wire detector was put in which sounds an alarm, indicates the spool position by a light and records the break by number on a position counter and on a master counter. If any one spool shows excessive defects it is rejected.

General Physical Characteristics

Pulp insulation is a new product and has certain inherent characteristics. These characteristics may be modified somewhat by choice of materials and methods of manufacture but they cannot be entirely controlled. A brief survey of these characteristics may be of interest to give a better picture of the possibilities and limitations of the product. This survey covers only 24 and 26 A.W.G. wire as these are the sizes which have been run almost exclusively to date. It should be noted, however, that wires ranging in a size from 19 to 28 A.W.G. have been covered successfully.

Some of the physical characteristics of the insulation are shown below in tabular form giving the possible range of values obtainable. They are controlled by the beating of the pulp, the amount of pulp fed to the machine, the dryness of the sheet in the polishers and the speed of drying.

Diameter of Insulated Wire, Inches	0.030 to 0.050 for 24 A. W. G.
	0.026 to 0.040 for 26 A. W. G.
Weight of Dry Pulp, Grams per Foot	0.045 to 0.12 for 24 A. W. G.
	0.040 to 0.095 for 26 A. W. G.
Density—Ratio of Fibre to Total Volume	35% to 55%—Independent of Gauge.

The tensile strength and flexibility of the insulation can be varied through rather wide limits by different treatments during manufacture. The elongation is quite comparable to that of ordinary paper and is not susceptible of much variation. The insulation is made sufficiently strong and flexible to withstand the various operations incident to cable fabrication and subsequent handling yet not so tough that it cannot be readily removed from the wire at the point of splicing.

The surface of the insulation has a rather rough blotting paper appearance, though some variation is possible by changes in the beating. The cross-section is circular with the conductor in the center in the ideal case, but because of limitations imposed by practical operating considerations there is a tendency toward some eccentricity and flattening of the insulation.

Pulp Insulated Cables

Design

The smallest wires now used in commercial telephone cables are 24 and 26 A.W.G. and it has been found that pulp is particularly suitable for insulating such fine wires. Here it is in direct competition with non-wood content strip paper that has been giving satisfactory quality performance. To displace the old standard, pulp must meet this competition and give a greater return for the money invested.

Telephone cable circuits are normally subjected to only a low dielectric stress which permits their being placed in close proximity to one another and the primary requirement of the insulation is that it be distributed in a thin layer of uniform application, with the wire well centered so that each conductor when packed into a cable is completely insulated from its neighbors throughout its length. The mean radial thickness of the pulp insulation for the 26 A.W.G. wire which is in common use is less than one hundredth of an inch and for 24 A.W.G. which is the next larger size of wire usually used for telephone cables this value is about 0.011 inch. The pulp is prepared and applied to the conductor in such a manner that the fibres pack together to form a cover with sufficient strength and elasticity to withstand the handling the insulated wire must receive and yet be as light as possible in weight per unit volume in order to obtain the best electrical characteristics.

At the time this development was started 24 A.W.G. wire was the finest regularly used and the earlier pulp cables were confined to this gauge. Pulp insulated wire is structurally more like textile insulated wire than air-spaced paper ribbon insulated wire. The insulation is firm with no appreciable air gap between it and the wire, and bundles of wires nestle together differently when grouped into a given space. Furthermore, it was found that when pairs of conductors were stranded together in the usual manner of concentric layers each reversed in direction, the unit thus formed was considerably less flexible than the present standard construction. This is apparently caused by the greater frictional resistance between layers sliding over each other as the cable is bent, thus causing sharp kinks for even moderate bends. While this feature is less pronounced for small cables, it is, of course, objectionable and an improvement in the handling qualities is effected by stranding several layers in the same direction rather than employing the single reverse layer construction. For the large size cable, a design whereby the pairs are first grouped into units of fifty-one or one hundred and one, all the pairs in these units being stranded in the same direction and the units then stranded together into a cable, gives a construction which seems to offer the most satisfactory arrangement. Thus, for example, a 1212 pair cable is made up of 12 units of 101 pairs each, arranged with four units in the center and eight in a surrounding layer, and an 1818 pair cable is laid up with two units in the center surrounded by six units in the first layer and ten units in the second layer. Fig. 7 shows a short section of 1818 pair 26 A.W.G. cable with the units separated. One might expect these rather large units would not group themselves together into a circular shape without poor utilization of the space they occupy but it has been found that by properly constructing the individual units and by suitable arrangement of the cable layup, a cross-section is obtained with the groups keystoning together nicely and presenting no noticeable voids.

The cable core must also have a certain firmness or density to give

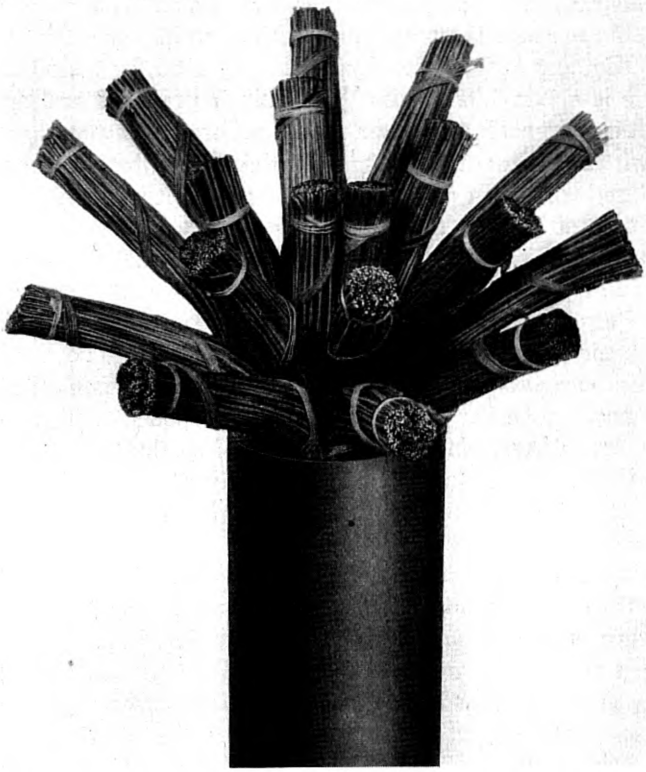


Fig. 7—Section of 1818 pair 26 A.W.G. cable showing units separated.

the best support to the sheath and insure satisfactory handling as the cables are being installed. With ribbon paper insulation the ratio of the amount of insulation to the non-copper space in a cable was found to be a fairly good criterion of the firmness required. With the fundamentally different physical characteristics of the pulp insulated wire this relationship was altered and experimental trials were therefore necessary to determine the approximate size of pulp insulated wire most suitable for the space it was to occupy in cable form. There is some latitude here in the distribution of a given amount of fibre but taking into account both the mechanical and electrical requirements, the diameter for the insulated conductor finally selected as the most satisfactory for the series of standard cables of 24 A.W.G. was 0.041 inch and for 26 A.W.G.—0.033 inch, and the aim in manufacture is to produce an insulation as uniformly close to these dimensions as possible. These diameters are measured by a volume displacement method. Short samples, as representative as possible of the wire under consideration, are inserted for a given distance into a small bore tube of mercury and the displacement noted. The gauge is calibrated so that mean diameters are read directly on the scale.

The above specific sizes of pulp insulated conductors apply only to cables designed for a particular set of characteristics. As in the case of ribbon paper cables, the amount of insulation for a given gauge of conductor may be varied within reasonable limits, so as to produce cables of other characteristics.

Electrical Characteristics

It was reasoned that pulp insulated cables would probably be inherently higher in mutual capacitance than similar sizes of paper ribbon cables because, considering the insulated wire itself, in the case of helically applied strip insulation the volume of air beneath the paper is about equal to the volume of the paper itself, while for pulp insulation there is very little air space between the insulation and the wire. This fundamental difference could be somewhat compensated for, however, by the introduction of more air into the spaces between the fibres of the pulp insulating medium than is found in the paper ribbon itself, but it was not expected that it would entirely neutralize the effect of lack of air space next to the wire. It was appreciated, however, that the aim should be to get as low density insulation as possible still consistent with obtaining a continuous, flexible and strong covering on the wire and emphasis was placed on this phase from the start of the development.

The very first experimental cables manufactured compared favorably in mutual capacitance with corresponding sizes of strip paper cables. The wire was insulated in a manner resulting in an apparently well centered, round insulation and the covering was low in weight of fibre per unit volume. The insulation after being formed around the wire was quickly dried in a hot tube resulting in less shrinkage and tightening down around the wire while the moisture was being driven out than if slowly dried. The insulation was unsatisfactory, however, from a continuity and tensile strength standpoint and could not be considered as suitable for commercial cable.

Special effort was then directed towards producing an insulation better mechanically, with the result that the early commercial cables were satisfactory in this regard but were from 20 to 25 per cent higher in mutual capacitance than the standard ribbon paper cable. This impairment in transmission efficiency was considered prohibitive for cables to be used for interoffice trunks and was definitely objectionable for any class of service. However, the indicated savings in cable first cost warranted continuing the development and over a period of years marked progress has been made in reducing this excess of capacitance and yet retaining an insulation sufficiently strong and

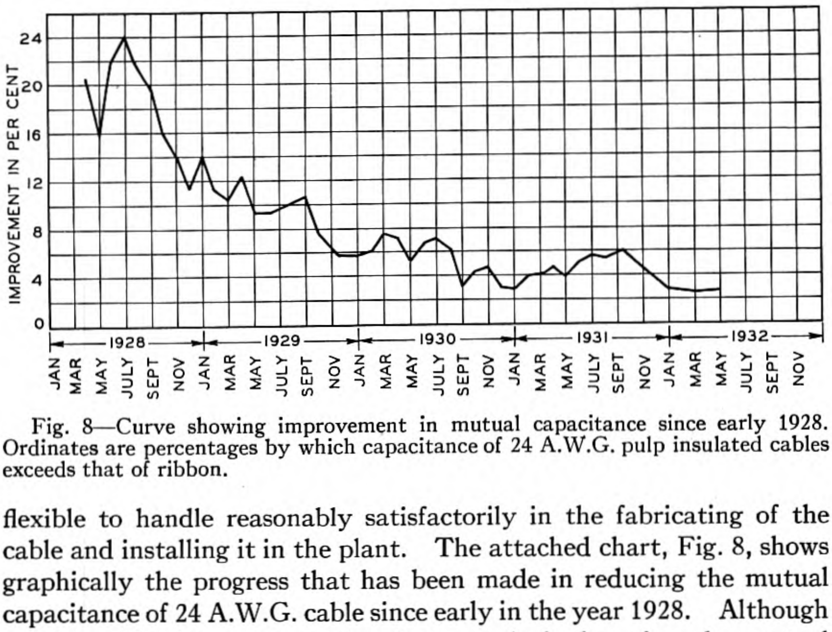


Fig. 8—Curve showing improvement in mutual capacitance since early 1928. Ordinates are percentages by which capacitance of 24 A.W.G. pulp insulated cables exceeds that of ribbon.

flexible to handle reasonably satisfactorily in the fabricating of the cable and installing it in the plant. The attached chart, Fig. 8, shows graphically the progress that has been made in reducing the mutual capacitance of 24 A.W.G. cable since early in the year 1928. Although a substantial improvement has been made in lowering the mutual capacitance to within less than 4 per cent of the corresponding ribbon paper cable, a further reduction would have considerable value warranting more effort in that direction. For 26 A.W.G. cable the excess in capacitance is even less than for 24 A.W.G. and furthermore it is not so objectionable from a transmission standpoint as in the case of the larger gauge.

The principal factors which have brought about this reduction in capacitance are improvements in the treatment of the pulp itself, refinements in machinery operation to permit the use of a lower density covering on the wire, the more rapid drying out of the moisture from the pulp resulting in less shrinkage of the insulation on the conductors and the producing of more nearly round and better centered insulation. Of these factors perhaps the one having the greatest effect on lowering the mutual capacitance was that of improving the out of roundness of the insulated conductors. In studying this phase of the problem, advantage was taken of the effect of flatness of the insulation, on the component parts which make up the mutual capacitance. The mutual capacitance of a pair of wires is composed of the direct capacitance between the two wires augmented by a series arrangement of two other direct capacitances, one from each of the two wires to the grounded group consisting of all other wires and sheath. As two wires with oval shape insulation are twisted, there is a

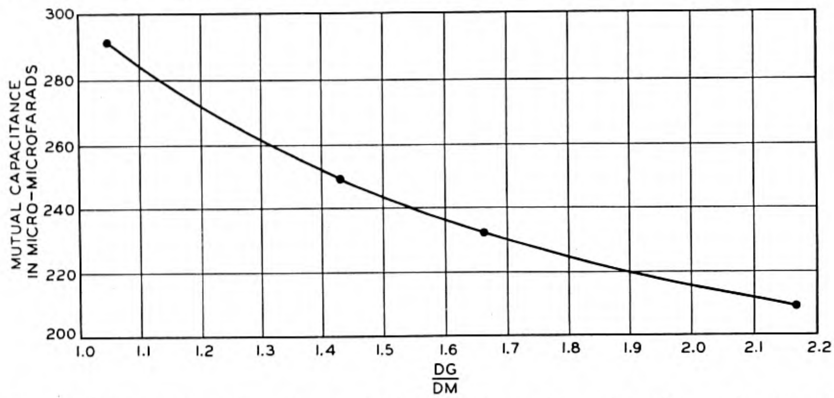


Fig. 9—Curve showing mutual capacitance versus direct capacitance to ground divided by direct capacitance to mate.

decided tendency for two flat sides to stay together resulting in the average separation of wire and mate being less than where circular sections are involved. To determine accurately the degree of out of roundness representing the average condition throughout a length of cable by mechanical means is next to impossible, whereas the direct capacitance between wire and mate automatically integrates this condition. Measurements therefore are made of the component direct capacitances and their ratio used as a sensitive indicator of the effect of flatness of the insulation on the mutual capacitance. By using the ratio of capacitances the cable length error is eliminated and accurate determination can readily be made on short lengths of cable.

As illustrative of the above relation there are given in the following table and curve, Fig. 9, data which were obtained on four short lengths of pulp insulated cables which so far as was known differed only as regards the lack of symmetry of the insulation.

MUTUAL CAPACITANCE VS. $\left\{\frac{\text{DIRECT CAPACITANCE TO GROUND}}{\text{DIRECT CAPACITANCE TO MATE}}\right.$
AVERAGE VALUE IN M.M.F.

Sample	Mut.	D_M	D_G	D_G/D_M
1	292	187	201	1.07
2	250	144	207	1.42
3	233	126	209	1.66
4	206	101	221	2.19

The alternating current mutual conductance follows the trend of the capacitance, resulting in the ratio of conductance to capacitance at a frequency of 900 cycles per second being somewhat higher than the standard ribbon paper cable, but not of a magnitude such as to introduce any serious transmission loss for these fine gauge circuits. The direct current insulation resistance is of the same order as that of strip paper cables.

The dielectric strength of the insulation is ample, being somewhat higher on the average than that of similar strip paper cables. A rather extensive series of mechanical tests comparing pulp and ribbon types of insulated cable under controlled conditions simulating those met with in actual installation, showed that the pulp insulated cables remained superior to the ribbon cables as regards dielectric strength but that under extreme loads they would not withstand quite as much stretch as the ribbon insulated cable without mechanical damage to the insulation.

Installation Features

No new features are involved in installing pulp insulated cable except in the splicing of the conductors after the lengths as supplied from the factory have been placed in position in the plant. This operation, however, is a considerable factor in the total time of the installation procedure because in a not unusual run of a mile of an 1818 pair cable, there may be as many as 40,000 joints to be made involving the stripping of twice that number of ends of insulated wire preparatory to joining the copper conductors.

Immediately upon removing the lead sheath from the ends of the cables thus exposing the dry insulation to the atmosphere, absorption of moisture rapidly takes place. It is customary, therefore, to boil out the ends of cable with paraffin wax before starting the splicing operation. With strip insulation this wax also aids in preventing the insulation from unfurling. It was found that even the most flexible pulp insulation so far produced, when impregnated with unmodified paraffin would not withstand satisfactorily the handling incident to splicing at low temperatures. A softer and more lubricating type of compound is required and a suitable combination has been found by adding paraffin oil to the paraffin wax. Different proportions of oil and wax are used depending upon the temperature at the time of installation and the compounding is done at the point of splicing. At an atmospheric temperature of about 75° F. no oil is required and below 10° F. about half oil and half wax makes a suitable compound with proportionate amounts of oil for intermediate temperatures.

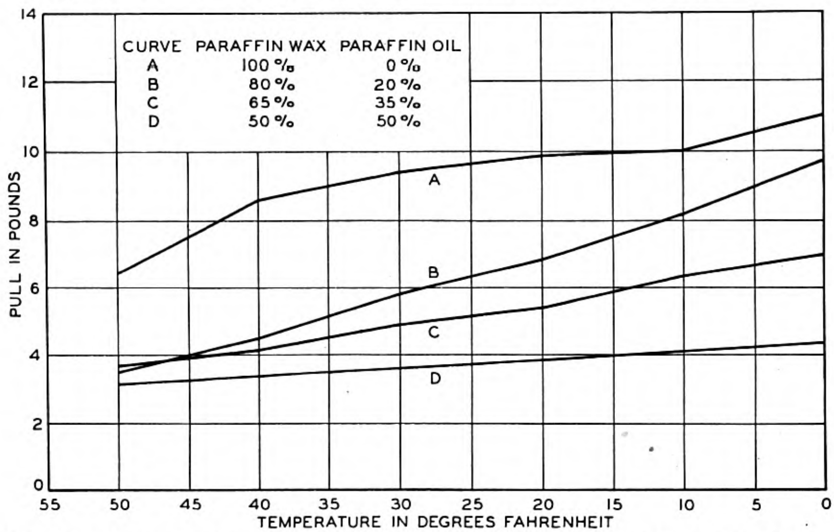


Fig. 10—Curve showing effect of temperature on pull required to strip insulation impregnated with various wax and oil mixtures.

In starting to make a splice, the insulated conductors are brought together in proper position, given a sharp crossover, the wires cut off so as to give several inches of free end, the insulation broken at the crossover and then stripped off the ends. Thus the ideal insulation is one which when waxed, can readily be parted at the crossover and when broken will slip freely along the wire, yet will withstand considerable bending and folding at other places in the splice without breaking. Pulp insulation tends to cling to the conductor somewhat more than a paper tube of strip insulation and although there is considerable variation in this characteristic in the product as now manufactured, it is sufficiently under control so that with a small amount of experience a splicer applying his usual technique is able to handle even 26 A.W.G. wire with little breaking of the conductors. Fig. 10 shows the stripping characteristics of typical pulp insulation on 24 A.W.G. conductors impregnated with compounds of different proportions of paraffin wax and oil. The pull required to strip the insulation from a few inches of wire is plotted against atmospheric temperature and shows the benefit of the higher percentage of oil particularly at the lower temperatures. There is, of course, with pulp no raveling of the insulation, and the cotton sleeves which are used to insulate the joint slip over the ends of the wires rather more readily than for the spirally applied paper. Thus the overall time required for joining a given number of pairs is practically the same for the two types of insulation.

An unbleached pulp is used and the natural brownish color of the Kraft stock results in less sharp color distinction for the different groupings of pairs than where ribbon insulation is used. However, by simplifying the color code so as to require only red, blue, and green, besides the natural color, sufficient contrast in the shades is obtained so that there is no difficulty in distinguishing colors in the splicing operation.

Possible Applications

This work was undertaken primarily to develop an insulation for use in exchange area cables and efforts have been confined largely to this phase of the study. It is possible to vary the characteristics widely by changes in the raw materials, process and subsequent treatment and other fields of use are being considered.

Pulp insulation is being used as sleeving for lead-in wires in some apparatus at the present time. For this purpose the insulation is made on 19 A.W.G. wire, stripped from the wire and cut in short lengths. It has proved quite superior to the old paper sleeves rolled by hand over mandrels.

Preliminary tests have indicated that there may be a field for use for this type of insulation with certain modifications for switchboard wiring, terminating cables and some kinds of coils.

Economies

Preliminary cost figures indicated that this process offered the possibility of a considerable saving over the ribbon process. These predictions have been verified by actual machine operation extending over a period of more than three years. The savings are made possible by the low cost of Kraft pulp as compared with manila paper and by the elimination of the intermediate paper making, paper slitting and handling operations.

Conclusions

A new type of insulated wire which is considerably cheaper than paper ribbon insulation has been developed. The insulation is formed from paper pulp directly on the conductor by a special type of paper making equipment. This equipment is not critical to the kind of pulp used but for the purposes of durability, strength and economy a Kraft wood pulp has been used in telephone cables. The process has progressed through the development stage and is now in continuous operation in the commercial production of all the principal exchange area cables of 24 and 26 A.W.G. conductors used in the Bell System. Thousands of miles of lead encased pulp insulated cables, ranging in size from the smallest consisting of 11 pairs to the largest consisting of 1818 pairs, are now giving satisfactory service and because of the substantial economies which the construction promises for the finer wire cables, attention is being directed toward its possible application to larger gauge cable conductors and to its use as an insulating medium for other electrical circuits.

A Recording Transmission Measuring System For Telephone Circuit Testing

By F. H. BEST

A number of types of measurement are made on telephone circuits to determine their transmission performance, these measurements being made with manually operated devices. This paper describes a transmission measuring system which automatically records the results of many of these measurements.

THE making of transmission measurements on telephone circuits is essentially a delicate operation. However, with the aid of vacuum tubes and, more lately, copper-oxide rectifiers, devices have been developed for measuring the various important transmission characteristics of telephone circuits, including transmission losses and gains for single frequencies, speech volume and noise, all of these measurements being made with meters as are measurements of the performance of electric power systems.

There has now been developed an experimental model of a system not only for indicating but also for recording the results of transmission measurements on telephone circuits. It was developed particularly for the purpose of automatically plotting curves of transmission loss versus frequency, this characteristic of telephone circuits being a very important index of the ability of the circuit to transmit speech clearly. It is, however, also suitable for making various records of performance as a function of time, including transmission loss, speech volume and noise.

The essential elements of the automatic recording system are shown in Fig. 1 as they are used in making a transmission-frequency run on a telephone circuit. At one end of the circuit is an adjustable frequency oscillator which generates testing power, a sending panel for supplying this power to the circuit and adjusting it to the proper value and a synchronous motor for varying the oscillator frequency. At the other end is a receiving panel which amplifies the weak received testing power and converts it to direct current which causes the pointer of the recording meter to move. The meter is calibrated to record the transmission efficiency of the circuit directly in decibels. The heavily outlined parts are those used for recording work only, the remainder being parts already in use in the field in the making of ordinary transmission measurements.

The general operation is as follows: Constant testing power is supplied to one end of the circuit by the adjustable frequency oscillator, the frequency generated being varied continuously from one end of the range to the other by slowly turning the frequency control dial with the synchronous motor. While this takes place the recording meter at the other end of the circuit makes a record of the received power on a strip of paper, which is moved steadily by a synchronous motor, the resulting curve being a graph of the variation of the transmission efficiency of the circuit with respect to frequency. The purpose of the tuned circuit shown in Fig. 1 is to cause a mark to be made on the paper in the recording meter when a particular frequency is received. This mark serves as a reference point for applying a frequency scale to the record after the curve has been made.

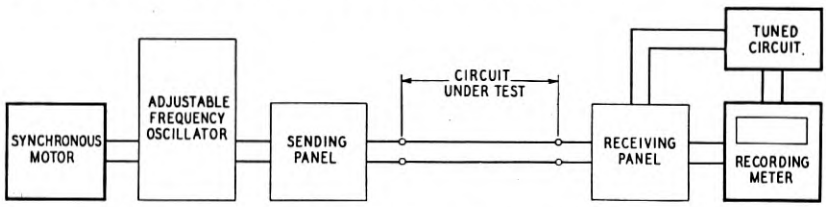


Fig. 1—Schematic arrangement of recording system.

If it is desired to obtain a record of transmission efficiency with respect to time, the same arrangement is used without the motor at the sending end, the oscillator frequency being fixed. The recording meter will then draw a line showing how the received power, and therefore the loss introduced by the circuit, changes with respect to time. If it is desired to record noise on the circuit instead of transmission loss the oscillator is disconnected from the circuit and the amplification at the receiving end increased until the very small noise currents are sufficient to cause readings on the meter. If the receiving apparatus is connected across a working telephone circuit it will serve as a recording speech volume indicator.

The oscillator, amplifier and other parts of the system have great stability and when left in continuous operation will maintain adjustments over long periods so that they may be connected to and used in the same manner as an ordinary voltmeter.

Figure 2 shows an experimental setup of the oscillator used at the sending end of a circuit and the recorder and associated parts at the receiving end. The motor-driven oscillator is at the left and the recorder at the right. Directly above the recording meter is the receiving panel which amplifies and rectifies the current received from the line. The tuned circuit associated with the frequency marking device is mounted on the rear of the panel below the meter.

The recording meter is a new design developed by the Weston Electrical Instrument Corporation in accordance with specifications drawn

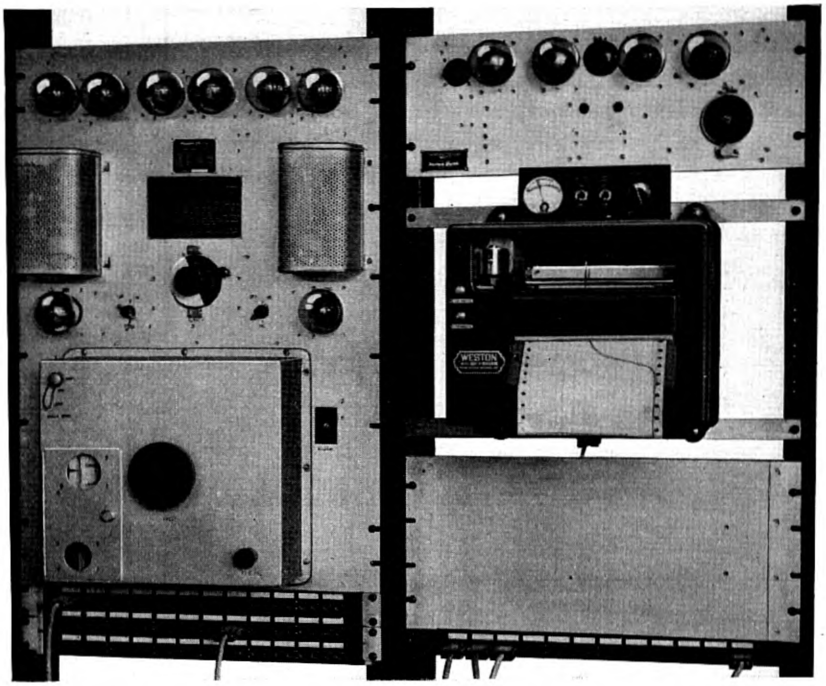


Fig. 2—Experimental setup of recording system.

up by Bell System engineers to meet the special needs of telephone circuit testing. It is extremely fast in operation, the moving system responding to fluctuating currents in about the same manner as the moving system of a fast d-c. voltmeter. Complete transmission frequency runs on circuits may be made in as short a time as one minute when the transmission loss is changing rapidly with frequency and in much less time with less rapid transmission loss variations. The ballistics of the moving system are such that the recorder may be used as a recording volume indicator although for this purpose the readings at some parts of the scale are not exactly the same as those of the nonrecording meters used in the standard volume indicators. Records of telephone circuit noise, which sometimes fluctuates in magnitude, can also be recorded. This high recording speed is made possible by making use of the fact that a record can be made on heat-sensitive paper without actual contact between the heat source and the paper and, therefore, without friction between the paper and the moving system which carries the heat source. Of particular importance is the fact that there is no static friction between these parts so that the power required to turn the moving system is only that necessary to overcome inertia, restoring spring force, damping and pivot friction, as is the case with an ordinary indicating meter.

Figure 3 illustrates the general principles of this recorder. Heat-sensitive paper is drawn over a straight bar which is at right angles to

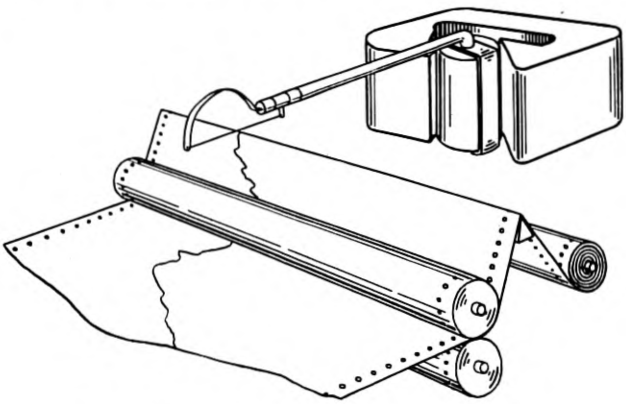


Fig. 3—Diagram illustrating recording principle.

the direction of paper movement, the bar being shaped so that only a line of paper is directly below the pointer of the moving system. A fine straight electrically heated wire is placed on the end of the pointer so that as the current through the moving system is varied the hot wire travels at approximately right angles to the line of the exposed paper and only a small spot of the paper is affected by the heat at any instant. With this arrangement the plot obtained has rectangular coordinates, which is a very desirable feature.

The heat-sensitive paper is a colored paper coated with white wax and before exposure is nearly pure white. The application of heat causes the wax to melt and be absorbed by the paper, making a distinct colored trace. The rapidity of action is dependent upon the amount of heat and the rate of movement of the heated wire with respect to the paper. The temperature of the wire is regulated to suit conditions; however, the maximum heat used is insufficient to char the paper even when it is not in motion. This method of recording is particularly satisfactory from a maintenance standpoint. A record is made almost the instant the current is turned on and there is no danger of failure of recording when the meter is not in continuous operation.

The reliability is so great that it is not necessary for the attendant making the test to see the recording meter while it is in use. Because of this and the stability of the associated sending and receiving apparatus, it should be possible to locate an instrument of this type at some central point in an office and have it used by testers some distance away. For such an installation it would of course, be necessary to have auxiliary circuits for enabling any tester to determine if the system is available for use, to indicate when a test has been completed and to enable the recording meter and oscillator to be started from remote points. Either the oscillator or the recording meter can be set in motion by the operation of a key and, if desired, each device can be made to stop automatically when the test has been completed.

Circuit characteristics, such as transmission efficiency, speech volume, and noise are all measured in terms of the unit of transmission—the decibel, referred to as the db—and the meters used in making these measurements are calibrated in db. An ordinary voltmeter or ammeter which has a uniform voltage or current scale will have a logarithmic db scale since current changes corresponding with db changes have a logarithmic relation. The logarithmic db scale is not suitable for maintenance work as some of the divisions are unnecessarily large and others too small to be read accurately. The range of the recording meter is about 26 db and the scale is divided into 2 db divisions. Ten of these divisions have been made approximately equal by a special design of the magnetic circuit of the instrument. In the conventional moving coil instrument the moving coil rotates in an airgap of uniform width and great effort is made to have the flux distribution in this gap uniform. In the recording meter the airgap is not constant but increases in width with the deflection of the coil. With suitable shaping of the pole faces of the permanent magnet and the iron core around which the moving coil turns, the magnetic flux distribution in the gap causes the angular movement of the coil to be approximately proportional to the current change expressed in db.

Figure 4 is a view of the moving system and the recording mechanism swung out of the case and shows the moving coil and the large magnet associated with it. It will be noted that a small magnet is mounted near the large one. This magnet induces eddy currents in a vane which is attached to an extension of the pointer, thereby acting as a brake to control the damping of the moving system. The ordinary meter with a uniform airgap does not need an auxiliary damping attachment as suitable damping can be obtained by eddy currents induced in the moving coil as it turns in the airgap. The non-uniform airgap gives a variable damping and the external damping device is provided to equalize this variation.

The heat-sensitive paper is also sensitive to friction and can be marked by pressure with a small wire, a characteristic which is utilized

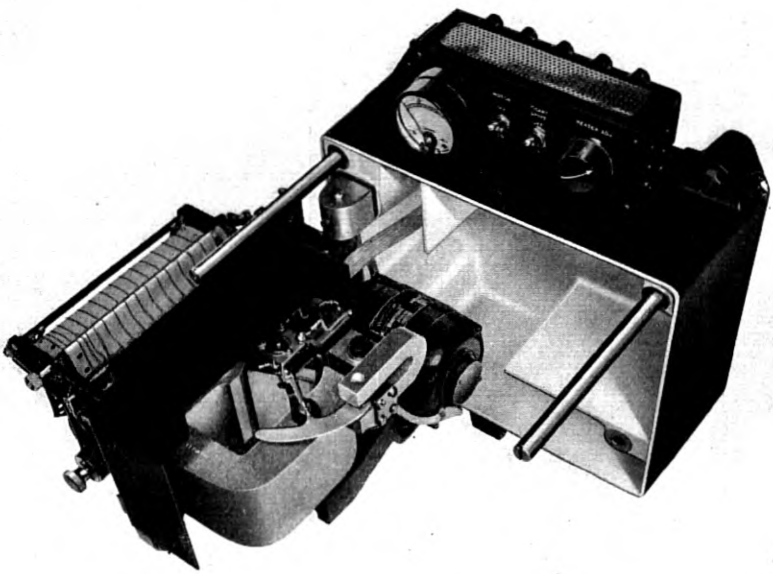


Fig. 4—Recorder mechanism showing moving system.

to make each recorder rule its own db scale as a record is made. Cheap plain unruled paper is used and a high accuracy of calibration is obtained by making the ruling devices adjustable. Fig. 5 shows the ruling devices which consist of loops of spring wire. While the paper used in the recorder is sensitive to both heat and friction it will stand handling without injury.

The paper is 6 inches in width. Two rates of paper movement are used in ordinary testing—10 inches per minute for transmission frequency measurements where speed is important and 6 inches per hour for long-period observations. The paper moving mechanism is made so that each curve can be torn off as soon as made, the paper coming out of a slot in the front of the case shortly after it has passed over the point of recording. The mechanism will accommodate a 400-foot roll of paper, which is sufficient for about 400 transmission frequency runs or for one month's operation at the slow speed. A new roll can be inserted in a very short time.

As previously stated, the deflection of the meter in db is plotted against frequency for some classes of measurements and against time for others. Since the same meter is used for many types of test it is

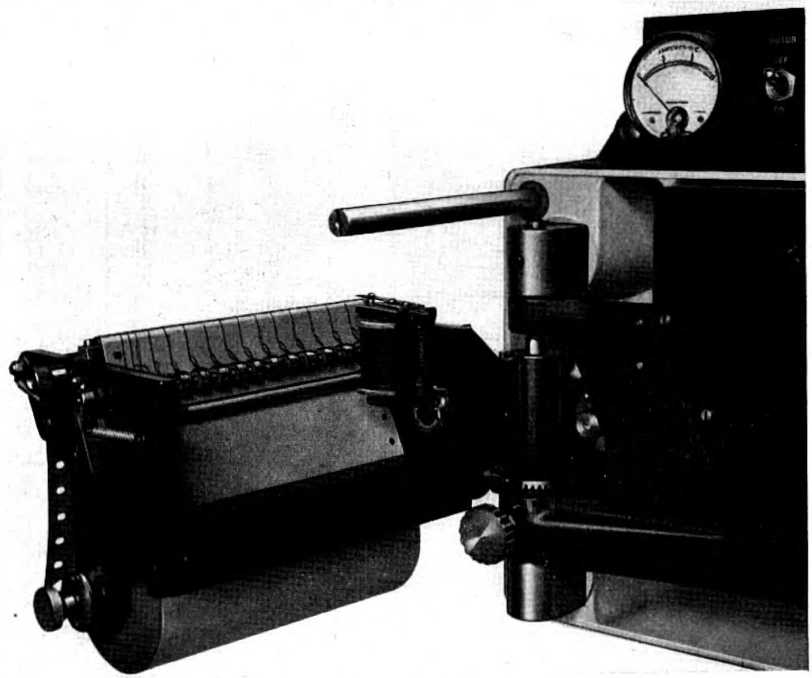


Fig. 5—Ruling and marking features of recorder.

preferable not to have the paper ruled for either frequency or time but to apply frequency or time markings after the record has been made. This is done by making reference marks on the margin of the paper as it goes through the recorder, and using them as indices to correlate the frequency or time and the record. As the paper passes over the bar, the marks are made by means of the electro-magnetic device shown in Fig. 5 at the right of the paper roll. As previously mentioned, when transmission-frequency characteristics are measured a tuned circuit causes a mark to be made when a particular frequency is received. Knowing the time frequency characteristics of the oscillator the entire frequency range can then be added by means of a rubber stamp. When time markings are desired the marking device may be operated by an external time clock. There is, of course, nothing in the design of the meter which would prevent using ruled paper in case this should be desirable.

The oscillator of the recording system is of the heterodyne type which uses a single dial for frequency adjustment, the frequency being varied continuously from one end of the range to the other as the dial is turned. When transmission-frequency curves are made a motor is connected to the dial, turning it at a uniform rate. The time-frequency scale of the oscillator is neither uniform nor logarithmic, as will be

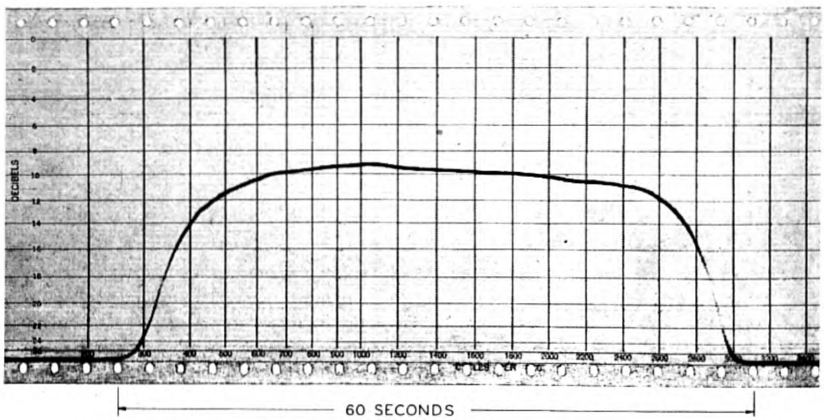


Fig. 6—Transmission frequency characteristic of message telephone circuit A.

noted in Figs. 6 to 8, being a compromise which gives sufficient space on the record to all parts of the range which are of particular interest.

A number of typical curves made by the recording system are shown in Figs. 6 to 12. Figures 6 and 7 are transmission-frequency characteristics of two telephone message circuits, each curve having been made in about one minute, using a paper speed of 10 inches per minute. Fig. 8, which is a transmission-frequency curve for a wide-band program circuit, was made in 30 seconds. Although a wide frequency range is covered by this curve, the absence of rapid transmission variations in the program circuit permitted a rapid change of the oscillator frequency.

Figures 9 and 10 were made with the slow rate of paper feed and are records of 24-hour continuous measurements on message telephone

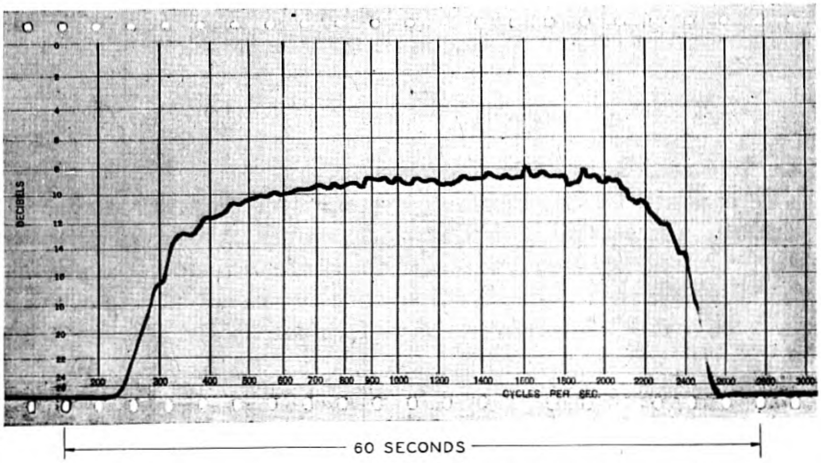


Fig. 7—Transmission frequency characteristic of message telephone circuit B.

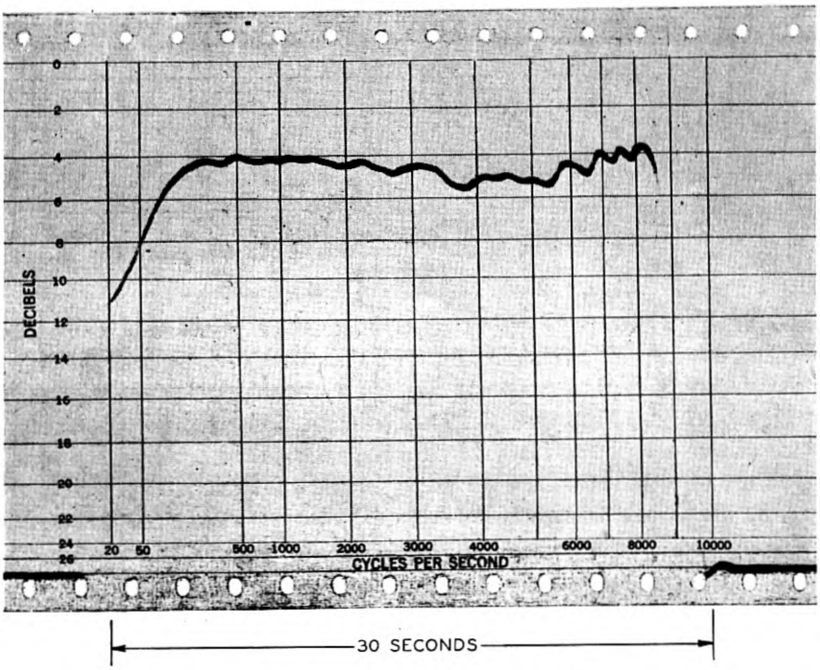


Fig. 8—Transmission frequency characteristic of program circuit.

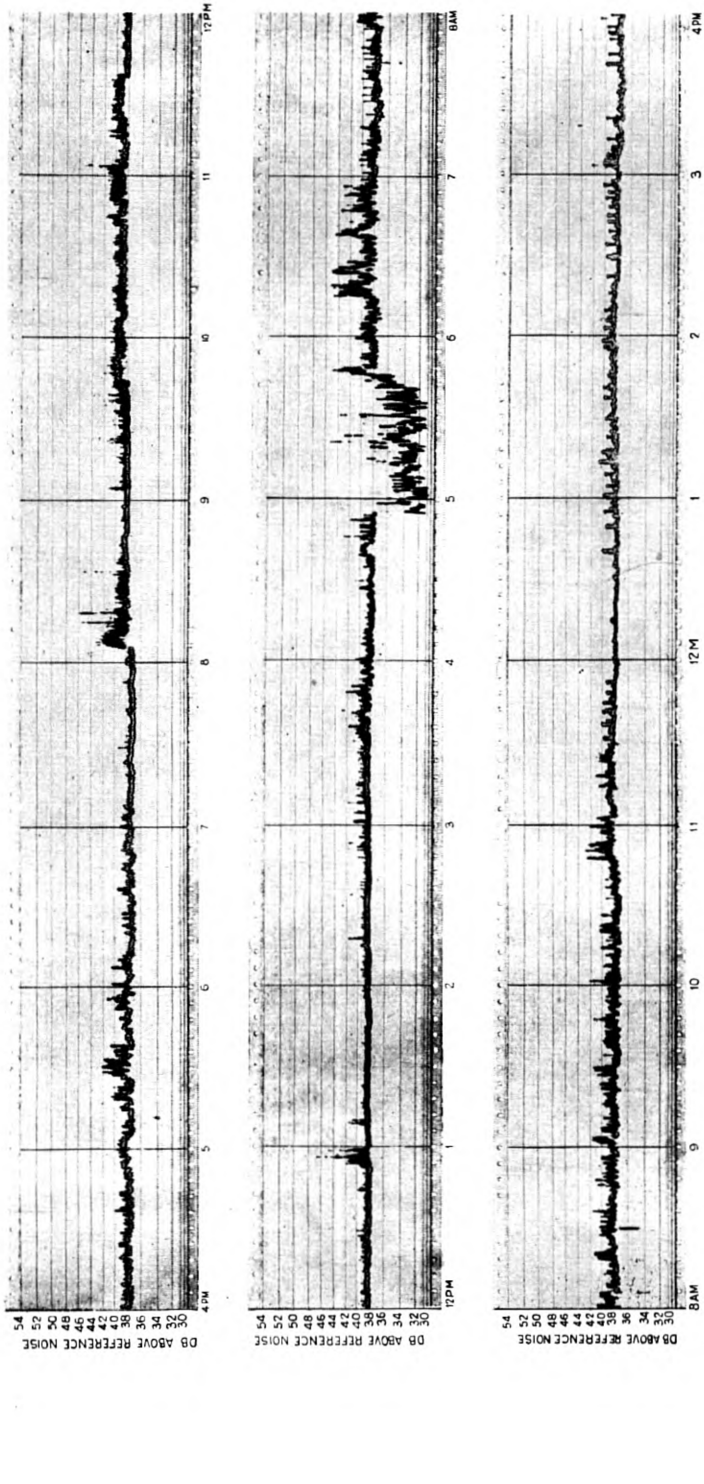


Fig. 9—24-hour record of noise on noisy open-wire phantom circuit.

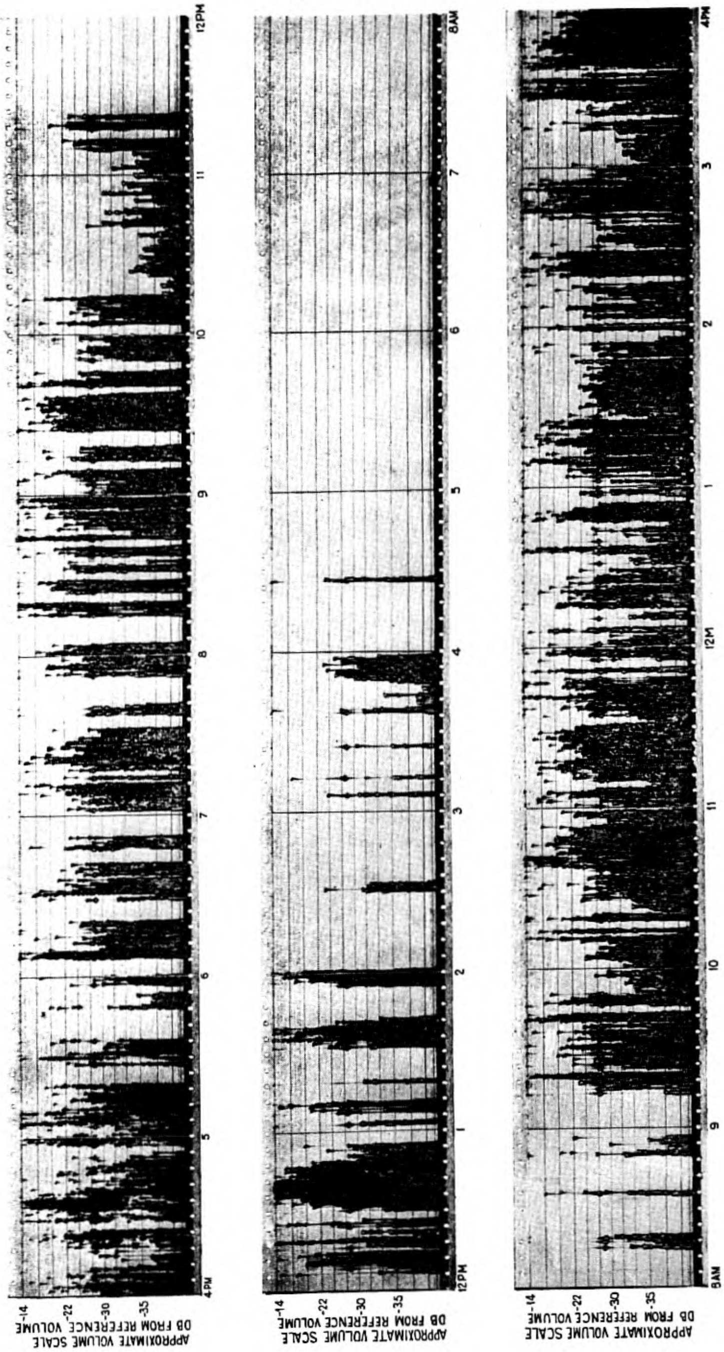


Fig. 10—24-hour volume indicator record on working telephone circuit

circuits. Figure 9 is a record of noise and Fig. 10 is a record of variations in speech volume on a working circuit. The speech volume record is of particular interest in showing graphically the variation of load on the circuit during the different periods of the day and also the extreme variations in the volume of different talkers. The recording system was bridged on one end of the circuit so that a difference of several db in volume level between the talkers at the two ends of the circuits is to be expected. Figure 11 is a short-period record of speech volume made at the high rate of paper feed.

It will be noted that the width of the mark made by the heated pointer is much greater in the case of the slow speed records than in the case of the high speed records. In the slow speed records illus-

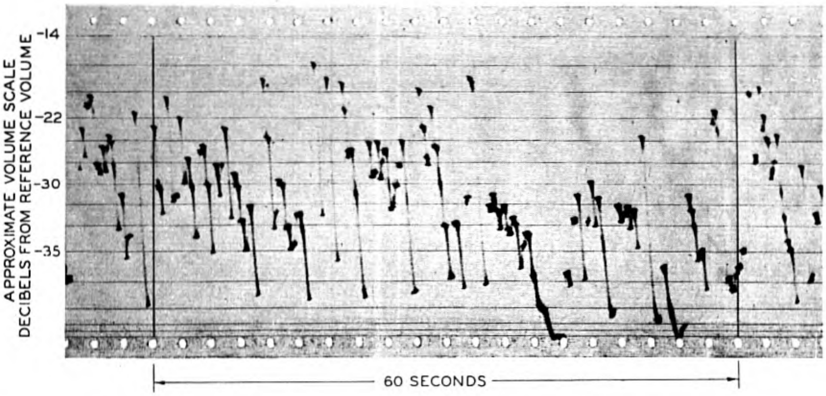


Fig. 11—Volume indicator record at high rate of paper movement.

trated the points of interest are the peaks which the pointer reaches frequently and the heat is adjusted so that a good record is made of these peaks. The movement of the pointer is so rapid that no trace is made between the peaks and the zero line. This feature is an advantage rather than a disadvantage since even with such a high speed recorder the movement of the pointer is slightly behind the electrical impulse which energizes it and for such tests as measurements of speech volume the record between the zero line and the peaks or between any two peaks would not be extremely accurate. The exact center of the broad line is directly under the heated wire. This point is clearly distinguishable in the broad trace made by slow speed records.

It is expected that recording transmission measuring systems will be of considerable value in locating intermittent troubles of very short duration which are not easy to locate with manual arrangements.

Fig. 12 is a record of the 1,000-cycle loss of a long four-wire cable circuit which was removed from service for purposes of trouble location. The small jogs in the curve were caused by the normal functioning of the automatic transmission regulators. The sudden change occurring at 8:50 a.m. was due to a trouble which momentarily decreased the transmission loss. The other large jog in the record was caused by an attendant making a routine adjustment. Evidently a trouble condition can not only be detected but located

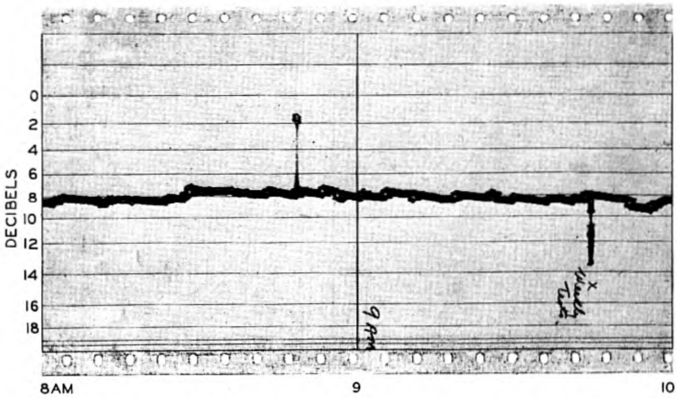


Fig. 12—Transmission loss—time record made on long four-wire cable circuit while locating a trouble.

by connecting transmission recorders at a number of different points along a circuit and making simultaneous records.

Transmission-frequency measurements on circuits and repeaters can be made with the recording system in less than one-tenth the time required with manually operated measuring apparatus. When the system has been completely developed and applied generally in the field, material time savings should result, particularly in the larger offices. Also, it is to be expected that the continuous records obtained with the recorders as compared to measurements at only a few points with manually operated apparatus will materially assist in disclosing abnormal circuit conditions.

Probability Theory and Telephone Transmission Engineering

By RAY S. HOYT

Part I of this paper contributes methods, theorems, formulas and graphs to meet a previously unfilled need in dealing with certain types of two-dimensional probability problems—especially those relating to alternating current transmission systems and networks, in which the variables occur naturally in complex form and thus are two-dimensional. The paper is concerned particularly with "normal" probability functions (distribution functions) in two dimensions, which are analogous to the familiar "normal" probability functions in one-dimensional probability problems. It supplies a comprehensive set of graphs for the probability that a "normal" complex chance-variable deviates from its mean value by an amount whose magnitude (absolute value) exceeds any stated value; in other words, the probability that the chance-variable lies without any specified circle centered at the mean value in the plane of its "scatter-diagram," that is, in the complex plane of the chance-variable. It gives a comprehensive treatment of the distribution-parameters of the "normal" complex chance-variable, and convenient formulas for the necessary evaluation of these parameters. For use in various portions of the paper, as well as for various possible outside uses, it supplies a considerable number of formulas and theorems on "mean values" ("expected values") of complex chance-variables.

Part II of the paper makes application of Part I to some important problems in telephone transmission systems and networks involving chance irregularities of structure and hence requiring the application of probability theory.

INTRODUCTION

IN telephone transmission engineering a frequent problem is that of determining the effects of random manufacturing variations upon the value of some characteristic (for instance, a transfer admittance, or a driving-point impedance, or a current-ratio) of a transmission system or network.[1] In certain cases, such effects may be of great or even controlling importance in the performance of the system and hence must be fully taken into account when designing the system and when making calculations for predicting its performance.

For example, in a multi-pair telephone cable the crosstalk between any two pairs is directly proportional to (strictly, a linear function

[1] Such problems have in the past been handled by various approximate methods the most satisfactory of which for many purposes was that described in a paper by George Crisson, entitled, "Irregularities in Loaded Telephone Circuits," published in this Journal for October, 1925. The method given in the present paper, while necessarily more involved than approximate methods, yields more precise results; and this additional precision is expected to be of importance in practice. Moreover, there has been an increasing need for a comprehensive paper covering the entire ground, and it is hoped that the present paper meets this need to a measurable extent.

In Crisson's paper references will be found to various engineers in the Bell System who had previously contributed to specific probability problems of the type dealt with in Part II of the present paper.

of) the deviations of certain internal parameters from their nominal values. Another example is furnished by two telephone lines connected by the usual type of two-way telephone repeater: If the two lines and their associated apparatus could be made identically alike, a state of perfect balance would exist at the repeater and there would be no tendency for the repeater to sing; however, as a result of manufacturing variations, perfect balance is unattainable and thus the practicable amplification obtainable from the repeater is limited by the manufacturing deviations of the lines and associated apparatus—particularly the deviations in the inductances and spacings of the loading coils, in case the lines are loaded.

Such examples may furnish at least the three types of probability problems described in the following three paragraphs:

Before the construction of the system there may arise the "direct" problem of calculating the characteristic to be expected, corresponding to the known (or assumed) ranges of the manufacturing variations in the elements. Before the elements are manufactured, the deviation of any element from its nominal value is of course unknown; moreover, such deviation is not completely predictable, since from its very nature it depends on chance. The deviation is a variable in the sense that it can take any value within a certain possible range. But it is a particular sort of variable, namely a chance-variable, in the sense that there exists a certain chance or probability that the deviation will lie within any stated range of values, with the chance depending of course on this range and on the specific probability law of deviation for the kind of element under consideration. Correspondingly the deviation of the contemplated transmission characteristic of the proposed system is a chance-variable, whose probability law depends of course on the probability laws for the deviations of the elements and on the functional formula connecting the contemplated transmission characteristic with the elements.

Before the elements of the system have been manufactured there may, on the other hand, arise the "inverse" problem of setting such restrictions on the manufacturing deviations of the elements as to insure that the contemplated characteristic of the proposed transmission system will have a preassigned probability of lying within a certain specified range. As might be expected, this "inverse" problem is more difficult than the "direct" problem, and often it can be solved only by successive tentative solutions of the corresponding "direct" problem.

Finally, after the system has been constructed and tested, there may arise the question as to whether its elements have been correctly connected together when installed. Assuming that the elements themselves are known, from previous individual measurements on them, to fall within their specified ranges of allowable variation, a comparison of the measured value of the contemplated characteristic with the calculated value to be expected on the basis of probability theory will give some indication as to whether some of the elements are incorrectly connected. Further, when there is present not merely a single system but a large number of systems which are nominally alike (for instance, the various pairs in a multi-pair telephone cable), measurement of the contemplated transmission characteristic of each of the systems and comparison of the statistical distribution of these measured values with their calculated theoretical distribution will give a more conclusive indication as to whether some of the elements are incorrectly connected.

Any particular problem to be solved can be handled most conveniently and advantageously if the general problem is first formulated analytically. Let us suppose, therefore, that H denotes the specific transmission characteristic under consideration (for instance, a transfer admittance, or a driving-point impedance, or a current-ratio), and $K_1, \cdots K_n$ the internal parameters on which H depends; and let the functional formula for H be

$$H = F(K_1, \cdots K_n), \tag{I}$$

where, of course, H and the K's are in general complex (on the supposition that the usual complex quantity method of treating alternating-current problems is being employed). As we shall be particularly concerned with the deviations of the various quantities from their nominal values it will be convenient to suppose that H and the K's denote the nominal values of the corresponding quantities, and that any actual set of values are denoted by $H + h$ and $K_1 + k_1, \cdots K_n + k_n$, so that h and $k_1, \cdots k_n$ will denote the corresponding complex deviations of these quantities from their nominal values. Then the general functional formula for h will of course be

$$h = F(K_1 + k_1, \cdots K_n + k_n) - F(K_1, \cdots K_n). \tag{II}$$

Since h may be regarded as causally dependent on the k's, it may naturally be called the "resulting" chance-variable.

Usually the k's will be so small compared with the K's that the right side of (II) can be replaced, as a good approximation, by the first-order terms of a Taylor expansion; thus, approximately,

$$h = D_1k_1 + \cdots + D_nk_n, \tag{III}$$

where

$$D_r = \partial F(K_1, \cdots K_n)/\partial K_r, \qquad (r = 1, 2, \cdots n). \qquad \text{(IV)}$$

Before the physical elements are manufactured the k's are chance-variables, in the sense already defined; for it is not possible to predict the value which any one, say k_r, will have, but only to state the chance that it will lie within any specified range, this chance being calculable from the known (or assumed) probability law $p_r(k_r)$. Hence h is also a chance-variable, whose probability law $p(h)$ depends on the functional formula for h and on the individual probability laws $p_1(k_1)$, $\cdots p_n(k_n)$. In the general case, the "direct" problem is to calculate from $p(h)$ the probability that h will have a value lying within any specified region of the h-plane.

In the types of problem contemplated in the present paper, the probability law $p(h)$ of h may usually be assumed to be approximately "normal" (Subsection 1.2). Moreover, the specified region in the h-plane will usually be a circle, since in such problems we are usually concerned only with the magnitude of h, not with its angle. For crosstalk, this is obviously true. For the usual type of two-way telephone repeater operating between lines whose impedances do not balance each other, it is true as a good approximation when the unbalance is not too large, since then the practicable amplification depends (approximately) only on the magnitude of the unbalance, not on its angle.

Unfortunately the complete solution of the problem for a circular region is sufficiently difficult and laborious, particularly as regards numerical evaluation, that apparently there has not heretofore been sufficient incentive to lead to its being carried through—at least so far as I am aware.[2] The present paper includes the needed solution, in convenient form for practical applications, by means of the comprehensive set of graphs described in Subsection 1.3, supplemented by Subsection 1.2 defining and formulating the "normal" complex chance-variable, and further supplemented by Section 2 giving general methods and formulas for evaluating the distribution-parameters of the "normal" complex chance-variable; and by Section 3, which applies Section 2 to the case where, as is usual, the contemplated "resulting" complex chance-variable is (at least approximately) a linear function of other complex chance-variables.

Section 4, which is somewhat in the nature of an appendix, supplies a considerable number of formulas and theorems on "mean values"

[2] As well-known to those familiar with the literature of the subject, the solution is quite easy for regions having certain other shapes, notably for an equiprobability ellipse and for a rectangle lying parallel to a principal axis of such an ellipse. However, those solutions are of no help in the case of a circular region.

("expected values") of complex chance-variables. These formulas and theorems find frequent and important uses in the present paper; and outside of the paper they may well find varied uses.

The method of treatment characterizing the present paper will now be very briefly indicated in the remainder of this Introduction.

As a preliminary step toward this objective we shall now return to the functional formulas (II) and (III) with the remark that, if the K's and k's were all real quantities and if these formulas were such that h also were a real quantity, then the "direct" problem would be to calculate the probability that h would lie within any stated linear range, say h_a to h_b; thus the probability problem would then be one-dimensional, and the well-known existing probability theory for real quantities would be immediately applicable, including the corresponding known methods and formulas for evaluation of the distribution-parameters.

When, as in the present paper, the K's and k's are in general complex quantities, the corresponding probability problem is inherently two-dimensional. The distribution-parameters, which naturally are more numerous than in the one-dimensional case, could be evaluated in a roundabout way by an extensive process of resolution into rectangular components; but it is believed that very superior advantages are possessed by the probability methods and formulas contributed by the present paper, for dealing with complex chance-variables in a more direct manner, as set forth at some length in Sections 2 and 3, extensively utilizing Section 4. The advantages of this method for evaluating the distribution-parameters are perhaps particularly marked whenever there is involved a summation of propagated effects, as in transmission lines; for then, as will appear more concretely in the applications in Part II, the necessary summations can be accomplished much more easily and the resulting expressions are much more compact and manageable than if a method employing rectangular resolutions were used.

Regardless of which method is used for evaluating the distribution-parameters, the new material contributed by Subsection 1.3 is necessary for the complete numerical solution of the problem in any specific case where the "resulting" complex chance-variable h is "normal." It may be recalled that this will be the case when h is a linear function of the k's and the k's themselves are "normal." Even when these two conditions are rather far from being fulfilled, however, it is known from certain rather broad theoretical considerations that in many practical problems h will be approximately "normal"; it may perhaps be recalled that one of the most important among a set of sufficient conditions for approximate "normality" is that the k's be numerous (n a large number).

As stated in the Synopsis, Part II makes application of Part I to some important problems in telephone transmission systems and networks involving chance irregularities in structure. One of these problems, namely that in Section 5, is the general problem already outlined in connection with the equations in this Introduction.

Part I: Theory

1. PROBABILITY OF THE DEVIATION OF A NORMAL COMPLEX CHANCE-VARIABLE FROM ITS MEAN VALUE

Toward the end of the Introduction it was stated that in many problems of the types contemplated in the present paper the distribution of the "resulting" complex chance-variable is approximately "normal."

To meet a previously unfilled need in the solution of such problems, this Section of the paper supplies (in Subsection 1.3) a comprehensive set of graphs for the probability that a "normal" complex chance-variable deviates from its mean value by an amount whose magnitude (absolute value) exceeds any stated value; that is, the probability that the chance-variable lies without any specified circle centered at the mean value in the plane of its "scatter-diagram." These graphs are accompanied by sufficient explanation to enable them to be understood and used without any necessity for studying the formulas from which they were computed—which, because of their length and complexity, have not been included in this paper.[3]

To furnish the necessary precise basis for the graphs, Subsection 1.3 describing them is preceded by Subsection 1.2 giving analytical definitions of the normal complex chance-variable and its distribution-parameters; and these quantities are discussed at moderate length there.

To lead up to the normal complex chance-variable, it is preceded by a brief review of the normal real chance-variable (Subsection 1.1), which is more familiar.

1.1. *The Normal Real Chance-Variable*

In order to lead up to the normal complex chance-variable (which is 2-dimensional) it will be recalled that a real chance-variable (which

[3] The formulas are given (with derivations) in an unpublished Appendix (Appendix A). Another unpublished Appendix (B) gives various concepts and definitions employed in two-dimensional probability theory, and also gives various analytical and graphical ways of representing probability. Still another (C) treats a problem of crosstalk in a telephone cable.

is 1-dimensional) is defined as "normal" if its probability law, or distribution function, can by the proper choice of origin be written in the form

$$P_u = \frac{1}{\sqrt{2\pi} S_u} \exp\left(-\frac{u^2}{2S_u^2}\right), \tag{1}$$

where, by definition of the term "probability law," $P_u du$ represents in general the probability that the unknown value u' of a random sample consisting of a single value of the chance-variable lies between u and $u + du$; or, what is ultimately equivalent, the probability that u' lies in the differential range $u \pm du/2$, namely in the differential range du containing the point u. S_u is a distribution-parameter called the "standard deviation" of u and defined by the equation

$$S_u^2 = \overline{(u - \bar{u})^2} = \overline{u^2} = \int_{-\infty}^{\infty} u^2 P_u du, \tag{2}$$

the superbar connoting the "mean value," or "mean," of any chance-variable to which it is applied. In this paper the term "mean value" is used as an alternative for "expected value," namely the "weighted average value" with the weighting in accordance with the probability of occurrence of each particular possible value of the variable. (Section 4 supplies a considerable number of formulas and theorems on mean values of complex chance-variables—and hence of real chance-variables, by specialization.)

From the foregoing definitions, it is easily verified that

$$\int_{-\infty}^{\infty} P_u du = 1,$$

which corresponds to taking unity as the measure of certainty.

It will be recognized that the chance-variable u in equation (1) is related to the original given chance-variable, which will be denoted by x, by the equation $u = x - \bar{x}$. Hence $\bar{u} = 0$, as has already appeared in equation (2); thus the origin is at the "center" c of the distribution, namely the point u_c with respect to which as origin the "mean value" of the chance-variable is zero, that is, such that $\overline{u - u_c} = 0$, whence $u_c = \bar{u} = 0$. If, in terms of the original variable x, the position of c is denoted by x_c, then $\overline{x - x_c} = 0$ and hence $x_c = \bar{x}$. Since $u = x - \bar{x}$, it is seen from (2) that

$$S_u^2 = \overline{(x - \bar{x})^2} = S_x^2. \tag{3}$$

The probability that the magnitude (absolute value) $|u'|$ of a random sample u' of u is less than any stated value r will be denoted by $p(|u'| < r)$. Then

$$p(|u'| < r) = \int_{-r}^{r} P_u du = \frac{2}{\sqrt{2\pi} S_u} \int_0^r \exp\left(-\frac{u^2}{2S_u^2}\right) du. \tag{4}$$

Evidently the number of parameters can be reduced from one (which is S_u) to none by taking as chance-variable the ratio u/S_u, which may be called the "reduced" chance-variable. Thus, with $|u'|$ denoted by r' and with r'/S_u and r/S_u denoted by R' and R respectively, equation (4) becomes

$$p(|u'| < r) = p(R' < R) = \text{erf}\,(R/\sqrt{2}), \tag{5}$$

where erf ( ) is the so-called "error function" defined, for any variable z, by the equation

$$\text{erf}\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\,(-\lambda^2) d\lambda \tag{6}$$

and extensively tabulated [4] for real values of z. For some purposes it is more convenient to employ the "error function complement," defined by the equation

$$\text{erfc}\,(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp\,(-\lambda^2) d\lambda \tag{7}$$

and hence related to erf (z) by the equation

$$\text{erf}\,(z) + \text{erfc}\,(z) = 1. \tag{8}$$

If u_3 denotes any fixed value of u, and if U_3 denotes u_3/S_u, then

$$p(u' > u_3) = \int_{u_3}^{\infty} P_u du = \frac{1}{2} \text{erfc} \frac{U_3}{\sqrt{2}}. \tag{9}$$

If u_1 and u_2 denote any two fixed values of u such that $u_1 < u_2$, and if U_1 and U_2 denote u_1/S_u and u_2/S_u respectively, then

$$p(u_1 < u' < u_2) = \frac{1}{2}\left(\text{erfc}\frac{U_1}{\sqrt{2}} - \text{erfc}\frac{U_2}{\sqrt{2}}\right). \tag{10}$$

[4] To avoid possible confusion, it may be well to remind the reader that there has also been extensively tabulated, for real values of z, the closely related function

$$\frac{1}{\sqrt{2\pi}} \int_0^z \exp\,(-\lambda^2/2) d\lambda,$$

which is more convenient for some purposes, though less convenient in the present paper.

If, with a view to generalizing (5), we inquire as to the probability $p(|u' - u_0| < r)$ that u' deviates from any fixed value u_0 of u by an amount whose magnitude is less than any stated value r, and if now we let r' and r_0 denote $|u' - u_0|$ and $|u_0|$ respectively and R, R', R_0 denote r/S_u, r'/S_u, r_0/S_u respectively, then

$$p(|u' - u_0| < r) = p(R' < R) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{R_0 + R}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{R_0 - R}{\sqrt{2}}\right)\right]. \tag{11}$$

When $u_0 = 0$ this formula correctly reduces to (5).

1.2. *The Normal Complex Chance-Variable*

Before proceeding to the "normal" complex chance-variable it should be remarked that, although any 2-dimensional chance-variable can be represented either as a complex chance-variable $z = x + iy = \mu \exp(i\eta)$ or as a pair of real chance-variables (x,y) or (μ,η), nevertheless the two modes of representation, though of course mutually equivalent, are not always equally advantageous. For the types of problems contemplated in the present paper, the complex representation has important advantages resulting from the fact that the chance-variable when so represented is formally a single entity and subject to the laws and transformations of complex algebra. In Sections 2, 3, 4 of Part I and also in Part II, the complex representation possesses very great advantages. In the present Subsection, however, which is mainly concerned with formulations of the 2-dimensional "normal" probability law (distribution function), the representation in terms of a pair of real variables is the more advantageous. In this Subsection, therefore, the complex representation is used only in those places where it is particularly conducive to brevity and sharpness of statement, and to simplicity and clearness of correlation with the remainder of the paper where the complex representation is mainly used.

The normal complex chance-variable (which of course is 2-dimensional) may be defined in several mutually-equivalent ways. Here a complex chance-variable z will be defined as "normal" if its probability law can, by the proper choice of a pair of rectangular axes u,v in the plane of the "scatter-diagram" of z, be written in the form

$$P_{u,v} = \frac{1}{2\pi S_u S_v} \exp\left(-\frac{u^2}{2S_u^2} - \frac{v^2}{2S_v^2}\right) = P_u P_v, \tag{12}$$

u and v being the pair of coordinates of any point of the scatter-diagram with respect to the u,v-axes. P_u and S_u have the values already defined by equations (1) and (2) respectively, and P_v and S_v are defined by those same two equations after changing u to v throughout; S_u and S_v are distribution-parameters called the "standard deviations" of u and v respectively.

It will be recognized that the u,v-axes are the "central principal axes," namely that pair of rectangular axes which have their origin at the "center" c of the scatter-diagram of z, and hence of $w = u + iv$, and are so oriented in the scatter-diagram that $\overline{uv} = 0$. By the "center" c of the scatter-diagram of any complex chance-variable z is meant that point z_c with respect to which as origin the "mean value" (Section 4) of the chance-variable is zero, that is, such that $\overline{z - z_c} = 0$; thus, $z_c = \bar{z}$. In the case of the chance-variable $w = u + iv$, whose origin is the center of the scatter-diagram, so that $w_c = 0$, it is thus seen that $\bar{w} = 0$; the fact that the u,v-axes have their origin at c may conveniently be indicated by designating them as the ucv-axes.

Instead of taking S_u and S_v as the distribution-parameters it will be found preferable to take b and S, defined by the equations [5]

$$b = \frac{S_u^2 - S_v^2}{S_u^2 + S_v^2} = \frac{1 - (S_v/S_u)^2}{1 + (S_v/S_u)^2}, \tag{13}$$

$$S^2 = S_u^2 + S_v^2 = \overline{u^2} + \overline{v^2} = \overline{|w|^2}. \tag{14}$$

It is convenient, and fairly natural, to call S the "resultant standard deviation" of [6] u and v. More explicit formulas for b and S^2 are (37) and (38) established in Section 2.

Equation (12) shows that the equiprobability curves of the complex chance-variable $w = u + iv$ are a set of similar ellipses centered at the center c of the scatter-diagram; and that the axes of these ellipses coincide with the principal axes of the scatter-diagram and have lengths proportional to S_u and S_v, and hence proportional to $\sqrt{1 + b}$ and $\sqrt{1 - b}$ respectively, since, from (13) and (14),

$$2S_u^2 = (1 + b)S^2, \qquad 2S_v^2 = (1 - b)S^2.$$

Thus, when $S_v = S_u$ and hence when $b = 0$, the ellipses degenerate to circles. When $S_v = 0$ or $S_u = 0$ and hence when $b = +1$ or

[5] A parameter which itself is simpler than b is $a = S_v/S_u$; but if a were used instead of b most of the formulas in the unpublished Appendix A, mentioned in footnote 3, would be rendered considerably longer and more complicated.

[6] It is to be noted that $\overline{|w|^2}$ is not equal to $S^2_{|w|}$ if, as is natural, this is defined by the equation

$$S^2_{|w|} = \overline{(|w| - \overline{|w|})^2} = \overline{|w|^2} - \overline{|w|}^2.$$

$b = -1$ respectively, the ellipses degenerate to superposed straight line segments coinciding with the u-axis or the v-axis respectively; owing to this superposition of the straight line segments the "probability density" on the resulting straight line locus is not constant but varies in accordance with the 1-dimensional normal law, as expressed by equation (1).

With the object of reducing the number of parameters from 2 to 1 and of dealing with variables that are independent of units, it will be preferable not to deal directly with the original chance-variable $w = u + iv$, which is referred to the central principal axes ucv, but rather to deal with the "reduced" chance-variable $W = U + iV$ defined by the equation

$$W = w/S = u/S + iv/S = U + iV, \tag{15}$$

which is referred to the central principal axes UCV coinciding with the central principal axes ucv (Fig. 1), so that the position of any point T

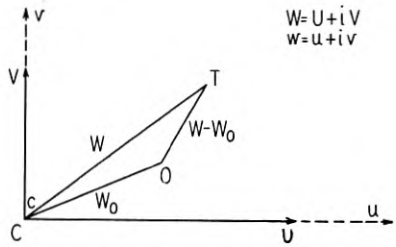


Fig. 1

in the W-plane will be represented by $W = U + iV$. Thus we shall be directly concerned with the scatter-diagram of $W = U + iV$ instead of with that of $w = u + iv$.

From (12) it is easily found that the probability law, say $P_{U,V}$, for $W = U + iV$ is

$$P_{U,V} = \frac{1}{\pi\sqrt{1-b^2}} \exp\left(-\frac{U^2}{1+b} - \frac{V^2}{1-b}\right), \tag{16}$$

which contains only the one parameter b, defined by (13), while moreover the variables U and V are independent of units. Thus the "reduced" complex chance-variable $W = U + iV$ given by (15) is defined as "normal" if its probability law can by the proper choice of a pair of rectangular axes UCV in the plane of its scatter-diagram be written in the form (16); the UCV-axes are the "central principal axes" of the scatter-diagram of $W = U + iV$; and the "mean value" of W is then zero, that is, $\overline{W} = 0$.

1.3. *Graphs for the Probability of the Deviation of a Normal Complex Chance-Variable from its Mean Value*

Before taking up the technical description of the graphs presented in this Subsection, some indication of their field for practical use will be furnished by the statement that the chance-variable $w = u + iv$ of the next paragraph may, for instance, be identified with the chance-variable h given by equation (II) of the Introduction, in case h is "normal" and is of zero "mean value," so that $\bar{h} = 0$; in case $\bar{h} \neq 0$, then w would be identified with $h - \bar{h}$. On referring to equation (II), it will be seen that h there denotes the deviation of any transmission characteristic from its nominal value; more generally, h may be any complex chance-variable which is "normal"—or approximately "normal."

The graphs here to be presented and described relate directly to the "reduced" complex chance-variable $W = U + iV$ given by equation (15) in terms of the original chance-variable $w = u + iv$ and the parameter S defined by equation (14). Assuming w to be "normal" and of zero "mean value" ($\bar{w} = 0$), it has the probability law formulated by equation (12); and hence $W = U + iV$ is normal and of zero mean value ($\overline{W} = 0$), and has the probability law formulated by (16), with the parameter b defined by (13).

With W' denoting the unknown value of a random sample consisting of a single value of the chance-variable W, the graphs herewith represent the probability that the magnitude $R' = |W'|$ of W' exceeds[7] any stated value R; that is, the probability that W' lies without a circle of radius R whose center coincides with the center C (Fig. 1) of the scatter-diagram of W, so that the center of the circle is at $W = 0$. This probability will be denoted by $p_b(R' > R)$, the subscript b implying dependence on the parameter b. The complementary probability will be denoted by $p_b(R' < R)$; this is of course the probability that R' is less than the stated value R; or, what is equivalent, the probability that W' lies within a circle of radius R centered at C. Of course the sum of the two foregoing probabilities is unity, that is,

$$p_b(R' > R) + p_b(R' < R) = 1. \tag{17}$$

[7] In engineering applications it is usually preferable to deal with the relatively small probability of exceeding, rather than with the complementary probability, nearly equal to unity, of being less than a preassigned rather large value of R.

Moreover,

$$p_b(R_1 < R' < R_2) = p_b(R' > R_1) - p_b(R' > R_2) \qquad (18)$$
$$= p_b(R' < R_2) - p_b(R' < R_1), \qquad (19)$$

where R_1 and R_2 denote any two stated values of R such that $R_1 < R_2$.

From (13) the total possible range of b is seen to be from -1 to $+1$, corresponding to the total possible range of S_v/S_u from ∞ to 0, with $b = 0$ corresponding to $S_v/S_u = 1$. However, it will evidently suffice to consider for b the range 0 to 1, corresponding to the range 1 to 0 for S_v/S_u, which will be secured by choosing S_u as the greater and hence S_v as the smaller of the two "standard deviations" (with the *ucv*-axes chosen correspondingly, of course).

The graphs in Figs. 2 and 3 show the relation between R and $p_b(R' > R)$ with b as parameter; similarly, Figs. 4 and 5 show the relation between R and the quantity $p_{b,0}(R' > R)$ defined by the equation

$$p_{b,0}(R' > R) = p_b(R' > R) - p_0(R' > R). \qquad (20)$$

Here $p_0(R' > R)$, being a particular value of $p_b(R' > R)$, plays the part of a reference value. It is a natural reference value, being the value for $b = 0$; and it can be evaluated immediately and accurately, since its exact formula is merely

$$p_0(R' > R) = \exp(-R^2). \qquad (21)$$

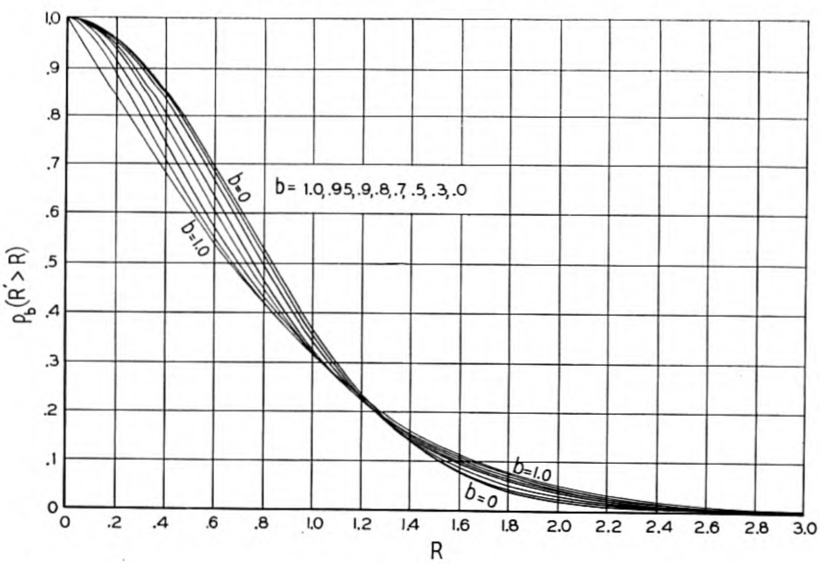


Fig. 2

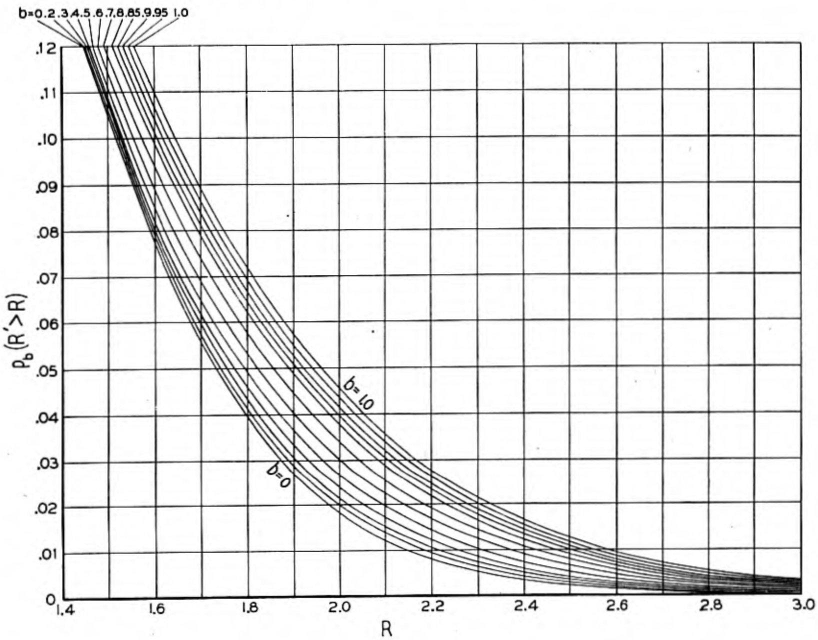

b=0 .2 .3 .4 .5 .6 .7 .8 .85 .9 .95 1.0
$p_b(R'>R)$
b=1.0
b=0
R

Fig. 3

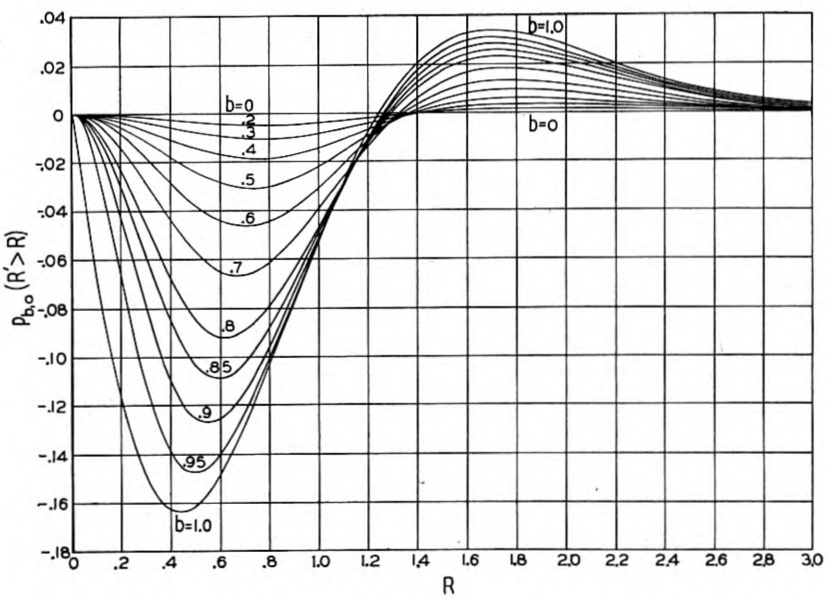

$p_{b,o}(R'>R)$
b=1.0
b=0
.2
.3
.4
.5
.6
.7
.8
.85
.9
.95
b=1.0
b=0
R

Fig. 4

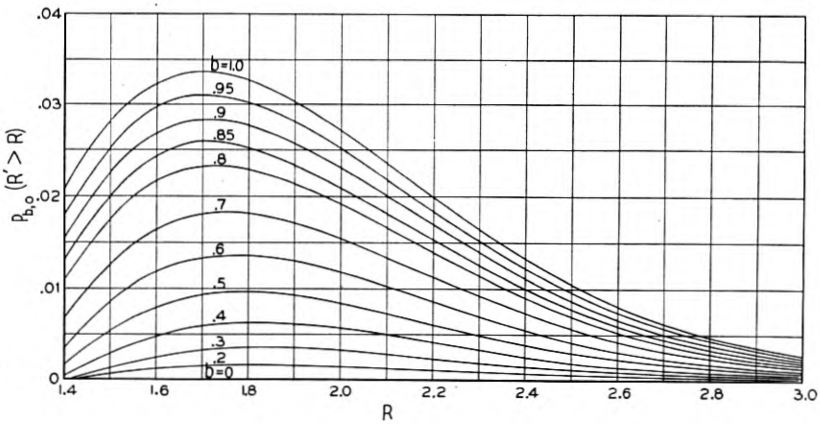


Fig. 5

The curves in Fig. 2 are chiefly useful for showing the form and range of the relations rather than for the reading-off of individual values; however, for the lower range of R ($R < 1$, say), they can be read with very fair accuracy. Fig. 3 is merely an enlarged plot of Fig. 2, over the R-range of about 1.5 to 3. The curves in Fig. 3 are accurately readable except in the upper part of this R-range; and the deficiency there is compensated by the curves of Fig. 5 described in the next paragraph.

The curves in Figs. 2 and 3 were plotted by aid of the much more accurately readable curves in Figs. 4 and 5, namely curves of R versus the quantity $p_{b,0}(R' > R)$ defined by equation (20); thus, by aid of (21),

$$p_b(R' > R) = p_{b,0}(R' > R) + \exp(-R^2). \qquad (22)$$

Fig. 5 is merely an enlarged plot of Fig. 4, over the R-range of 1.4 to 3.0.

The material of Fig. 2 is represented in alternative forms, which are more convenient for some purposes, by Figs. 6 and 7, the former giving curves of $p_b(R' > R)$ versus b with R as parameter, the latter giving curves of b versus R with $p_b(R' > R)$ as parameter.

The material of Fig. 4 is represented in one alternative form by Figs. 8 and 9 each of which gives curves of $p_{b,0}(R' > R)$ versus b with R as parameter.

Returning to Fig. 2, it will be noted that the curves cross each other, but not at a common point; they cross rather diffusely in the neighborhood of $R = 1.2$. In the lower range of R, $p_b(R' > R)$ decreases ·ith increasing b; while in the upper range of R, it increases with increasing b. Quantitatively these relations are shown more clearly and accurately by Figs. 6 and 7.

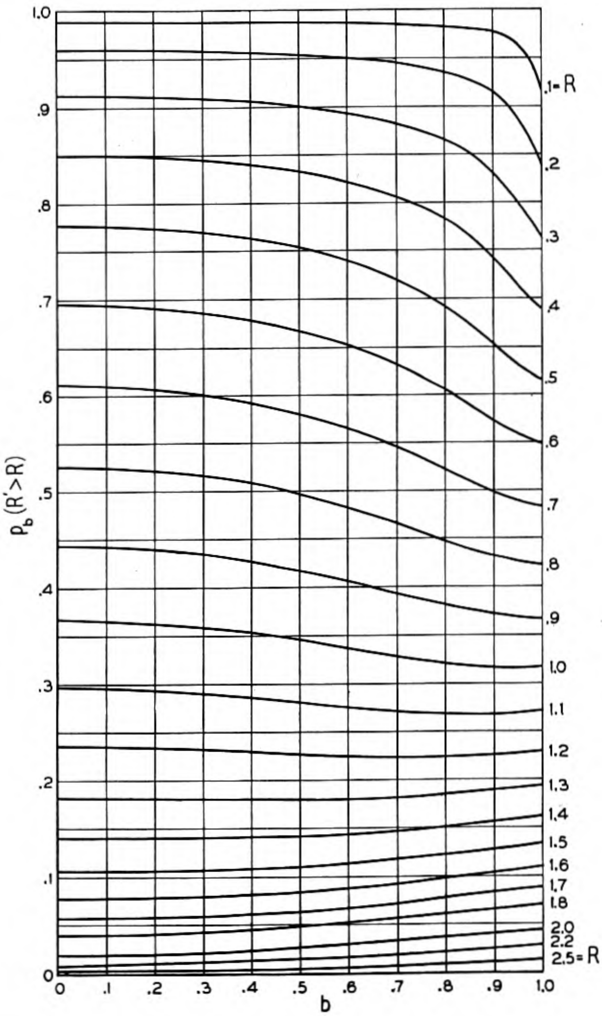


Fig. 6

Correspondingly in Fig. 4 the curves of $p_{b,0}(R' > R)$ cross each other rather diffusely in the neighborhood of[8] $R = 1.2$; thus, $p_{b,0}(R' > R)$ changes sign in this neighborhood. $p_{b,0}(R' > R)$ is nega-

[8] Except for values of b very nearly equal to 0; but in such cases $p_{b,0}(R' > R)$ is very small, so that the exception would be unimportant in most practical applications. A corresponding qualification applies, of course, to the discussion of Fig. in the preceding paragraph.

tive in the lower range of R and positive in the upper range; and the magnitude of $p_{b,0}(R' > R)$ always increases with increasing b. Since the value of R at which $p_{b,0}(R' > R)$ changes sign depends somewhat on b it will be denoted by R_b. Fig. 4 shows that R_1 is equal to about 1.24; and that R_b, when $1 > b > 0$, is greater than R_1 but only slightly greater except when b is very nearly zero. (See also Figs. 8 and 9.)

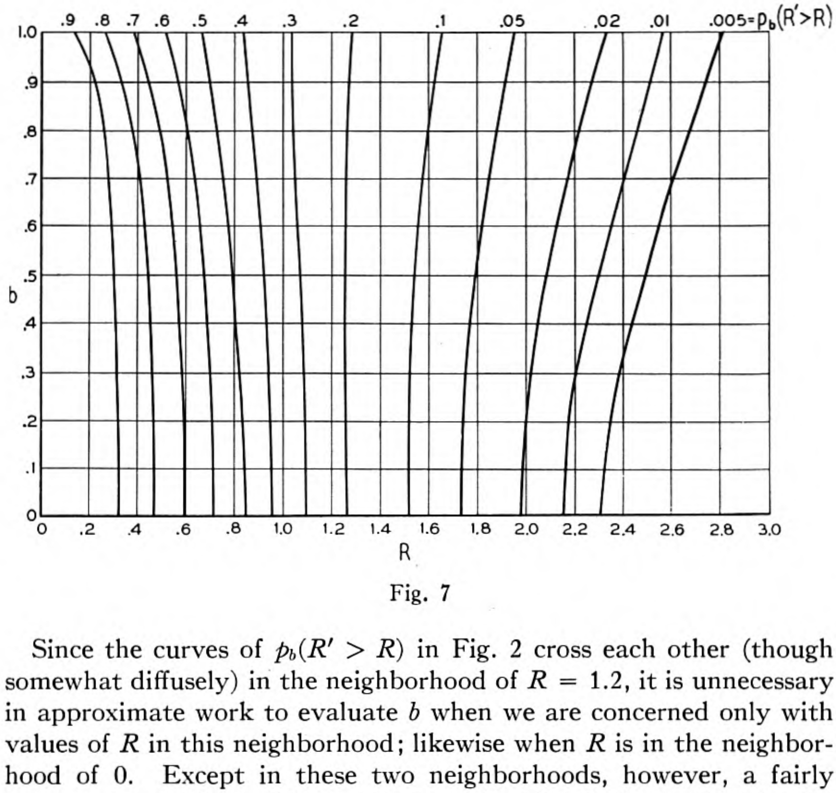


Fig. 7

Since the curves of $p_b(R' > R)$ in Fig. 2 cross each other (though somewhat diffusely) in the neighborhood of $R = 1.2$, it is unnecessary in approximate work to evaluate b when we are concerned only with values of R in this neighborhood; likewise when R is in the neighborhood of 0. Except in these two neighborhoods, however, a fairly accurate evaluation of b is necessary; for Fig. 2 shows that, in the upper R-range, $p_b(R' > R)$ depends very greatly on b, while even in the lower R-range the dependence on b is considerable. Thus the error resulting from assuming a value for b (in order to avoid the considerable labor of its actual evaluation) would usually be large. Quantitatively these facts are indicated more clearly and accurately by Figs. 6 and 7.

The computations underlying the graphs have proved to be so difficult and laborious that it has been deemed advisable to preserve he fundamental results in tabular form herewith (Table I), chiefly to enable the graphs to be replotted to a larger and more finely-divided scale by anybody so desiring. The values for $b = 0$ and $b = 1$ were omitted from the table, as being unnecessary because

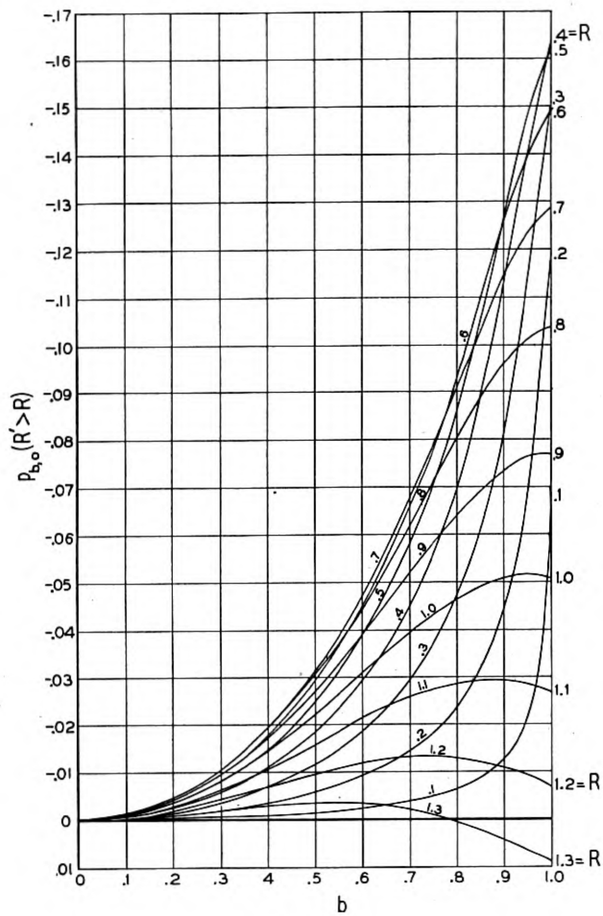


Fig. 8

$p_{b,0}(R' > R)$ is identically zero for $b = 0$, while for $b = 1$ it is given by the simple and exact formula

$$p_{1,0}(R' > R) = \text{erfc } (R/\sqrt{2}) - \exp{(-R^2)}.$$

Although in many of the computed values in Table I the last digit (the third significant figure) cannot be regarded as reliable, it is thought that the tabulated values are accurate to about one per cen or better, which of course is quite adequate for all practical purpose

TABLE I

VALUES OF $p_{b,0}(R' > R)$

R	$b = .2$	$b = .3$	$b = .4$	$b = .5$	$b = .6$	$b = .7$	$b = .8$	$b = .85$	$b = .9$	$b = .95$
.1	−.000204	−.000469	−.000902	−.00154	−.00246	−.00391	−.00637	−.00862	−.0128	−.0217
.2	−.000773	−.00180	−.00340	−.00577	−.00922	−.0145	−.0236	−.0313	−.0440	−.0678
.3	−.00161	−.00377	−.00705	−.0119	−.0189	−.0295	−.0467	−.0604	−.0809	−.1120
.4	−.00257	−.00598	−.0111	−.0186	−.0293	−.0450	−.0691	−.0867	−.1101	−.1394
.5	−.00348	−.00808	−.0149	−.0247	−.0385	−.0577	−.0851	−.1036	−.1252	−.1471
.6	−.00419	−.00970	−.0178	−.0291	−.0446	−.0653	−.0922	−.1085	−.1256	−.1399
.7	−.00459	−.0106	−.0193	−.0311	−.0467	−.0665	−.0899	−.1025	−.1144	−.1235
.8	−.00464	−.0106	−.0192	−.0304	−.0447	−.0617	−.0796	−.0883	−.0957	−.1012
.9	−.00431	−.00978	−.0175	−.0272	−.0391	−.0520	−.0641	−.0692	−.0735	−.0765
1.0	−.00368	−.00829	−.0145	−.0221	−.0309	−.0395	−.0463	−.0486	−.0506	−.0518
1.1	−.00283	−.00630	−.0108	−.0160	−.0215	−.0260	−.0284	−.0291	−.0294	−.0287
1.2	−.00187	−.00411	−.00683	−.00953	−.0120	−.0132	−.0125	−.0117	−.0110	−.00920
1.3	−.000925	−.00196	−.00300	−.00353	−.00350	−.00214	.000771	.00241	.00451	.00670
1.4	−.0000796	−.0000620	.000327	.00152	.00342	.00654	.0108	.0130	.0154	.0178
1.5	.000603	.00147	.00295	.00536	.00853	.0127	.0176	.0200	.0227	.0252
1.6	.00110	.00256	.00477	.00792	.0118	.0165	.0217	.0242	.0269	.0297
1.7	.00141	.00320	.00581	.00933		.0181	.0232	.0258	.0284	.0309
1.818	.00157	.00353	.00631	.00969	.0137	.0181	.0227	.0250	.0275	.0299
2.0	.00144	.00321	.00562	.00849	.0118	.0154	.0192	.0210	.0231	.0252
2.222	.00102	.00225	.00392	.00588	.00814	.0107	.0133	.0146	.0162	.0176
2.5	.000507	.00112	.00197	.00296	.00413	.00554	.00702	.00782	.00872	.00963
2.857	.000145	.000329	.000612	.000915	.00134	.00188	.00245	.00282	.00320	.00362
3.333	.0000187	.0000432	.0000868	.000140	.000203	.000334	.000458	.000559	.000657	.000798

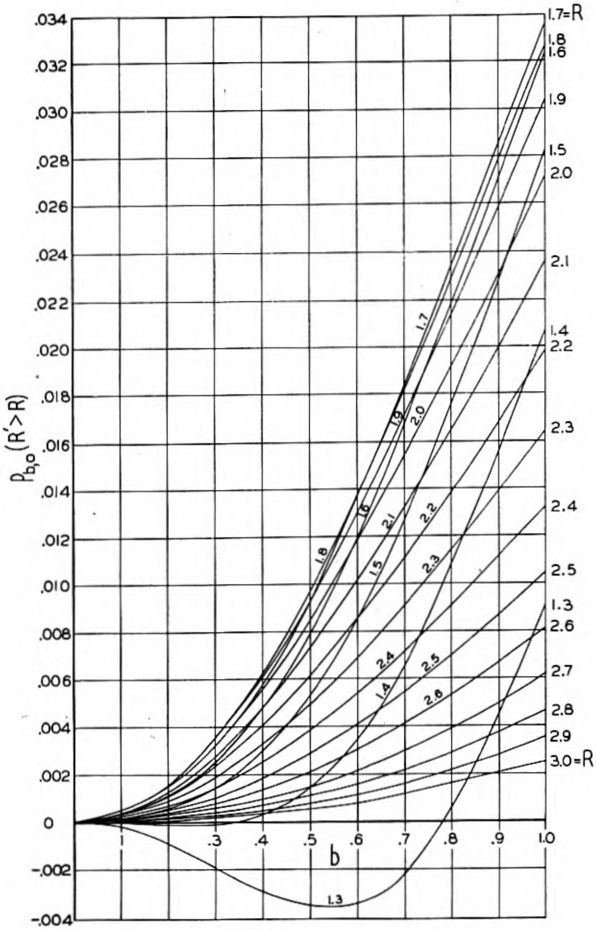


Fig. 9

2. THE LEADING DISTRIBUTION-PARAMETERS OF ANY COMPLEX CHANCE-VARIABLE

By the "leading distribution-parameters" of any complex chance-variable will here be meant a certain set of distribution-parameters (specified below) which would be sufficient for completely fixing the distribution if it were "normal." Even when the distribution is not "normal" these parameters are usually present among the other parameters in the distribution-function; indeed they are often th most important of the distribution-parameters.

In order to define and formulate the "leading distribution-para eters" of any complex chance-variable $Z = X + iY$ in an exp manner, conformably to the implicit definition in the preceding paragraph, we could proceed in a purely analytical manner, as outlined in Subsection 2.2 below. However, in recognition of the very substantial aid to thought and description furnished by the concept of the "scatter-diagram," for graphically representing any two-dimensional distribution, this concept will here be invoked in framing the definitions and in deriving the desired formulas.

Proceeding on this basis, it will be found that three of the "leading distribution-parameters" are certain "average values" pertaining to the scatter-diagram of the contemplated chance-variable; for any "average value" pertaining to the scatter-diagram is equal to the corresponding "mean value" ("expected value") pertaining to the chance-variable, when the "mean value" is defined as just after equation (2). It will be recalled that there a superbar applied to the symbol denoting any chance-variable was used to connote the "mean value" of the chance-variable. In the present Section (2), owing to the above-noted relation, the superbar may interchangeably be regarded as connoting either an "average value" pertaining to the scatter-diagram or the corresponding "mean value" pertaining to the chance-variable.

Having in mind the definition of the "scatter diagram" of any complex chance-variable $Z = X + iY$, let XAY (Fig. 10) designate

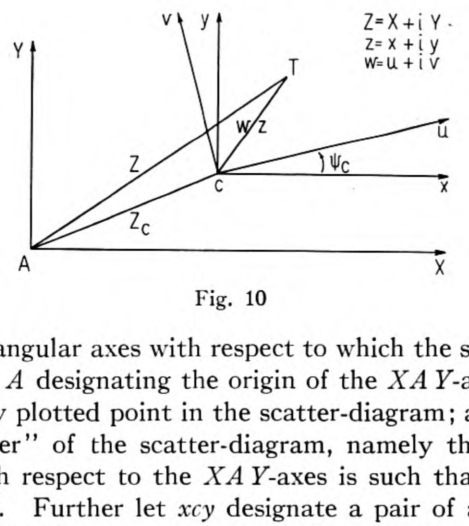


Fig. 10

the pair of rectangular axes with respect to which the scatter-diagram of Z is plotted, A designating the origin of the XAY-axes. Also, let T designate any plotted point in the scatter-diagram; and let c designate the "center" of the scatter-diagram, namely the point whose position Z_c with respect to the XAY-axes is such that $\overline{Z - Z_c} = 0$, whence $Z_c = \overline{Z}$. Further let xcy designate a pair of axes through c parallel to the XAY-axes, and ucv any other pair of rectangular axes through c; and let $w = u + iv$ represent the position of the point T with respect to the ucv-axes, the position of T with respect to the xcy-axes being represented by $z = x + iy$ and with respect to the

XAY-axes by $Z = X + iY$, whence $z = Z - Z_c$. Any pair of axes, such as ucv, through the center c are called "central axes"; ψ_c denotes their orientation-angle with respect to the xcy-axes, and hence with respect to the XAY-axes. When ψ_c has such a value ψ_c' that $\overline{uv} = 0$, the central axes ucv are called "principal central axes"; the corresponding values of $\overline{u^2}$ and $\overline{v^2}$ are denoted by S_u^2 and S_v^2 respectively, and S_u and S_v are called the "principal standard deviations" pertaining to the chance-variable $w = u + iv$.

Conformably to the implicit definition in the first paragraph of this Section, we may now state that the "leading distribution parameters" of any complex chance-variable $Z = X + iY$ are the four quantities Z_c, ψ_c', S_u, S_v defined and named in the preceding paragraph; it will be recognized that these four quantities would be sufficient for fixing the distribution if it were "normal."

(Still referring to Fig. 10, it may be noted that an alternative set of four parameters fixing the distribution of any "normal" complex chance-variable consists of Z_c, Π_{xy}, S_x, S_y, where $\Pi_{xy} = \overline{xy}$, $S_x^2 = \overline{x^2}$, $S_y^2 = \overline{y^2}$. The set Z_c, ψ_c', S_u, S_v was chosen as being much preferable for this paper.)

With a view to formulating precise definitions of the various additional technical terms needed, and to establishing general formulas from which to deduce the desired formulas for the last three of the "leading distribution parameters" Z_c, ψ_c', S_u, S_v, consider Fig. 11,

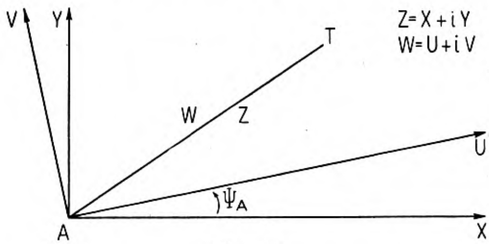


Fig. 11

which is a partial reproduction of Fig. 10, with the addition of the axes UAV, which are any pair of rectangular axes through A, so that $W = U + iV$ represents the position of any point T with respect to the UAV-axes, the position of T with respect to the XAY-axes being represented by $Z = X + iY$, of course. Then it can be shown (Subsection 2.1) that when the orientation-angle Ψ_A of the UAV-axes (Fig. 11) with respect to the XAY-axes has either of the values Ψ_A' given by the equation [9]

[9] In this paper, if Z denotes any complex quantity, then $ag\,Z$ denotes its angle, $|Z|$ its absolute value, $\hat{Z}$ its conjugate, $Re\,Z$ its real part, and $Im\,Z$ its imaginary part (that is, the cofactor of i when Z is written in rectangular form).

$$2\Psi_A' = ag(\pm \overline{Z^2}), \tag{23}$$

then the "mean" of the product UV vanishes, that is,

$$\overline{UV} = 0, \tag{24}$$

and the mean of $\overline{U^2}$ and the mean of $\overline{V^2}$ have the values expressed by the equations

$$2\overline{U^2} = \overline{|Z|^2} \pm |\overline{Z^2}|, \tag{25}$$

$$2\overline{V^2} = \overline{|Z|^2} \mp |\overline{Z^2}|, \tag{26}$$

and these values are extremum values in the sense that one is a maximum and the other a minimum when Ψ_A has either of the values Ψ_A' given by (23). Regarding the double signs in equations (23), (25), (26), it is hardly necessary to remark that the upper signs go together as one set, and the lower signs as another set. However, the presence of the double signs is a triviality; for the UAV-axes (Fig. 11) with respect to which equations (23), (24), (25), (26) are fulfilled are unique except merely as to their designations (U versus V, with signs), the values of Ψ_A' differing only by a multiple of $\pi/2$. (In numerical applications it will usually be convenient to choose the upper set of signs, so that $\overline{U^2}$ will be the maximum quantity and $\overline{V^2}$ the minimum.)

The particular UAV-axes (Fig. 11) for which equation (24) is fulfilled and for which Ψ_A therefore has a value Ψ_A' given by equation (23) are called the "principal axes" through A; and the corresponding mean squares $\overline{U^2}$ and $\overline{V^2}$ given by (25) and (26) are called the "principal mean squares." It will therefore be natural. and will be found convenient, to call Ψ_A', $\overline{U^2}$, $\overline{V^2}$ the "principal parameters" pertaining to the point A; they are seen to depend only on $\overline{Z^2}$ and $\overline{|Z|^2}$.

More generally, when the point A in Fig. 11 is not restricted to being the origin of the scatter-diagram of the given complex chance-variable but is any point in that scatter-diagram and when the XAY-axes and the UAV-axes are any two pairs of rectangular axes through A, it is readily seen that the formulas (23), (24), (25), (26) remain unchanged, although of course Z no longer represents the given chance-variable but now represents merely the position of any point T with respect to the XAY-axes, while W represents the position of T with respect to the UAV-axes. The quantities Ψ_A', $\overline{U^2}$, $\overline{V^2}$ given by equations (23), (25), (26) will naturally continue to be called the "principal parameters" relating to the point A, which is now any point. Thus the "principal parameters" are more general than the last three (ψ_c', S_u, S_v) of the "leading distribution-parameters," to which the "principal parameters" reduce when A coincides with the "center" c.

Continuing to regard A in Fig. 11 as any point in the scatter-diagram, it can be shown that in the degenerate case characterized by $\overline{Z^2} = 0$ all pairs of rectangular axes through A are "principal axes"; for when $\overline{Z^2} = 0$, equation (24) is fulfilled for all values of Ψ_A (as will be shown in the last paragraph of Subsection 2.1). Furthermore the mean squares with respect to all pairs of rectangular axes through A are then equal, as is shown by the fact that equations (25) and (26) reduce to

$$2\overline{U^2} = 2\overline{V^2} = \overline{|Z|^2} = \overline{X^2} + \overline{Y^2}. \tag{27}$$

Since A in Figs. 10 and 11 can be any point, the desired formulas for the last three of the "leading distribution-parameters" Z_c, ψ_c', S_u, S_v, relating to the point c in Fig. 10, are now seen to be immediately obtainable from formulas (23), (25), (26) for the "principal parameters" relating to the point A, by merely letting A coincide with c, the XAY-axes with the xcy-axes and the UAV-axes with the ucv-axes; for then Ψ_A', U, V, Z become ψ_c', u, v, z respectively; whence, after writing S_u^2 and S_v^2 for $\overline{u^2}$ and $\overline{v^2}$, the desired formulas are seen to be

$$2\psi_c' = ag(\pm \overline{z^2}), \tag{28}$$

$$2S_u^2 = \overline{|z|^2} \pm |\overline{z^2}|, \tag{29}$$

$$2S_v^2 = \overline{|z|^2} \mp |\overline{z^2}|, \tag{30}$$

where, as will be recalled, $z = Z - Z_c = Z - \overline{Z}$ represents (Fig. 10) the position of any point T of the scatter-diagram of Z with respect to the axes xcy through the center c parallel to the XAY-axes, which latter are there the axes of Z; thus $\bar{z} = 0$, though of course $\overline{Z} \neq 0$ in general. In accordance with (28), (29), (30) the last three of the leading distribution-parameters of $Z = z + Z_c = z + \overline{Z}$, which are the same as the last three of the leading distribution-parameters of z, are completely determined by the two mean values $\overline{z^2}$ and $\overline{|z|^2}$.

In order to represent explicitly the last three of the leading distribution-parameters of Z as depending on $Z - \overline{Z}$, it seems worth while to rewrite (28), (29), (30) in the following equivalent forms:

$$2\psi_c' = ag(\pm \overline{[Z - \overline{Z}]^2}), \tag{31}$$

$$2S_u^2 = \overline{|Z - \overline{Z}|^2} \pm |\overline{[Z - \overline{Z}]^2}|, \tag{32}$$

$$2S_v^2 = \overline{|Z - \overline{Z}|^2} \mp |\overline{[Z - \overline{Z}]^2}|, \tag{33}$$

which are completely determined by the two mean values $\overline{[Z - \overline{Z}}$

and $\overline{|Z - \overline{Z}|^2}$, though each of these depends on $\overline{Z}$, which plays the part of a reference value.

The foregoing formulas, by aid of (88) and (89) in Subsection 4.2, can be written also in the forms:

$$2\psi_c' = ag(\pm[\overline{Z^2} - \overline{Z}^2]), \tag{34}$$

$$2S_u^2 = \overline{|Z|^2} - |\overline{Z}|^2 \pm |\overline{Z^2} - \overline{Z}^2|, \tag{35}$$

$$2S_v^2 = \overline{|Z|^2} - |\overline{Z}|^2 \mp |\overline{Z^2} - \overline{Z}^2|, \tag{36}$$

which are completely determined by the three mean values $\overline{Z}$, $\overline{Z^2}$, $\overline{|Z|^2}$.

S_u and S_v are termed the "*principal* standard deviations," obviously because they relate to the "*principal* central axes," namely the particular *ucv*-axes corresponding to $\psi_c = \psi_c'$ (Fig. 10). They are special values of the "standard deviations" S_x and S_y, which latter relate to any specified central axes, *xcy*, and are defined by the equations $S_x^2 = \overline{x^2}$ and $S_y^2 = \overline{y^2}$.

By aid of the pairs of equations (29), (30) and (32), (33) and (35), (36), the parameters b and S defined by equations (13) and (14) can now be written in the following more explicit forms:

$$b = \frac{|\overline{z^2}|}{\overline{|z|^2}} = \frac{|\overline{(Z - \overline{Z})^2}|}{\overline{|Z - \overline{Z}|^2}} = \frac{|\overline{Z^2} - \overline{Z}^2|}{\overline{|Z|^2} - |\overline{Z}|^2}, \tag{37}$$

$$S^2 = \overline{|z|^2} = \overline{|Z - \overline{Z}|^2} = \overline{|Z|^2} - |\overline{Z}|^2. \tag{38}$$

Returning now to the general case in which point A in Fig. 11 is any point in the scatter-diagram of the given complex chance-variable, it will be recalled that formulas (23), (25), (26) give the values of the "principal parameters" relating to the point A. Let it now be required to formulate the principal parameters relating to any other point, a, in terms of quantities relating to the point A. With this purpose, consider Fig. 12. Here the XAY-axes are any rectangular axes through A; but the UAV-axes are the principal axes through A, as implied by the symbol Ψ_A' for their orientation-angle. The *xay*-axes are merely a pair of auxiliary axes through a drawn parallel to the XAY-axes; and the *uav*-axes are the principal axes through a. Z, W, z, w represent the position of any point T with respect to the axes XAY, UAV, *xay*, *uav* respectively; and Z_a represents the position of point a with respect to the XAY-axes. Then, corresponding to

(23), (25), (26), the formulas for the principal parameters relating to the point a are, of course,

$$2\psi_a' = ag(\pm \overline{z^2}), \tag{39}$$

$$2\overline{u^2} = \overline{|z|^2} \pm |\overline{z^2}|, \tag{40}$$

$$2\overline{v^2} = \overline{|z|^2} \mp |\overline{z^2}|. \tag{41}$$

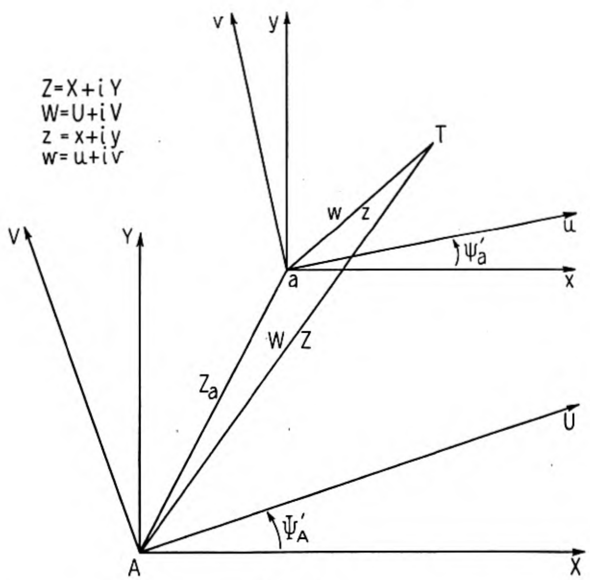


Fig. 12

But, since the xay-axes are parallel to the XAY-axes,

$$z = Z - Z_a. \tag{42}$$

Squaring (42) and taking the mean of the result gives

$$\overline{z^2} = \overline{Z^2} + Z_a^2 - 2\overline{Z}Z_a. \tag{43}$$

Multiplying (42) by its conjugate [9] and taking the mean of the result gives

$$\overline{|z|^2} = \overline{|Z|^2} + |Z_a|^2 - 2\text{Re}\,(\overline{Z}\hat{Z}_a). \tag{44}$$

Substituting (43) and (44) into (39), (40), (41) yields the desired formulas expressing the principal parameters relating to the point a (Fig. 12) in terms of quantities relating to the point A.

In particular, the formulas (34), (35), (36) for the last three of the "leading distribution parameters" of the original given chance-variable Z are immediately obtainable by merely letting the point a (Fig. 12) coincide with the center c; for then equations (42), (43), (44) reduce to

$$z = Z - Z_c = Z - \overline{Z}, \tag{45}$$

$$\overline{z^2} = \overline{Z^2} - \overline{Z}^2, \tag{46}$$

$$\overline{|z|^2} = \overline{|Z|^2} - |\overline{Z}|^2. \tag{47}$$

2.1. *Proofs of Formulas (23), (25), (26)*

With $W = U + iV$ here denoting any complex quantity,[9] formulas (23), (25), (26) will be proved by starting with the three identities [10]

$$2UV = \text{Im } W^2, \tag{48}$$

$$2U^2 = |W|^2 + \text{Re } W^2, \tag{49}$$

$$2V^2 = |W|^2 - \text{Re } W^2. \tag{50}$$

In order to apply these identities in proving formulas (23), (25), (26), which relate to Fig. 11, we evidently must identify the W appearing in these identities with the W in Fig. 11, and also must introduce the relation existing between W and Z in Fig. 11, namely

$$W = Z \exp(-i\Psi_A). \tag{51}$$

To prove (23) we substitute (51) into (48) and take the mean value of the result, thus getting

$$2\overline{UV} = |\overline{Z^2}| \sin(\text{ag } \overline{Z^2} - 2\Psi_A). \tag{52}$$

For the general case in which $|\overline{Z^2}|$ is not zero, this equation shows that the necessary and sufficient condition for $\overline{UV}$ to be zero is that Ψ_A shall have any of the special values Ψ_A' satisfying the following equation, in which n is real:

$$\text{ag } \overline{Z^2} - 2\Psi_A' = n\pi, \qquad (|n| = 0, 1, 2, 3, \cdots). \tag{53}$$

[10] These are equivalent to the identities

$$i4UV = W^2 - \hat{W}^2,$$
$$4U^2 = W^2 + \hat{W}^2 + 2W\hat{W},$$
$$-4V^2 = W^2 + \hat{W}^2 - 2W\hat{W},$$

ıich are immediately obtainable from the pair of simpler identities $2U = W + \hat{W}$ 1 $i2V = W - \hat{W}$. However, formulas (48), (49), (50) can be readily verified by ·ely substituting $W = U + iV$.

Hence

$$2\Psi_A' = \mathrm{ag}\,\overline{Z^2} - n\pi = \mathrm{ag}\,(\pm\overline{Z^2}), \tag{54}$$

which is (23). Evidently there are only two geometrically distinct values of Ψ_A', namely that for even n and that for odd n; and even this duality is a triviality, in the sense indicated in the latter part of the paragraph containing equations (25) and (26).

To prove (25) and (26) and at the same time to show that they are extrema, we substitute (51) into (49) and (50) and take the mean value of each result, thus getting

$$2\overline{U^2} = \overline{|Z|^2} + |\overline{Z^2}| \cos(\mathrm{ag}\,\overline{Z^2} - 2\Psi_A), \tag{55}$$

$$2\overline{V^2} = \overline{|Z|^2} - |\overline{Z^2}| \cos(\mathrm{ag}\,\overline{Z^2} - 2\Psi_A). \tag{56}$$

For the general case in which $|\overline{Z^2}|$ is not zero, these two equations show that when Ψ_A is varied, $\overline{U^2}$ and $\overline{V^2}$ have extremum values when Ψ_A has any of the special values Ψ_A' satisfying (53) and hence satisfying (23). Substitution of (53) into (55) and (56) gives (25) and (26), which are thus proved.

In the degenerate case characterized by $\overline{Z^2} = 0$, the unrestricted equation (52) shows that (24) will be fulfilled for all values of Ψ_A. This remark serves to prove the statement made in the paragraph containing equation (27).

2.2 *Outline of a Purely Analytical Treatment of the Leading Distribution-Parameters*

This Subsection is supplied, in accordance with the second paragraph of Section 2, in order to show that the leading distribution-parameters can be equivalently defined and formulated in a purely analytical manner, that is, without the aid of the "scatter-diagram" concept.

With $Z = X + iY$ denoting the given chance-variable, let Z_c denote that particular value of Z determined by the equation $\overline{Z - Z_c} = 0$, so that $Z_c = \overline{Z}$, the superbar connoting the "mean value" ("expected value") of Z, as defined just after equation (2). On account of the restriction of the present Subsection to pure analysis, Z_c cannot here be consistently called the "center of the scatter-diagram"; instead it will be called the "central value" of Z.

Next let $z = x + iy$ and $w = u + iv$ be the auxiliary chanc variables defined by the equations

$z = Z - Z_c,$ (57) $w = z \exp(-i\psi_c),$ (

where, however, ψ_c is arbitrary, so that w is not determined until ψ_c is assigned. Also let ψ_c' be such a value of ψ_c that $\overline{uv} = 0$; and let S_u^2 and S_v^2 denote the corresponding values of $\overline{u^2}$ and $\overline{v^2}$ respectively, that is, the particular values taken by $\overline{u^2}$ and $\overline{v^2}$ when $\psi_c = \psi_c'$, so that $\overline{uv} = 0$.

The formulas (28), (29), (30) for ψ_c', S_u, S_v can now be established in a purely analytical manner in just the same way as the more general formulas (23), (25), (26) were established in Subsection 2.1.

3. FORMULAS FOR THE LEADING DISTRIBUTION-PARAMETERS OF A LINEAR FUNCTION OF COMPLEX CHANCE-VARIABLES

To meet the needs in dealing with problems of the type handled in Part II, namely problems involving linear functions of complex chance-variables, the present Section furnishes formulas for the "leading distribution-parameters" of any complex chance-variable Z which is a linear function of any number n of complex chance-variables $Z_1, \cdots Z_n$, so that

$$Z = a + b_1Z_1 + \cdots + b_nZ_n, \tag{59}$$

where $a, b_1, \cdots b_n$ are any constants, complex in general.

It will be recalled that the "leading distribution-parameters" of any complex chance-variable Z are the quantities Z_c, ψ_c', S_u, S_v defined and formulated in Section 2.

Since, in general, $Z_c = \overline{Z}$, application of Theorem 3 of Subsection 4.2 to (59) gives

$$\overline{Z} = a + b_1\overline{Z}_1 + \cdots + b_n\overline{Z}_n, \tag{60}$$

so that here $\overline{Z}$ is not zero even when $\overline{Z}_1, \cdots \overline{Z}_n$ are all zero.

The formulas for ψ_c', S_u, S_v are (28), (29), (30), where $z = Z - Z_c$; or the equivalent formulas (31), (32), (33) or (34), (35), (36).

With a view to using formulas (28), (29), (30), which have the advantage of compactness, we introduce the quantities z and z_r defined by the equations

$$z = Z - Z_c = Z - \overline{Z}, \tag{61}$$

$$z_r = Z_r - \overline{Z}_r, \qquad (r = 1, \cdots n), \tag{62}$$

which show that $\bar{z} = 0$ and that

$$\bar{z}_r = 0, \qquad (r = 1, \cdots n). \tag{63}$$

Subtracting (60) from (59) and then substituting (61) and (62) into he result gives

$$z = b_1z_1 + \cdots + b_nz_n, \tag{64}$$

hich has the advantage of not involving a.

Formulas (28), (29), (30) involve $\overline{z^2}$ and $\overline{|z|^2}$. To evaluate $\overline{z^2}$ we square z and take the mean value of the result; to evaluate $\overline{|z|^2}$ we multiply z by its conjugate $\hat{z}$ and take the mean value of the result. We thus obtain from (64) the formulas

$$\overline{z^2} = b_1^2\overline{z_1^2} + \cdots + b_n^2\overline{z_n^2} + \cdots + 2b_s b_t \overline{z_s z_t} + \cdots, \tag{65}$$

$$\overline{|z|^2} = |b_1|^2\overline{|z_1|^2} + \cdots + |b_n|^2\overline{|z_n|^2} + \cdots + 2\text{Re}\, b_s\hat{b}_t\overline{z_s\hat{z}_t} + \cdots, \tag{66}$$

where $s = 1, \cdots n - 1$ and $t = s + 1, \cdots n$. These two formulas can also be written

$$\overline{z^2} = \sum_r^{1\cdots n} b_r^2\overline{z_r^2} + 2\sum_{s<t}^{1\cdots n} b_s b_t\overline{z_s z_t}, \tag{67}$$

$$\overline{|z|^2} = \sum_r^{1\cdots n} |b_r|^2\overline{|z_r|^2} + 2\text{Re}\sum_{s<t}^{1\cdots n} b_s\hat{b}_t\overline{z_s\hat{z}_t}, \tag{68}$$

corresponding respectively to formulas (94) and (95) in Subsection 4.3.

When the subscripted Z's are independent, and hence the subscripted z's are independent, equations (65) and (66) respectively reduce to

$$\overline{z^2} = b_1^2\overline{z_1^2} + \cdots + b_n^2\overline{z_n^2}, \tag{69}$$

$$\overline{|z|^2} = |b_1|^2\overline{|z_1|^2} + \cdots + |b_n|^2\overline{|z_n|^2}, \tag{70}$$

on account of Theorem 1 in Subsection 4.1 together with equation (63).

4. SOME FORMULAS AND THEOREMS ON MEAN VALUES OF COMPLEX CHANCE-VARIABLES

The present Section supplies a considerable number of formulas and theorems on "mean values" ("expected values") [11] of complex chance-variables. Many of these formulas and theorems have already been used in Part I, and further use for them will be found in Part II; while outside of this paper they may well find varied other uses.

The theorems are word-statements of the simpler and more frequently useful of the formulas; the remaining formulas are more general and are not simple enough to be profitably expressed as theorems.

Theorems 1 and 2 regarding the mean of a product of complex chance-variables and Theorem 3 regarding the mean of a sum are generalizations of the corresponding known theorems for real chance-variables, are formally the same as the latter, and are susceptible of the same sort of proofs. These three theorems furnish a natural basis for the remaining theorems, besides having extensive other uses.

[11] Defined just after equation (2).

4.1. *Mean of a Product of Independent Complex Chance-Variables*

The following Theorems 1 and 2 relating to the mean of a product of complex chance-variables are very important notwithstanding their limitation to chance-variables which are independent.

Two discrete chance-variables are said to be "independent" (or "uncorrelated" or "non-correlated") if the probability that either takes any given value is independent of the value taken by the other.

Two continuous chance-variables are said to be "independent" if the probability that either lies close to any given value is independent of the value taken by the other.

THEOREM 1. *If any number of complex chance-variables are independent, the mean of their product is equal to the product of their individual means.*

That is, if the Z's are independent,

$$\overline{Z_1 Z_2 \cdots Z_n} = \bar{Z}_1 \bar{Z}_2 \cdots \bar{Z}_n. \tag{71}$$

THEOREM 2. *If the magnitudes (absolute values) of any number of complex chance-variables are independent, the mean of the magnitude of the product of these complex chance-variables is equal to the product of the means of their individual magnitudes.*

That is, if the $|Z|$'s are independent,

$$\overline{|Z_1 Z_2 \cdots Z_n|} = \overline{|Z_1|}\,\overline{|Z_2|} \cdots \overline{|Z_n|}. \tag{72}$$

For the validity of Theorem 2 it is not necessary that the angles of the chance-variables be independent, but only their magnitudes. Moreover, if $\phi_1, \cdots \phi_n$ denote the angles of $Z_1, \cdots Z_n$ and Φ the angle of their product, then, by Theorem 3,

$$\bar{\Phi} = \overline{\phi_1} + \cdots + \overline{\phi_n}, \tag{72a}$$

whether or not the ϕ's are independent.

4.2. *Mean of a Sum of Complex Chance-Variables*

The following Theorem 3 is of unlimited scope, in the sense that it involves no assumption as to independence of the chance-variables.

THEOREM 3. *Given any number of complex chance-variables, which need not be independent, the mean of their sum is equal to the sum of their individual means.*

That is, whether or not the Z's are independent,

$$\overline{Z_1 + \cdots + Z_n} = \bar{Z}_1 + \cdots + \bar{Z}_n. \tag{73}$$

Since the Z's in Theorem 3 need not be independent, the theorem will continue to be valid when the Z's are any functions of any number of other chance-variables $w_1, \cdots w_m$.

The following six simple and useful equations, in which $Z = X + iY$ denotes any complex [9] chance-variable, are immediately obtainable by means of Theorem 3.

$$\bar{Z} = \bar{X} + i\bar{Y}, \qquad (74) \qquad\qquad \bar{\hat{Z}} = \bar{X} - i\bar{Y} = \hat{\bar{Z}}, \qquad (75)$$

$$\overline{Z^2} = \overline{X^2} - \overline{Y^2} + i2\overline{XY}, \qquad (76)$$

$$\overline{|Z|^2} = \overline{Z\hat{Z}} = X^2 + \overline{Y^2}, \qquad (77)$$

$$\bar{Z}^2 = \bar{X}^2 - \bar{Y}^2 + i\, 2\bar{X}\bar{Y}, \qquad (78)$$

$$|\bar{Z}|^2 = \bar{Z}\hat{\bar{Z}} = \bar{X}^2 + \bar{Y}^2. \qquad (79)$$

The following eight equations can be obtained by solving the foregoing set of equations or by applying Theorem 3 to the appropriate identities.

$$\bar{X} = \overline{\text{Re}\, Z} = \text{Re}\, \bar{Z}, \qquad (80) \qquad\qquad \bar{Y} = \overline{\text{Im}\, Z} = \text{Im}\, \bar{Z}, \qquad (81)$$

$$2\overline{XY} = \text{Im}\, \overline{Z^2}, \qquad (82)$$

$$2\overline{X^2} = \overline{|Z|^2} + \text{Re}\, \overline{Z^2}, \qquad (83)$$

$$2\overline{Y^2} = \overline{|Z|^2} - \text{Re}\, \overline{Z^2}, \qquad (84)$$

$$2\bar{X}\bar{Y} = \text{Im}\, \bar{Z}^2, \qquad (85)$$

$$2\bar{X}^2 = |\bar{Z}|^2 + \text{Re}\, \bar{Z}^2, \qquad (86)$$

$$2\bar{Y}^2 = |\bar{Z}|^2 - \text{Re}\, \bar{Z}^2. \qquad (87)$$

Theorem 3 yields also the following two useful equations

$$\overline{(Z - \bar{Z})^2} = \overline{Z^2} - \bar{Z}^2, \qquad (88)$$

$$\overline{|Z - \bar{Z}|^2} = \overline{|Z|^2} - |\bar{Z}|^2. \qquad (89)$$

The first can be obtained immediately by squaring $Z - \bar{Z}$ and then applying Theorem 3; the second by expanding the product $(Z - \bar{Z})(\hat{Z} - \hat{\bar{Z}})$ and then applying Theorem 3 together with equation (75).

When, instead of a single chance-variable Z, there are n chance variables $Z_1, \cdots Z_n$, not restricted to being independent, equatio

(88) and (89) become

$$\overline{\sum (Z_r - \bar{Z}_r)^2} = \sum (\overline{Z_r^2} - \bar{Z}_r^{\,2}), \tag{90}$$

$$\overline{\sum |Z_r - \bar{Z}_r|^2} = \sum (\overline{|Z_r|^2} - |\bar{Z}_r|^2), \tag{91}$$

where each summation $\sum$ covers the set $r = 1, \cdots n$.

4.3. *Mean of a Squared Sum of Complex Chance-Variables*

With a view to arriving at Theorems 4 and 5 below, and also several formulas which are more general than the theorems but are not simple enough to be profitably expressed as theorems, let $Z_1, \cdots Z_n$ denote any complex chance-variables; and for brevity let W denote their sum, so that

$$W = Z_1 + \cdots + Z_n. \tag{92}$$

As indicated by its title, this Subsection will be concerned particularly with formulas for $\overline{W^2}$ and $\overline{|W|^2}$, but it will also include formulas for $\bar{W}^2$ and $|\bar{W}|^2$.

Squaring W, given by (92), and then applying Theorem 3 gives

$$\overline{W^2} = \sum_{r=1}^{n} \overline{Z_r^2} + 2 \sum_{h=1}^{n-1} \sum_{k=h+1}^{n} \overline{Z_h Z_k}, \tag{93}$$

or, in a briefer notation,

$$\overline{W^2} = \sum_{r}^{1\cdots n} \overline{Z_r^2} + 2 \sum_{h<k}^{1\cdots n} \overline{Z_h Z_k}, \tag{94}$$

the second $\sum$ in (94) thus denoting double summation.[12]

Taking the product of W and its conjugate $\hat{W}$ and then applying Theorem 3 gives

$$\overline{|W|^2} = \sum \overline{|Z_r|^2} + 2\mathrm{Re} \sum \overline{Z_h \hat{Z}_k}. \tag{95}$$

Applying Theorem 3 to (92) and then squaring the result gives

$$\bar{W}^2 = \sum \bar{Z}_r^{\,2} + 2 \sum \bar{Z}_h \bar{Z}_k. \tag{96}$$

Taking the product of $\bar{W}$ and $\hat{\bar{W}}$ gives

$$|\bar{W}|^2 = \sum |\bar{Z}_r|^2 + 2\mathrm{Re} \sum \bar{Z}_h \hat{\bar{Z}}_k. \tag{97}$$

[12] In (95), $\cdots$ (99) the summations evidently cover the same sets of values as in (94).

When the Z's are independent, so that Theorem 1 is applicable, equations (94) and (95) respectively reduce to

$$\overline{W^2} = \sum \overline{Z_r^2} + 2 \sum \overline{Z}_h \overline{Z}_k, \tag{98}$$

$$\overline{|W|^2} = \sum \overline{|Z_r|^2} + 2\text{Re} \sum \overline{Z}_h \hat{\overline{Z}}_k, \tag{99}$$

although (96) and (97) remain unchanged. Thus, when the Z's are independent, the following relations exist:

$$\overline{W^2} - \overline{W}^2 = \sum (\overline{Z_r^2} - \overline{Z}_r^2), \tag{100}$$

$$\overline{|W|^2} - |\overline{W}|^2 = \sum (\overline{|Z_r|^2} - |\overline{Z}_r|^2). \tag{101}$$

It is of interest to compare these with (90) and (91), which do not require the Z's to be independent.

When, further, not more than one of the Z's is of non-zero mean value, so that at least $n - 1$ are of zero mean value, that is, when [13]

$$\overline{Z}_r = 0, \qquad (r = 1, \cdots, j - 1, j + 1, \cdots, n), \tag{102}$$

then (98) and (99) reduce to

$$\overline{W^2} = \sum \overline{Z_r^2}, \tag{103}$$

$$\overline{|W|^2} = \sum \overline{|Z_r|^2}. \tag{104}$$

After substitution of the value of W from the defining equation (92), and with due regard to (102), equations (103) and (104), on account of their importance and simplicity, may profitably be expressed in the form of two theorems, respectively, as follows:

THEOREM 4. *If any number of complex chance-variables are independent and if not more than one is of non-zero mean value, then the mean of the squared value of their sum is equal to the sum of the means of their individual squared values.*

That is,

$$\overline{(Z_1 + \cdots + Z_n)^2} = \overline{Z_1^2} + \cdots + \overline{Z_n^2}, \tag{105}$$

provided the Z's are independent and not more than one is of non-zero mean value, in accordance with (102).

THEOREM 5. *If any number of complex chance-variables are independent and if not more than one is of non-zero mean value, then the mean of the squared magnitude (absolute value) of their sum is equal to the sum of the means of their individual squared magnitudes.*

[13] An important practical instance in which one of the Z's is of non-zero mean value will be found in connection with equation (120) in the problem treated Section 6.

That is,

$$\overline{|Z_1 + \cdots + Z_n|^2} = \overline{|Z_1|^2} + \cdots + \overline{|Z_n|^2}, \tag{106}$$

provided the Z's are independent and not more than one is of non-zero mean value, in accordance with (102).

Part II: Applications

The methods, theorems and formulas presented in Part I will now be applied to two important problems in telephone transmission engineering.[14] However, in each of these problems the solution is carried no further than to formulate the "leading distribution-parameters" in a form suitable for numerical evaluation in any specific case, since Subsection 1.3 of Part I has furnished the means of solving such problems when once these parameters have been evaluated and when the distribution is known to be approximately "normal."

The two problems mentioned above are treated separately in the following Sections 5 and 6. Section 5 sketches the solution of the general problem which was outlined in the Introduction (in Part I) in connection with the equations there; Section 6 deals somewhat fully with another problem, which, though specific, is yet of a rather broad type.

The problem in Section 6 has heretofore been handled by various approximate and less comprehensive methods, as indicated in the first footnote of the Introduction. The relative simplicity of the method described by Crisson in his paper there cited is due to his simplifying assumption (made just after his equations 26 and 27) which amounts to assuming that the scatter-diagram is circular instead of, as actually, elliptical.

5. Deviation of any Characteristic of a Transmission System or of a Network

This Section sketches an approximate solution of the general problem outlined in the Introduction, in connection with equations (I) and (II), which are the general functional formulas for the contemplated characteristic H and its deviation h, respectively; in general H and h are complex.

The present Section relates chiefly to formulas for the "leading distribution-parameters" of h when this is regarded as a chance-variable.

In accordance with Section 2 (in Part I) the leading distribution-parameters of h are completely determined by $\bar{h}$, $\overline{h^2}$, $\overline{|h|^2}$. Evidently

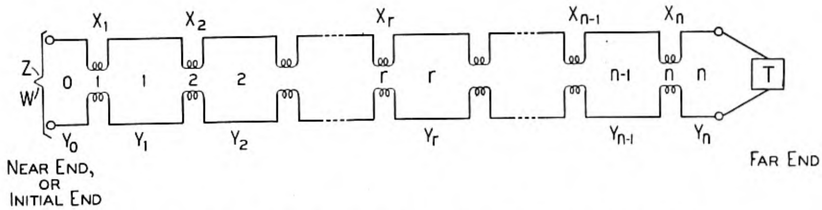
[14] An additional problem, crosstalk in a telephone cable, is treated in the unpublished Appendix C already mentioned in footnote 3.

the exact formulas for these three quantities must depend, in any specific case, on the corresponding specific form of the function F in equations (I) and (II) of the Introduction. However, general approximate formulas can be obtained when, as usual, the k's in (II) are small enough compared with the K's to enable the right side of (II) to be represented by the first-order terms of a Taylor expansion, so that h will be given by formula (III), as a good approximation. Since h, when so given, is a linear function of the chance-variables $k_1, \cdots k_n$, the formulas of Section 3 (in Part I) are directly applicable by setting $a = 0$ there, and identifying Z, b_r, Z_r there with h, D_r, k_r here, and hence z and z_r there with $h - \overline{h}$ and $k_r - \overline{k}_r$ here, respectively. Thus it is not necessary to write down here the formulas for $\overline{h}$, $\overline{h^2}$, $\overline{|h|^2}$.

When h is approximately "normal," the chance that the unknown value h' of a random sample consisting of a single value of h lies without a circle of specified radius centered at the mean value $\overline{h}$ of h can be found by application of the graphs presented and described in Subsection 1.3.

6. IMPEDANCE-DEVIATION AND REFLECTION COEFFICIENT OF A LOADED CABLE DUE TO LOADING IRREGULARITIES AND TERMINAL IRREGULARITY

As represented schematically by Fig. 13, the physical system considered in this problem consists of a periodically loaded cable whose loading-coil impedances and loading-section admittances, and also the



Z = IMPEDANCE OF SYSTEM: W = 1/Z = ADMITTANCE OF SYSTEM.
T = ADMITTANCE OF TERMINAL APPARATUS.
X_r = IMPEDANCE OF TYPICAL LOADING-COIL NO. r.
Y_r = ADMITTANCE OF TYPICAL WHOLE LOADING SECTION, NO. r.
X,Y = NOMINAL VALUES OF X_r, Y_r.
Y/2 = NOMINAL VALUE OF Y_0 AND Y_n.

Fig. 13

terminal admittance (T), deviate randomly from their nominal values, so that the deviations are complex chance-variables; however, the nominal value of the terminal admittance is not here restricted to equality with the iterative impedance of the loaded cable, since such a restriction would not correspond to the conditions usually existing in practice.

The resulting deviation in the impedance Z of the initial end of the system (Fig. 13) from the iterative impedance of the loaded cable is a complex chance-variable which is of much engineering importance in case the loaded cable is to constitute part of a transmission system containing a 2-way repeater, of the 22-type, connected between the initial end of the loaded cable and the remainder of the transmission system (not shown in Fig. 13); for, so far as the loaded cable is concerned, the practicable amplification obtainable from the repeater will depend approximately inversely on the impedance-deviation of the loaded cable; more precisely, it will depend inversely on the reflection coefficient defined, in terms of the impedance-deviation, by equation (107) below.

In Fig. 13 the loaded cable is represented as beginning with a half-section, and as ending with a half-section, and the latter as terminated with an admittance T. The formulas herein established are for this system. Analogous formulas for a system beginning and ending with half-coils, instead of with half-sections, can be obtained in an analogous manner, or even written down directly by analogy.

The important reflection coefficient mentioned at the end of the second paragraph, and to be denoted by ρ, is defined by the equation

$$\rho = -\frac{Z-h}{Z+h} = -\frac{(Z-h)}{2h+(Z-h)} = -\frac{(Z-h)/2h}{1+(Z-h)/2h}, \quad (107)$$

Z denoting the impedance of the system in Fig. 13, and h the mid-section iterative impedance of the loaded cable. Each of the forms in (107) is useful and significant. However, if $W = 1/Z$ denotes the admittance of the system, and $H = 1/h$ the mid-section iterative admittance of the loaded cable, the equation for ρ can be written in the equivalent forms

$$\rho = \frac{W-H}{W+H} = \frac{(W-H)}{2H+(W-H)} = \frac{(W-H)/2H}{1+(W-H)/2H}, \quad (108)$$

and these forms, instead of those in (107), will be the ones mostly used herein, because of their simpler and more direct relations to the corresponding current deviations. For, if an electromotive force E is impressed between the terminals of the system in Fig. 13, the current I there will be WE; and if I^0 denotes the value that I would have if W were equal to H, then $I^0 = HE$. Thus the reflection coefficient ρ defined in terms of W and H by equation (108) can be expressed in terms of I and I^0 by the equation

$$\rho = \frac{I - I^0}{I + I^0} = \frac{(I - I^0)}{2I^0 + (I - I^0)} = \frac{(I - I^0)/2I^0}{1 + (I - I^0)/2I^0}. \qquad (109)$$

If the system contained no internal irregularities within the loaded cable itself and also no terminal irregularity at the far end, ρ would of course be zero. There are three types of irregularities here to be considered: section-irregularities, coil-irregularities, and the terminal-irregularity. Each of these types will be considered separately, with the ultimate object of constructing, by superposition, an approximate formula for ρ in terms of all of the existing irregularities.

First, consider the typical section-irregularity, situated in section No. r and consisting in the admittance-deviation [15] $y_r = Y_r - Y$ of the admittance Y_r of this section from its nominal value Y. The admittance-increment y_r may evidently be regarded as situated anywhere within the section. However, for the present purpose it is most conducive to simplicity of thought to regard y_r as situated just beyond the nominal mid-point of the section, namely the point which is at a distance of half a normal, or "regular," section from the initial end of the section; for then it is immediately evident that the admittance of the portion of the system beyond the nominal mid-point will deviate from the mid-section iterative admittance H by an amount approximately [16] equal to y_r, and hence that the corresponding reflection coefficient ζ_r pertaining to that mid-point will, in accordance with (108), be given (approximately) by the formula

$$\zeta_r = \frac{y_r}{2H + y_r} = \frac{y_r/2H}{1 + y_r/2H}. \qquad (110)$$

Due to the presence of the internal admittance-increment y_r in section No. r, the admittance W of the whole system (Fig. 13) at its initial end will deviate somewhat from the mid-section iterative admittance H; the admittance-deviation W-H will be denoted by y_r', and the corresponding reflection coefficient of the system will be denoted by ζ_r', so that, in accordance with (108),

$$\zeta_r' = \frac{y_r'}{2H + y_r'} = \frac{y_r'/2H}{1 + y_r'/2H}. \qquad (111)$$

[15] Here $r = 1, 2, \cdots n - 1$; for of course the nominal values of Y_0 and Y_n are each $Y/2$, and hence $y_0 = Y_0 - Y/2$ and $y_n = Y_n - Y/2$. With these qualifications duly observed, formula (110) is valid for $r = 0$ and $r = n$ as well as for $r = 1, 2, \cdots n - 1$. As seen below, y_0 is to be regarded as situated at the initial end of section No. 0, and y_n at the far end of section No. n.

[16] "Approximately," because y_r is distributed; "exactly," if y_r were localized.

Then it can rather easily be shown that ζ_r' is related to ζ_r in accordance with the simple but exact equation

$$\zeta_r' = \zeta_r e^{-2r\Gamma} = \zeta_r Q^{2r}, \tag{112}$$

where

$$Q = e^{-\Gamma} = e^{-A}e^{-iB}, \tag{113}$$

$\Gamma = A + iB$ denoting the propagation constant and Q the propagation factor of the loaded cable, each per periodic interval. It is sometimes convenient to call ζ_r' the "propagated value" of ζ_r, though it is to be observed that the apparent propagation constant of ζ_r is 2Γ not Γ. Alternatively, ζ_r' may be called the "apparent value" of ζ_r as viewed from the initial end of the system.

Second, consider the typical coil-irregularity, situated in coil No. r and consisting in the impedance-deviation $x_r = X_r - X$ of the impedance X_r of this coil from its nominal value X. The impedance-increment x_r will be regarded as situated just beyond the nominal mid-point of the coil; and the corresponding reflection coefficient ξ_r pertaining to that mid-point will, in accordance with (107), be given by the following formula, in which K denotes the mid-coil iterative impedance of the loaded cable:

$$\xi_r = -\frac{x_r}{2K + x_r} = -\frac{x_r/2K}{1 + x_r/2K}. \tag{114}$$

Since ξ_r is situated at a distance of $r - 1/2$ periodic intervals from the initial end, it appears at that end as a reflection coefficient ξ_r' such that

$$\xi_r' = \xi_r Q^{2r-1}. \tag{115}$$

Third, consider the terminal-irregularity situated at the junction of the loaded cable with the terminal-admittance T and consisting in the admittance-deviation $t = T\text{-}H$ of the admittance T from the mid-section iterative admittance H of the loaded cable. The corresponding reflection coefficient τ pertaining to that point will be given by the formula

$$\tau = \frac{t}{2H + t} = \frac{t/2H}{1 + t/2H}. \tag{116}$$

This will appear at the initial end as a reflection coefficient τ' given by the formula

$$\tau' = \tau Q^{2n}. \tag{117}$$

Finally let all of the loading-section admittances differ from their nominal values, all of the loading-coil impedances from their nominal values, and the terminal-admittance T from the mid-section iterative admittance H. Then, when these deviations are not too large, the resulting reflection coefficient ρ at the initial end of the system will be approximately equal to the sum of the "propagated" or "apparent" values of the reflection coefficients arising from all of the individual irregularities, that is,

$$\rho = \sum_{r=0}^{n} \zeta_r' + \sum_{r=1}^{n} \xi_r' + \tau', \tag{118}$$

whence, by substitution of (112), (115), (117),

$$\rho = \sum_{r=0}^{n} \zeta_r Q^{2r} + \sum_{r=1}^{n} \xi_r Q^{2r-1} + \tau Q^{2n}. \tag{119}$$

Since ζ_r, ξ_r, τ are chance-variables, ρ is a complex chance-variable. In accordance with Section 2 (in Part I) the leading distribution-parameters of ρ are completely determined by $\overline{\rho}$, $\overline{\rho^2}$, $\overline{|\rho|^2}$; and these will completely determine the distribution of ρ if it is "normal." In the present problem, owing to the presence of τ in equation (119), $\overline{\rho}$ is not to be taken as zero; for, in accordance with the second half of the first paragraph of this Section, $\overline{\tau}$ would usually not be zero in practice. However, $\overline{\zeta_r}$ and $\overline{\xi_r}$ would usually be zero and will here be so taken. Hence, from (119),

$$\overline{\rho} = \overline{\tau} Q^{2n}. \tag{120}$$

Since the chance-variables ζ_r, ξ_r, τ are independent, and since only one of them, namely τ, has a non-zero mean value, Theorems 4 and 5 of Subsection 4.3 (in Part I) are applicable to (119). Assuming all of the loading-section deviations to be statistically alike, so that [17]

$$\overline{\zeta_r^2} = \overline{\zeta^2}, \qquad \overline{|\zeta_r|^2} = \overline{|\zeta|^2}, \qquad (r = 0, 1, 2, \cdots n), \tag{121}$$

and all of the loading-coil deviations to be statistically alike, so that

$$\overline{\xi_r^2} = \overline{\xi^2}, \qquad \overline{|\xi_r|^2} = \overline{|\xi|^2}, \qquad (r = 1, 2, \cdots n), \tag{122}$$

application of Theorems 4 and 5 to (119), followed by the execution of the indicated summations, gives the formulas

$$\overline{\rho^2} = \overline{\zeta^2}\frac{1 - Q^{4(n+1)}}{1 - Q^4} + \overline{\xi^2}\frac{1 - Q^{4n}}{1 - Q^4} Q^2 + \overline{\tau^2} Q^{4n}, \tag{123}$$

$$\overline{|\rho|^2} = \overline{|\zeta|^2}\frac{1 - q^{4(n+1)}}{1 - q^4} + \overline{|\xi|^2}\frac{1 - q^{4n}}{1 - q^4} q^2 + \overline{|\tau|^2} q^{4n}, \tag{124}$$

where q denotes the attenuation factor of the loaded cable per peri-

[17] The assumption represented by (121) is an approximation to the extent that, statistically, ζ_0 and ζ_n would usually differ somewhat from ζ_j, where $j = 1, 2, \cdots n - 1$.

odic interval, that is,

$$q = |Q| = e^{-A}, \tag{125}$$

A denoting the attenuation constant of the loaded cable per periodic interval, in accordance with equation (113).

When q^{4n} is small compared to unity, formulas (123) and (124) reduce approximately to

$$\overline{\rho^2} = \frac{\overline{\zeta^2} + \overline{\xi^2}Q^2}{1 - Q^4} + \overline{\tau^2}Q^{4n}, \tag{126}$$

$$\overline{|\rho|^2} = \frac{\overline{|\zeta|^2} + \overline{|\xi|^2}q^2}{1 - q^4} + \overline{|\tau|^2}q^{4n}. \tag{127}$$

When, further, q is nearly equal to unity, which by (125) will be the case when $2A$ is small compared to unity, then formula (127) reduces approximately to

$$\overline{|\rho|^2} = \frac{\overline{|\zeta|^2} + \overline{|\xi|^2}q^2}{4A} + \overline{|\tau|^2}q^{4n}. \tag{128}$$

Returning to the formulas (110) and (114), which give ζ_r and ξ_r in terms of $y_r/2H$ and $x_r/2K$ respectively, it may be said that for practical applications it is more convenient to express ζ_r and ξ_r in terms of the fractional deviations δ_r and ϵ_r and the coefficients D and G, defined by the following four equations:

$$\delta_r = y_r/Y, \quad (129) \qquad \epsilon_r = x_r/X, \quad (130)$$

$$D = Y/2H, \quad (131) \qquad G = X/2K. \quad (132)$$

With these substitutions, formulas (110) and (114) become

$$\zeta_r = \frac{D\delta_r}{1 + D\delta_r}, \quad (133) \qquad \xi_r = -\frac{G\epsilon_r}{1 + G\epsilon_r}. \quad (134)$$

It can be shown that D and G, defined by equations (131) and (132), are approximately equal and may be expressed approximately in each of the forms appearing in the equation

$$D = G = \sqrt{\frac{XY/4}{1 + XY/4}} = \sqrt{1 - 1/HK} = \tanh(\Gamma/2), \tag{135}$$

with H, K, Γ already defined in connection with equations (108), (114), (113) respectively. Equation (135) would be exact if the cable wires were perfectly conducting, since then each section-admittance Y could be regarded as effectively localized, so that the loaded cable would be effectively a ladder-type structure, for which equation (135) is known to be rigorously exact.

An Oscillograph for Ten Thousand Cycles

By A. M. CURTIS

Efforts to extend the frequency range of oscillographs have, for the most part, been directed toward increasing the natural frequency of the vibrating element, which has formed the upper limit of the useful range. This paper describes a new method of attack which consists in employing a vibrator strung to only a moderately high natural frequency, and in equalizing the response of the string by electrical circuits both up to and beyond the fundamental resonance frequency. Employing this method of equalization, a galvanometer element has been developed for the rapid record oscillograph which uses strings stretched to a natural frequency of 4500 c.p.s., and equalized to from ten to twelve thousand cycles. The paper concludes with a description of such a modified rapid record oscillograph and with an oscillogram illustrating its use.

In the past, oscillographs have been employed over a frequency range extending only to a little below the natural frequency of the vibrating element, and efforts to obtain a wider range have been directed toward raising the resonant frequency of the vibrator. In the present paper there is described a new method of attack that obviates many of the difficulties and restrictions previously encountered. In brief it consists in equalizing the natural characteristics of the string by electrical networks inserted in the circuit. One part of the network equalizes for the fundamental resonance F_0, and another equalizes the range above this frequency. Other factors enter to limit the upper frequency obtainable, but practically flat characteristics are secured up to about two and one-half times the fundamental frequency of the vibrator.

The new oscillograph arose from efforts to extend the frequency range of the rapid record oscillograph (Fig. 1) already described.[1] This instrument was of the string type, and before electrical compensation could be applied, a complete study of the string characteristics of the galvanometer was necessary.

If a measurement is made of the deflection of the string by an alternating current of constant value but variable frequency, it is found that the sensitivity increases enormously in the region of its fundamental resonance frequency (F_0) and that there are subsidiary resonance peaks occurring at approximately $3F_0$, $5F_0$, $7F_0$, and so on. No signs of resonance appear at even multiples of the fundamental fre-

[1] *Electronics*, August 1931, p. 70; *Jour. S.M.P.E.*, January, 1932, p. 39; and *Bell Laboratories Record*, August, 1930, p. 580.

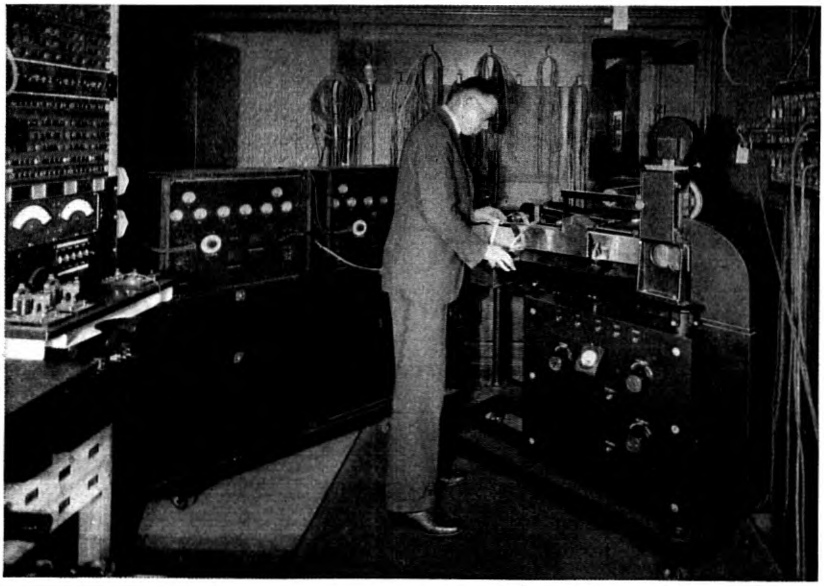


Fig. 1—The rapid record oscillograph.

quency. The odd numbered modes of vibration may not be exact multiples of the fundamental because their frequency is influenced by the beam stiffness of the string. With a relatively short, wide ribbon, for example, the third resonance peak may be considerably higher than $3F_0$. The increase in sensitivity in the neighborhood of the various resonance points is accompanied, as with other electrically

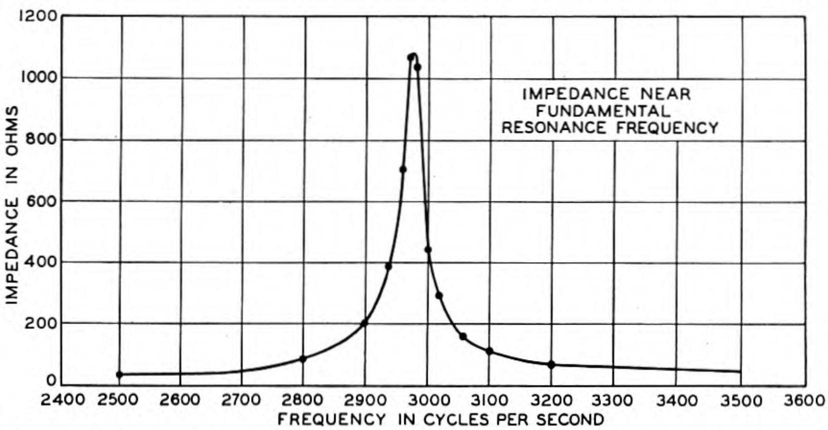


Fig. 2—Variation in impedance near fundamental resonance frequency with image amplitude constant at 2 mm., peak to peak.

driven vibrating systems, by marked variations in the electrical characteristics that the system presents. Measurements of impedance, resistance, and reactance of a rapid record oscillograph galvanometer tuned to 2970 cycles are shown in Figs. 2, 3, and 4, respectively.

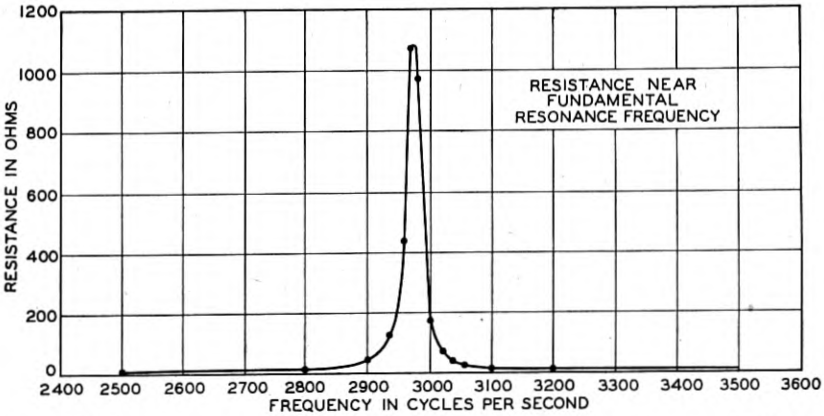


Fig. 3—Variation in resistance near fundamental resonance frequency with image amplitude constant at 2 mm., peak to peak.

If approximate equalization is desired only to a frequency a little below F_0, and if maximum sensitivity is not essential, it is sufficient to shunt the galvanometer with a suitable resistance. Four ohms is about the right value for the instrument under discussion, and gives a deflection vs. frequency curve as shown in Fig. 5. There is a decided peak of sensitivity at $3F_0$ with the result that when a current with a

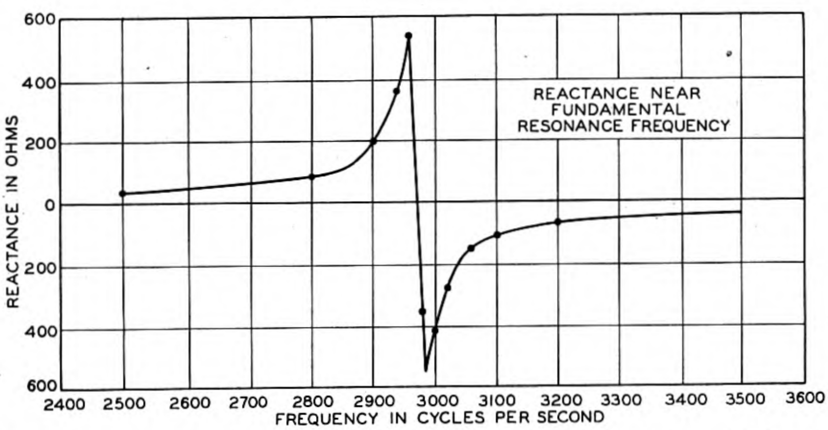


Fig. 4—Variation in reactance near fundamental resonance frequency with image amplitude constant at 2 mm., peak to peak.

square wave front is applied, a weak damped oscillation containing about one cycle of F_0 and many cycles of $3F_0$ will be superposed on the square wave record, as has already been reported by Professor H. B. Williams.[2] The effect of the third partial oscillation is usually of

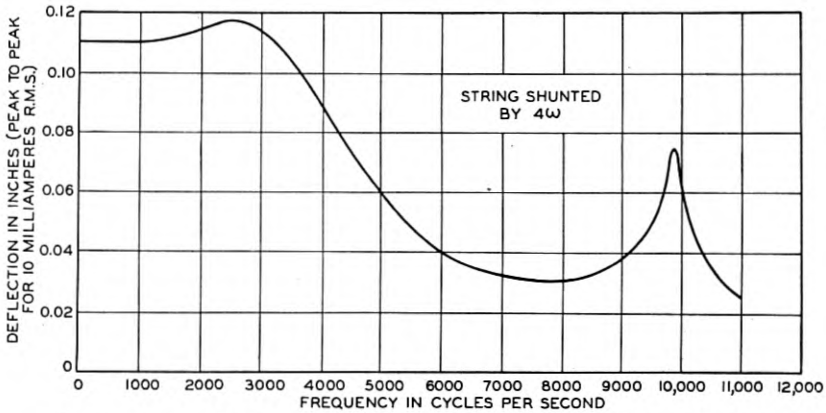


Fig. 5—Characteristics of instrument shunted with a resistance of 4 ohms.

minor importance. Its amplitude is less than the width of the string image, and its effect is noticeable principally as a slight blurring of the trace.

This method of resistance-shunt damping, used with the earlier form of the rapid record oscillograph, gives very satisfactory characteristics up to nearly F_0, but it does not develop maximum sensitivity, which for its attainment requires an equalizing network with characteristics inverse to those of the vibrating string, as described by J. T. Irwin.[3] An inductance in series with a capacitance, which resonates it to F_0, and a suitable resistance are sufficient The characteristics of a string shunted with such an equalizing element, in which for convenience the capacitance was made considerably less than the optimum value, is shown in curve A of Fig. 6. It will be noticed that the deflection for a 10 ma. current has been increased from 0.11 inch, obtained with resistance damping above, to about 0.36 inch—a sensitivity better than three times as great

This type of equalization alone, however, gives a sensitivity at $3F_0$ nearly as great as that at F_0. Because of this there is a greater $3F_0$ distortion with a resonant shunt when a square wave is impressed than with the resistance shunt. The sensitivity at $3F_0$, however, may be damped out by an additional shunt element, and when this is employed the characteristics are as shown by curve B of Fig. 6.

[2] *Jour. Optical Soc.*, September, 1926.
[3] U. S. Patent No. 1,324,054.

With this arrangement the sensitivity falls off rapidly beyond F_0, but recent advances in the art of designing equalizing networks have made it possible to combine with the string already equalized to its

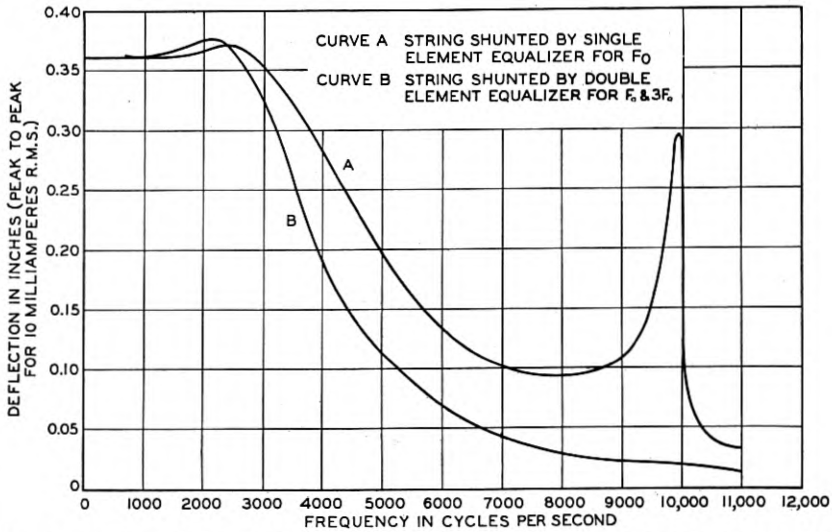


Fig. 6—Characteristics of galvanometer with resonant shunt, alone for curve A, and with an additional shunt to suppress the resonance at $3F_0$, for curve B.

natural frequency of vibration, a second equalizer, designed by E. L. Norton, which extends the range of frequencies through which the deflection is proportional to the current to a point considerably higher than F_0. This is, of course, accomplished at the expense of a corresponding reduction in sensitivity. While a variety of combinations of F_0 and equalizers is possible, a particular case in which a string was tuned to 4500 c.p.s and equalized to 10,000 is illustrated in Fig. 7, which shows the circuit of the equalizer and the characteristic obtained.

As has already been noted, former practice has required an increase of the natural frequency of the vibrator, and the employment of the galvanometer only up to this frequency. Such an increase in natural frequency may be obtained by increasing the tension of the string, by decreasing its mass, by shortening its free length, or by a combination of one or more of these modifications. There are rather severe restrictions to this method, however. Both the diameter to which the wire may be drawn and the stress that may be applied are limited. The string employed for both the earlier and present oscillographs, a duralumin wire 0.0008 inches in diameter, approaches the best available combination of mass and strength, and for the length employed, 6000 cycles is about at the upper limit of fundamental resonance obtainable.

It is possible, of course, to shorten the string and to employ a shorter pole face. Halving the string length might be expected to double the natural frequency, although it would reduce the sensitivity to one quarter its former value. The linearity between deflection and current, however, holds only so long as the deflection is small enough not

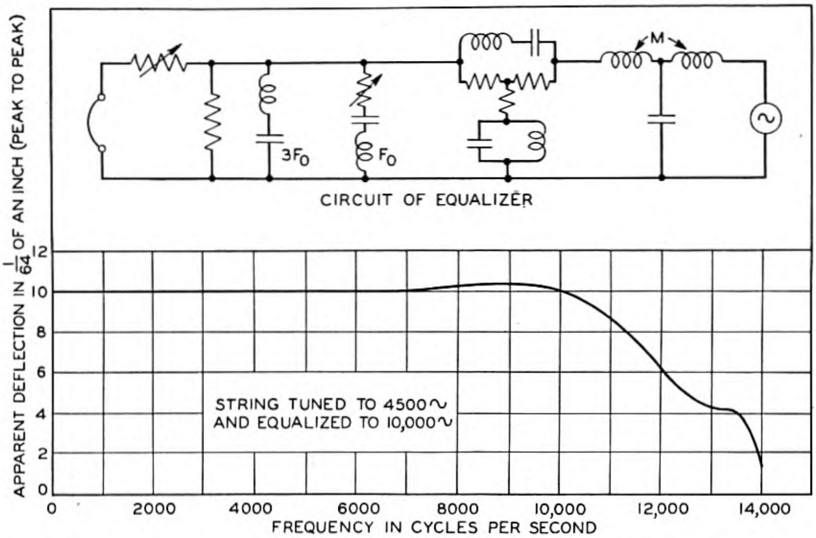


Fig. 7—Equalizing network employed with new oscillograph and the characteristic obtained.

to increase the tension appreciably, so that the shorter the string the less is the permissible deflection. To compensate for this and return to the original size of oscillogram requires an increase in optical magnification, which in turn reduces the transmitted light and thus the speed at which the paper can be exposed. It also increases the width of the string image, which thus becomes a larger proportion of the total deflection, but there is a compensation in that the sensitivity is somewhat increased. Such a design, although capable of responding to a higher frequency, and having a sensitivity greater than that which would be obtained from the shortened string without the additional optical magnification, is less capable of making a photographic record. Actually, with a given string material, light source, and magnetic field strength, there is a definite length of string that will give the widest frequency range both electrically and photographically. It turns out that by using the longer string and the methods of equalization already discussed, the overall sensitivity is about the same as that for the shortened string, and that the optical disadvantages are avoided.

An investigation of how far beyond F_0 the new method of equalization could be employed disclosed certain limitations. In general an increase in frequency range, either by shortening the string or by electrical equalization, reduces the sensitivity, with the result that more current must be passed through the string to secure the desired deflection. Since the heating of the string increases with the square of the current a limit of improvement is ultimately reached. With electrical equalization this limit has been found to be in the neighborhood of $2.5F_0$. A peculiar action of the string in the neighborhood of $3F_0$, described below, would also place an obstacle in the way of equalizing the galvanometer much beyond $2.5F_0$, were the limit not already set by the heating.

When a current remote from $3F_0$ is applied to a string, the deflection is found to be practically proportional to current for all values within the normal range. When a frequency near $3F_0$ is applied, however, the deflection is linear for very small deflections, but at a certain critical value becomes non-linear—increasing very rapidly to from two to three times its previous value. Beyond this point the deflection again becomes linear with current. As the current is decreased, the deflection decreases linearly to approximately the critical value and then decreases abruptly. It does not follow the curve of increasing current, however, but actually forms a hysteresis loop. The phenomenon is shown in Fig. 8. With a frequency of 2000 cycles the deflection

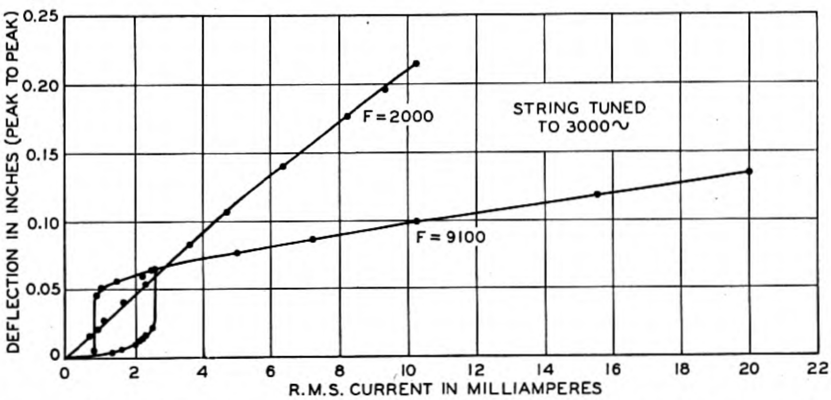


Fig. 8—Deflection-current characteristics for strings tuned to 3000 cycles for frequencies near and remote from $3F_0$.

is practically linear with current for all values, but for a frequency of 9100 cycles, approximately $3F_0$, the hysteresis loop occurs. This discontinuity is greatest at $3F_0$ but is detectable at frequencies several hundred cycles above or below that value. The change from low to high amplitude or vice versa, although apparently instantaneous, actually lasts about a hundredth of a second. Although no satisfactory explanation has been reached, this phenomenon may be associated with the method of supporting and stretching the string. Its study and elimination would become important, however, only should some means become available of permitting the string to carry several times the present maximum current without overheating—as might be possible if an alloy should be developed with the mechanical properties of duralumin and the conductivity of copper.

The amount of phase distortion present with these various methods of damping and equalizing is difficult to measure directly for frequencies much above F_0. A measurement up to about 4000 cycles was obtained with a two-string galvanometer arranged for somewhat shorter strings than those usually employed with an F_0 of 3200 cycles. One of the strings was stretched to an F_0 of 6000 cycles and left undamped except by air friction. Its phase distortion was computed and is plotted as the lower curve of Fig. 9. The other string was stretched

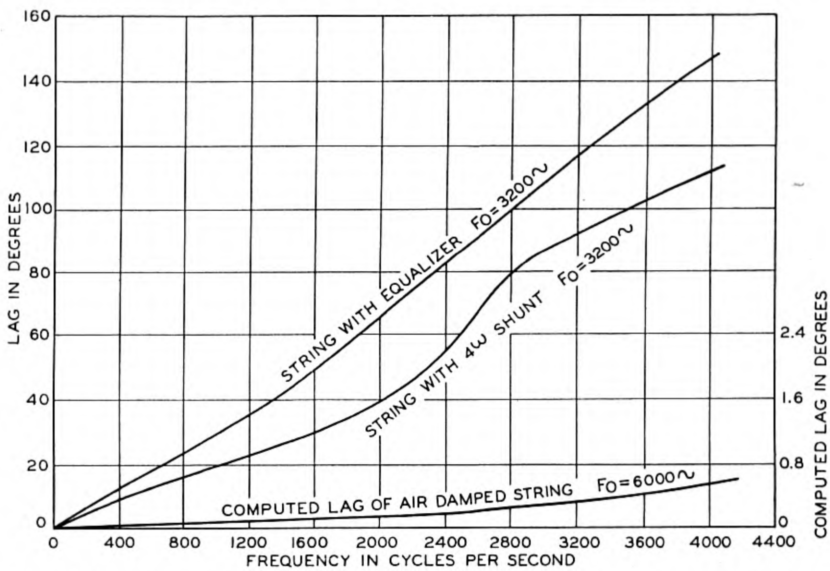


Fig. 9—Phase distortion with equalized and resistance damped strings up to about F_0.

to an F_0 of 3200 cycles, and equalized for the fundamental and third harmonic as already described. Both strings were fed from the same oscillator, and a series of oscillograms taken from 50 to 4000 cycles. The phase shift between the strings was then measured and is plotted as the upper curve of Fig. 9. As may be seen here, it was found to be nearly linear—with a maximum deviation of about 10°. When the experiment was repeated with a string that was resistance damped, considerable phase distortion was found as shown by the middle curve of the plot.

This method of measurement cannot be used for much higher frequencies because of the difficulty of stretching a string to appreciably more than 6000 cycles. The amount of the phase distortion in an instrument equalized to higher frequencies may be judged, however, by taking oscillograms of square-front flat-topped waves and noting the irregularities produced. The phase correction required was determined by making such oscillograms with a resistance-capacity phase corrector in the circuit, and adjusting the phase corrector to bring about a minimum amount of distortion. An electrical equivalent of this experimental network, giving the same phase correction but with negligible attenuation, is included as part of the equalizing circuit of the new oscillograph. Although it is realized that the resulting phase characteristics are not perfect up to 10,000 cycles, the oscillogram of Fig. 10 shows that there is no great amount of phase distortion present. It has been found that the amount of distortion indicated here is not usually of practical importance.

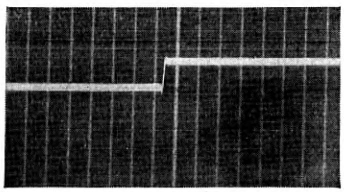


Fig. 10—Oscillogram of square front, flat top wave from an instrument tuned to 4500 cycles and equalized to 10,000. Abscissa divisions are .001 second.

A few years ago a recording oscillograph, of the string type, was developed by Bell Telephone Laboratories, which would satisfactorily record frequencies over the part of the voice range important in telephone work. It represented a distinct advance over the oscillograph of similar type developed during the war for locating enemy guns by sound ranging and improved subsequently for studying circuit phenomena. This earlier oscillograph [4] would record frequencies up to 200 cycles per second and had facilities for developing and fixing the paper record at the rate at which it was exposed, while the improved oscillograph increased the frequency range to 3000 cycles. Although

[4] *Bell Laboratories Record*, March 1927, p. 225.

it also provided for developing the paper immediately after exposure, the rate of development had to be slower than that of exposure because of the very high speed of the paper necessitated by the higher frequencies recorded.

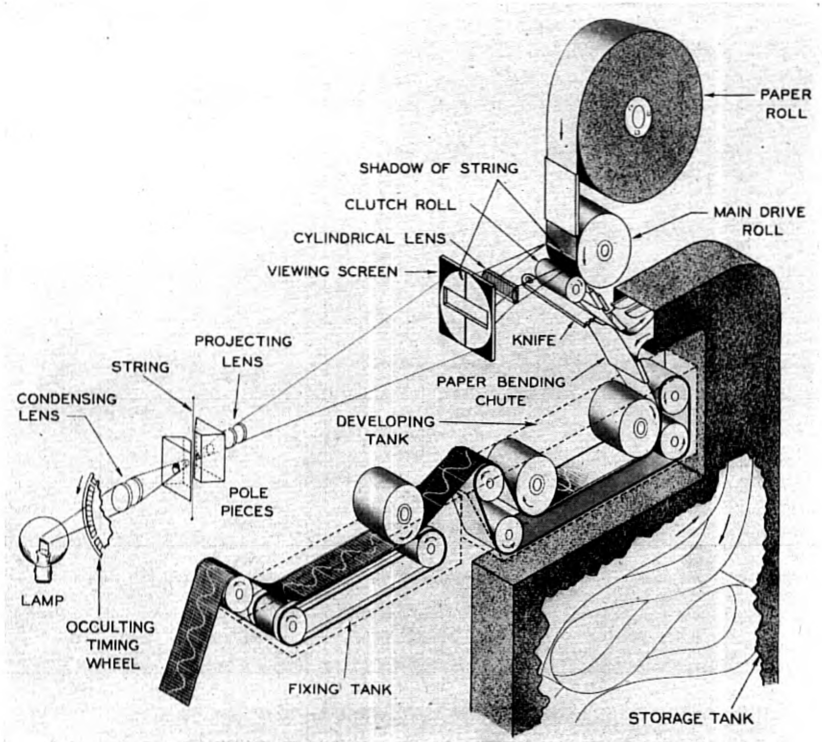


Fig. 11—Diagrammatic arrangement of the new rapid record oscillograph.

This improved instrument, christened the rapid record oscillograph, proved greatly superior to other available equipment and has been used extensively in the varied work of Bell Telephone Laboratories. Recently the galvanometer of this oscillograph has been redesigned, employing the electrical methods of equalization already discussed, and in its present form has a frequency range extending up to ten or twelve thousand cycles per second. With this new equipment most of the components of speech and music may be recorded.

Its arrangement is shown in the schematic photograph of Fig. 11. Light from the lamp at the left is focussed by the condensing lens on the strings of the instrument through the perforated pole piece. Only one of the two or three strings provided is shown on the diagram. The images of the strings are focussed by the projecting lens onto the sensitized paper used for the record, where they appear as shadows on a light background. An achromatic cylindrical lens in front of the paper further focusses the light into a narrow band with a width of a

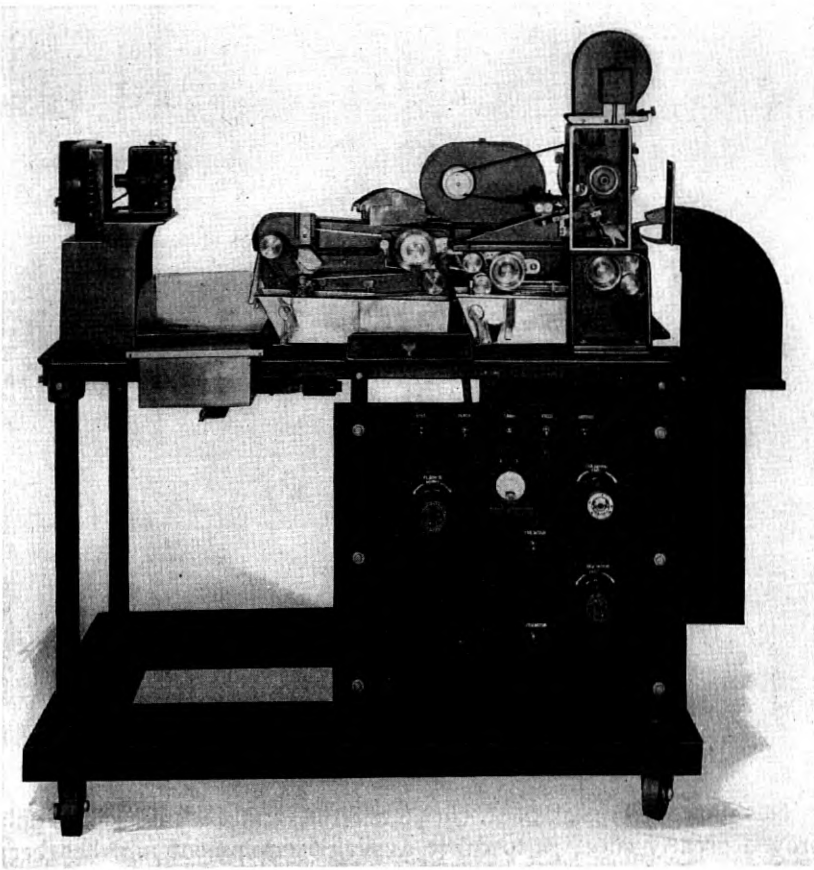


Fig. 12—Rapid record oscillograph. Front view with mechanism exposed. The developing and fixing tanks may be dropped in a few seconds when required.

few thousandths of an inch on which the shadows of the strings fall. As the paper is drawn through the machine, the shadows of the vibrating strings thus photograph on it a trace of the motion of the middle of the string.

Between the lamp and the condensing lens is a timing wheel whose rotation is controlled by an electrically driven tuning fork. Spokes of the wheel interrupt the light from the lamp every thousandth of a second and thus trace timing lines across the sensitized paper. Every tenth spoke is thicker than the intermediate ones to indicate with a heavier line the hundredths second divisions. Rulings on the cylindrical lens mark horizontal lines a twentieth of an inch apart on the exposed paper to give a convenient measure of the amplitudes of the oscillations.

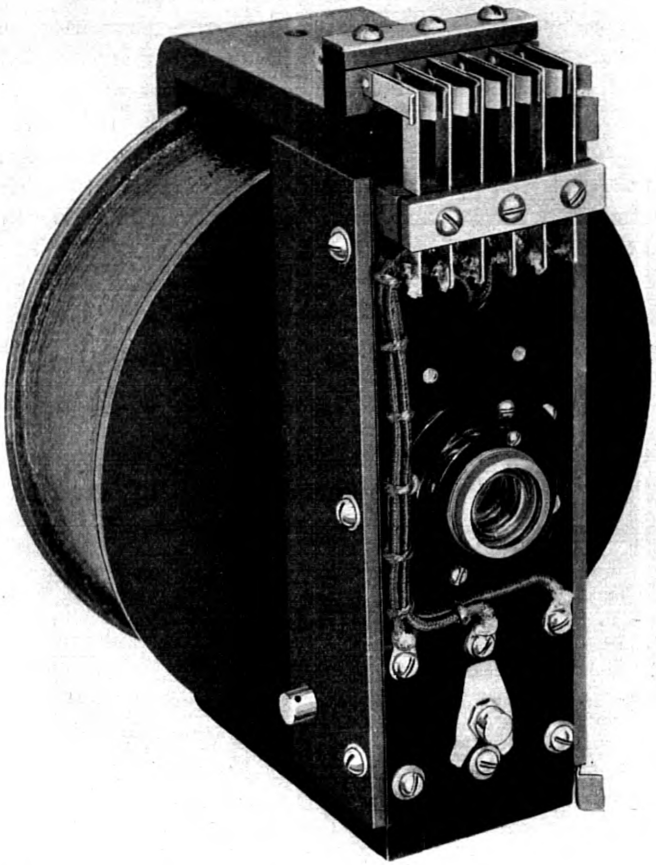


Fig. 13—Galvanometer element of rapid record oscillograph.

Two motors operate the exposing, and the developing and fixing mechanisms. One rotates the main drive roll, which pulls the strip of paper from the unexposed roll through the light beam, and pushes it into the developing tank. The second carries the paper through the developing and fixing tanks. Each is adjustable in speed and controlled separately. The speed of the main drive motor is adjusted to best exhibit the phenomena that are being observed. Maximum speed is about 130 inches per second, which gives a little over a hundredth of an inch between crests of a 12,000 cycle wave. The motor controlling the developing equipment is adjustable to give paper speeds from two to ten inches a second. The faster the speed at which the paper is exposed the more slowly will it be developed.

Since the paper being exposed is moving faster than that being developed, a storage reservoir for undeveloped paper is provided as indicated in the illustration. At the beginning of an oscillogram the paper is pushed by the main drive roll in between the drive rolls of the developing tank. Since these carry the paper at a lower speed than the main drive, a loop of paper is formed between the two drive rolls which passes into the storage tank. The amount of paper that can be stored depends on the speed of exposure, and varies inversely with it. At low speed the paper settles compactly in the tank and an entire 250 foot roll may be stored. At high speeds the paper does not have time to settle properly, and only about fifty-five feet, corresponding to about five seconds exposure, can be held.

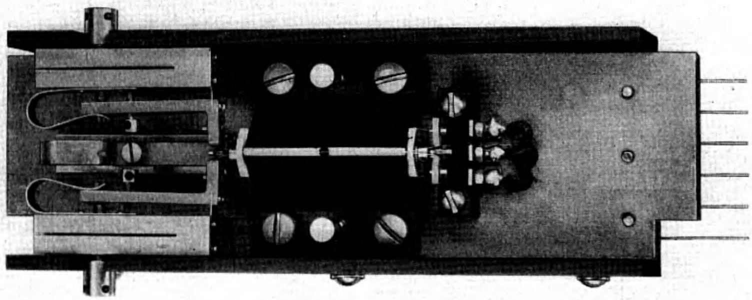


Fig. 14—Rapid record oscillograph. Front pole face of galvanometer. Terminals for the three-string elements are brought to knife contacts, which allows the string mounting and pole piece to be readily removed from the galvanometer.

Both motors having been started, operation of the oscillograph is commenced by pulling out a lever which withdraws a knife blade from the paper, and moves an idler pulley, which presses the paper against the main drive roll. The paper is then run through the storage tank, and the developing and fixing tanks as already described. After the

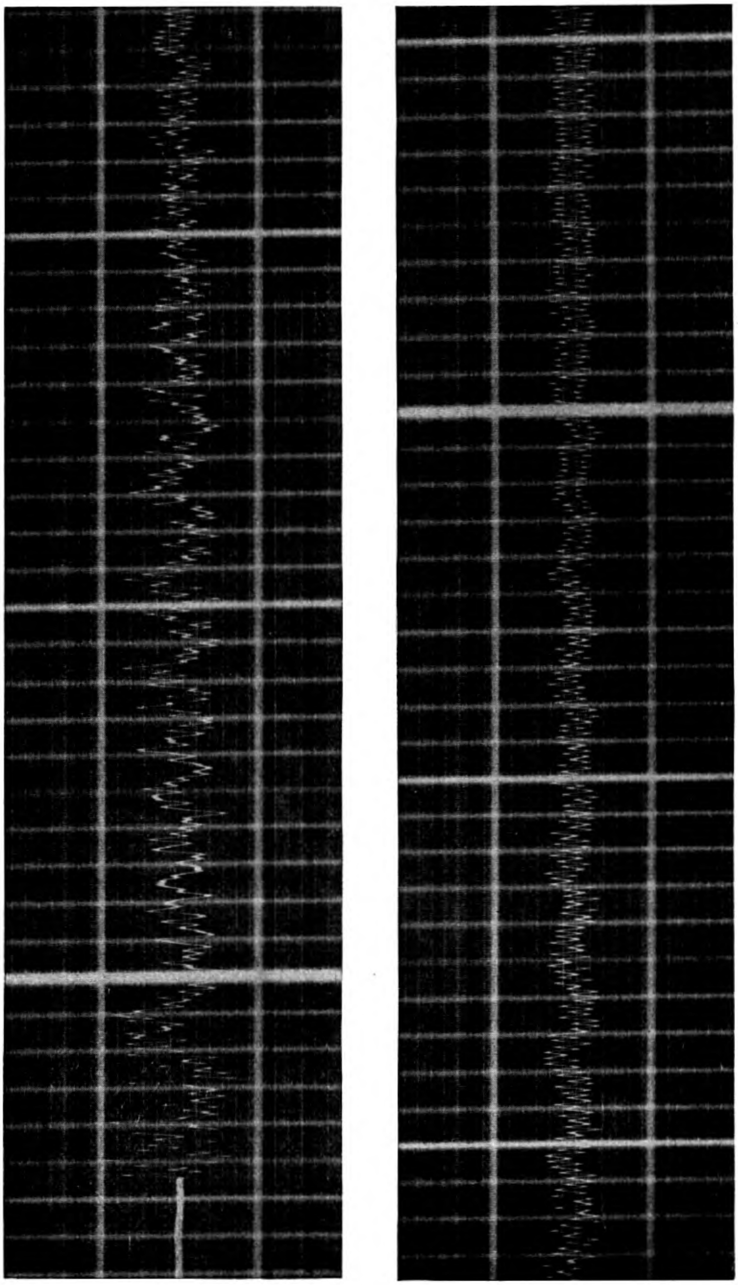


Fig. 15—Sound radiated by a gong struck once, recorded on rapid record oscillograph equalized to 7500 c.p.s. The 6000 c.p.s. tone continues for several seconds more. Abscissa divisions are .001 second.

events under observation have been recorded the starting lever is pushed back, withdrawing the idler pulley from the main drive roll, and releasing the knife, which cuts off the exposed section of paper. An electromagnetic brake, operated by a timing control circuit, is momentarily applied to the spinning roll of paper, and stops it in a fraction of a revolution. The exposed paper continues to pass through the developing and fixing tanks, and into the rinsing tank until it has all been developed. A view of the machine with the developing and fixing tanks dropped for inspection of the mechanism is shown in Fig. 12. A solenoid may be provided for operating the machine from a distance when desired.

The complete galvanometer element of the three-string model is shown in Fig. 13, and one pole piece and the string mounting, in Fig. 14.

As an illustration of the many uses of the new oscillograph, an oscillogram is given in Fig. 15, which shows the wave form of the sound radiated from the gong of a telephone ringer struck once by the clapper. The sound was picked up by a dynamic type microphone, and the resulting current was amplified and fed to a rapid record oscillograph tuned to 4000 cycles and equalized to 7500. The entire system was reasonably distortionless from 30 to 8000 cycles per second. It is interesting to note that the predominant frequency, about 6000 cycles per second, would not have been detected had the record been made with the older types of oscillographs.

Contemporary Advances in Physics, XXV

High-Frequency Phenomena in Gases, Second Part

By KARL K. DARROW

This article on high-frequency phenomena in gases, a continuation of the one which appeared in the preceding number of this Journal, is concerned with the self-sustaining high-frequency discharges. First come the conditions for establishment of the discharge, a spark or corona if the gas-pressure is high, a glow if it is low; then, the laws of the glow-discharge when established in rarefied gas, in tubes with internal or external electrodes. The complexity of the situation is such that fundamental theory is almost powerless as yet, the article thus consisting chiefly of descriptions of data and statements of empirical laws.

THE article preceding this one was devoted principally to the things which are observed when a high-frequency electric field, generally small in amplitude, is impressed upon a gas which by some other agency is populated with electrons. The gas may be, for instance, the vehicle of a self-sustaining direct-current glow-discharge, carrying a steady current-flow between two electrodes maintained at a constant potential-difference. It will then be rich with free electrons, and also with positive ions. To a part of this host of mobile charged particles circulating among neutral atoms, the high-frequency field is applied by means of a second pair of electrodes. Or, the gas may be flooded with free electrons supplied from a heated filament, and the high-frequency force will act upon these. In all these cases of Part I, the motions which the high-frequency field imposes on the corpuscles are held accountable for the phenomena. Predictions may then be made, out of our knowledge of the behavior of free electrons wandering through gases under constant fields; and on the whole, the observations agree with the predictions to an extent decidedly satisfactory, though enough remains unexplained to encourage further study.

Those phenomena of Part I are thus the high-frequency analogues of what happens, when a weak constant electric field is applied across a gas which is ionized or flooded with free electrons by some external agent: X-rays or beta-rays or the electrons from a hot filament, for example, or a stronger field simultaneously applied in a different direction and maintaining a glow-discharge. Now if such a feeble field be gradually increased in strength, these electrons themselves take up the rôle of ionizing agents; the ionization due to the external agent is "self-amplified," as I have elsewhere said. When the field is further strengthened, the amplification becomes more intense, the ionization more abundant, and there comes a point when the gas "breaks down." A luminous discharge occurs, which may be transitory (a spark) or durable (a glow or corona or arc). Breakdown and the subsequent discharge occur even when there is no external agent of ionization, apart from those feeble rays which constantly pervade the atmosphere and every gas not shut off from the atmosphere by heavy walls. Moreover, they occur with a high-frequency field, provided its amplitude is raised to a sufficient value. This second part of the present article is devoted to the conditions for breakdown by high-frequency fields, and the characteristics of the discharge which sets in thereafter.[12]

The discharge ensuing upon breakdown is as a rule enduring only if the gas is rarefied (to a pressure not more than a few hundredths as great as atmospheric) or one at least of the electrodes is sharply curved. Otherwise, it is a spark. Striking as is the contrast between these cases, one does well to disregard it while thinking about the processes which may lead up to breakdown, or observing the conditions under which this phenomenon occurs. What happens before the sudden transition may be controlled by laws quite other than what happens after it. Indeed, we know that the choice between spark and durable glow-discharge is not so important in principle. The choice between spark and glow is influenced, for instance, by the constants of the circuit—not merely by the E.M.F. available, but also by the resistance and inductance in series with the gas. It is advantageous, therefore, to think of the conditions for breakdown and the presumptive details of the process as forming a problem by themselves, apart from the problems of the state which follows.

General Remarks on Breakdown

Breakdown by "steady voltage" is brought about in either of two ways: by gradually increasing the voltage across a pair of electrodes separated by a stratum of gas, or by applying a fixed voltage and gradually changing the distance between the electrodes. It is detected either by the blaze of light attending the ensuing spark or glow, or by a sudden violent change in the reading of a voltage-measuring device shunted across the "gap," that is, connected across the electrodes. The figure given as the "breakdown-potential" is the last value of voltage recorded just before either of these events.

[12] The order of treatment is thus the same as is customary in treatises on direct-current phenomena, and as I have followed in my book "Electrical Phenomena in Gases," to which again reference is occasionally made: first the drifting and accelerations of electrons in gases exposed to weak fields, then the conditions for breakdown, finally the laws of the luminous discharges ensuing after breakdown. Equations, footnotes, and figures are numbered consecutively to those of Part I.

If the voltage between the electrodes is augmented rapidly instead of slowly, the breakdown-potential may be greater; it is as though the discharge were delayed for an appreciable time after the proper critical P.D. was reached, during which time the voltage is overshooting the mark and giving rise to error. I mention this because it has bearing on what follows.

If the voltage is supplied from a "source of high frequency" of one of the types customary before the development of the vacuum-tube oscillator—for instance, an induction-coil or an interrupter—it arrives as a sequence of highly-damped high-frequency wavetrains with longish intervals between. At the end of each interval, the P.D. between the plates rises suddenly and rapidly, and if it rises far enough, breakdown takes place. The difference between the rise which (were it not interrupted by breakdown) would end in the attainment of a thenceforward constant voltage, and the rise which (were it not interrupted by breakdown) would be followed by successive falls and smaller rises and alternations of direction, is practically small. True, breakdown might occur, in the latter case, during the second rise when it had missed the first; or after the completion of one damped wave-train, the gas might be left in an abnormal state lasting until the coming of the next and facilitating breakdown by the next. But this does not seem to happen in practice, and if it did, there would be obvious advantages in studying it with trains of undamped waves such as nowadays can be produced. For successions of damped wave-trains, therefore, I will merely quote the general result applicable to air at atmospheric pressure: the voltage producing sparkover, between definite electrodes at a definite distance, is almost if not quite independent of frequency up to such high values as a million cycles—what changes have been observed are generally increases and may be ascribed to the fact just mentioned, that when the voltage is increasing very rapidly it may overshoot the minimum value sufficient for sparkover before the spark gets started.

Turning now to sinusoidal wavetrains such as modern technique makes available: if such a one be applied while its amplitude is yet too small to cause a breakdown, and then the amplitude is gradually increased (or alternatively, the distance between the electrodes is diminished) it will gradually modify the gas by reproducing ionization in ever-increasing amount—the "self-amplifying" effect of the ionization, which I mentioned above; and this will eventually bring about breakdown. We know a great deal about this preliminary process for steady voltages, but as yet we can only infer it for alternating voltages. Thus for steady voltages, there must be at least two modes of ionization: the well-known action of free electrons striking molecules of the gas, and a complementary process, which may (for instance) be the ejection of fresh electrons from the cathode by positive ions striking that electrode.[13] Were it not for the latter process (or some other) the direct-current discharge could never develop; for though at a given instant there might be some electrons in the gas, they and all the other electrons which they might liberate would steadily drift off toward the anode, and ionization and current-flow would cease after all of them had reached it. Now if the voltage is oscillating instead of constant, the electrons in the gas may rush to and fro and ionize all parts of it, and the importance of the complementary process will be reduced; though it can never be annulled, since the electrons will sooner or later get to the anode or the walls, and must be replenished from the cathode.

Again, we know that a factor in the advent of breakdown by a steady voltage is the distortion of the field in the gas by space-charge, which arises chiefly near the cathode, because the positive ions formed there by electron-impact drift only slowly toward the cathode while the electrons which should balance their charge drift rapidly off toward the anode. If the voltage is oscillating there will also be a positive space-charge due, in the last analysis, to the fact that electrons drift faster than positive ions; but it will be distributed symmetrically about the middle of the gap.

These remarks may have given the impression that the differences between breakdown at high frequencies and breakdown by steady voltage have been successfully explained. As a matter of fact, there is no quantitative explanation, and I have little to say except to present the data.

Breakdown of Air at Atmospheric Pressure

For air at atmospheric pressure, for which breakdown is sparkover unless one or both of the electrodes be sharply curved, the latest data are those of Lassen.

Curves of sparking-potential versus distance, (V_s-vs-d curves), obtained with spherical electrodes of 2.5-cm. diameter, over the range of distances from 0.05 to 0.5 cm., appear in Fig. 12. The voltage was adjusted to a chosen value, and the distance gradually lessened until sparkover occurred. The straight line is fitted to the points obtained with frequency $1.1 \cdot 10^5$ and the points obtained with frequency 50. Fifty-cycle A.C. is practically the same, with regard to the processes

[13] "Electrical Phenomena in Gases," pp. 280–297.

leading up to breakdown, as steady voltage; these data therefore indicate that up at least to frequencies of the order of a hundred thousand, an oscillating voltage causes breakdown when its amplitude becomes the same as the steady voltage which can have the like effect; and this is in agreement with other observations.

The curves which depart from the straight line correspond to

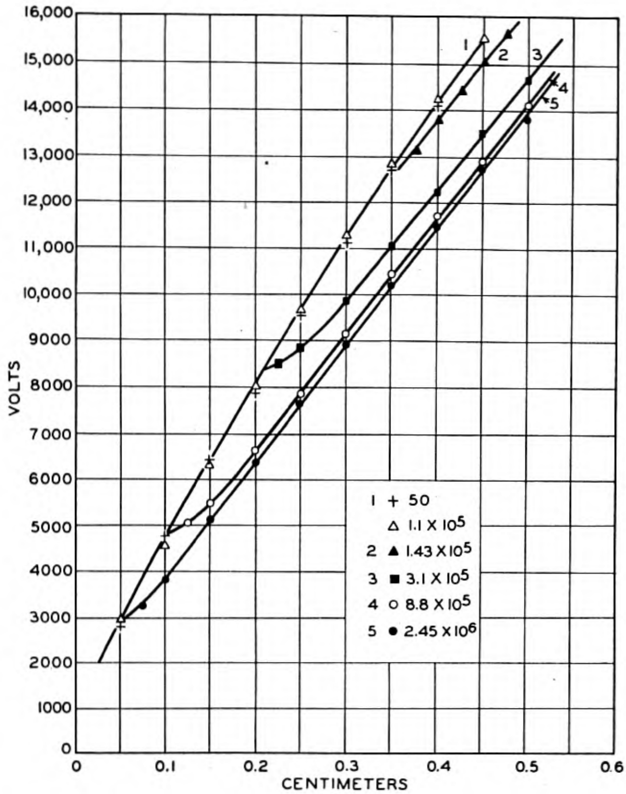


Fig. 12—Curves of sparking-potential versus distance, in atmospheric air, between spherical electrodes of 2.5-cm. diameter, at various frequencies. (Lassen, *Arch. f. Elektrotech.*)

various higher frequencies, indicated on the figure. The "critical distance" at which the departure occurs is inversely proportional to the 2/3 power of the frequency. If the data are plotted differently, sparking-potential versus frequency for the various gap-widths, each curve is parallel to the axis of abscissæ up to a "critical frequency" which increases with decrease of distance (Fig. 13). Beyond the critical frequency, each curve drops off, the ordinate sinking by fifteen to twenty per cent. On Lassen's curves (hollow circles of Fig. 13), there are indications that beyond the drop the curve again becomes horizontal; these are borne out by curves earlier obtained by Reukema with 6.25-cm. spheres (black dots of Fig. 13), although there are clashes between the two sets of data which may or may not be entirely due to the difference in the sizes of the spheres.

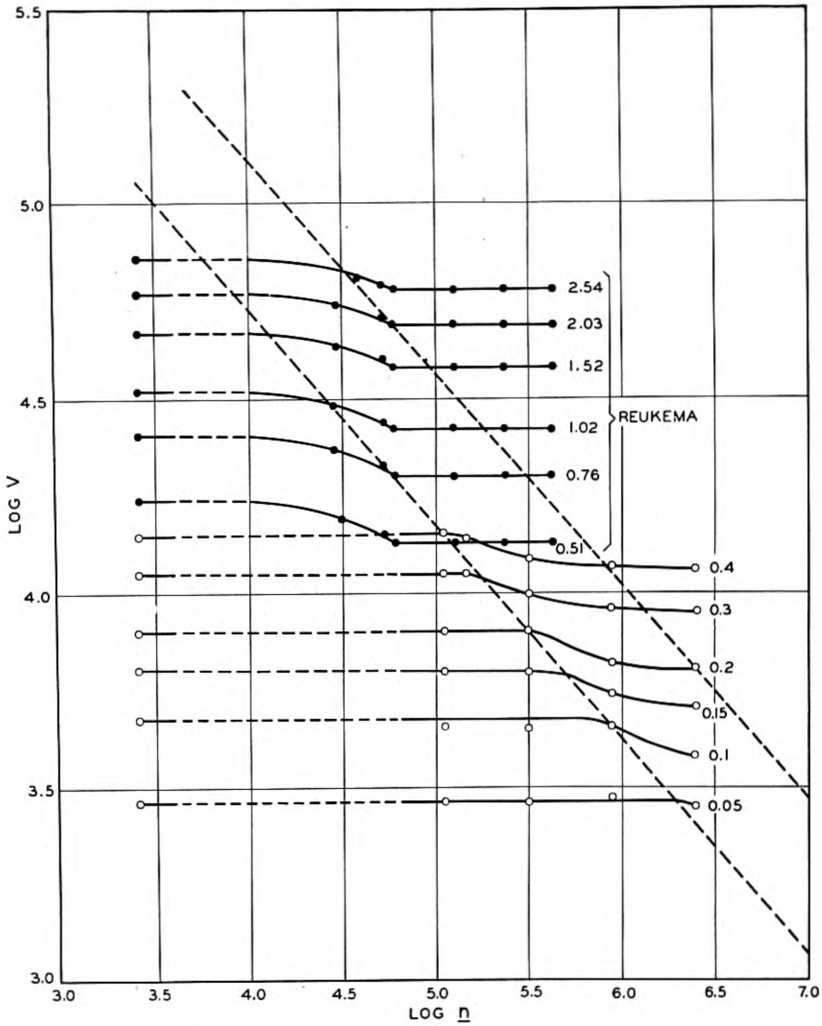


Fig. 13—Curves of sparking-potential vs. frequency, in atmospheric air, between spherical electrodes. (Data from Lassen and Reukema; the various curves correspond to the indicated gap-widths.)

The data which I have thus far cited pertain to gap-widths considerably smaller than the radii of curvature of the electrodes: fairly close approximations to the extreme case of infinite parallel planes. Experience with steady voltages suggests that what really counts is probably not the absolute value of gap-width, but its ratio to the radii of curvature (or to the smaller of the two, if the electrodes are not alike). The foregoing data then show that as this ratio increases, there is a diminution of breakdown-voltage at the higher frequencies,

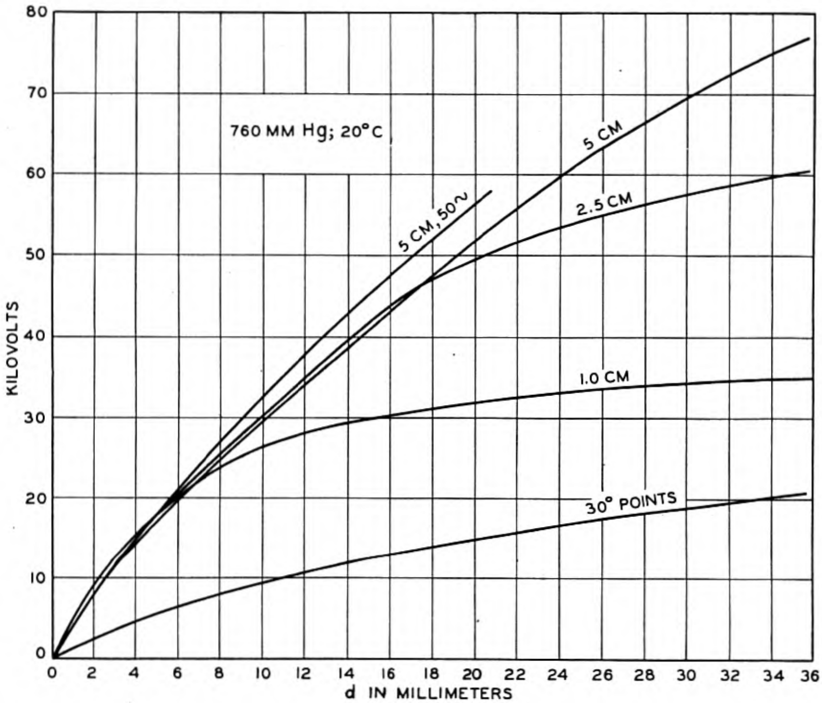


Fig. 14—Curves of sparking-potential vs. gap-width, in air, between spherical electrodes of the indicated radii, at frequencies of the order 10^5 (except the topmost). (Kampschulte, *Arch. f. Elektrotech.*)

setting in earlier the larger the ratio is. Continuing in this line of thought, we infer that as we approach the opposite extreme case of sharply-curved or pointed electrodes at distances many times as great as their radii of curvature, the diminution will begin at very low frequencies and will be considerable.

This occurs, and is illustrated by Figs. 14 and 15 (from Kampschulte) the former of which shows the breakdown-potentials between spheres of the indicated radii, over the range of distances indicated along the

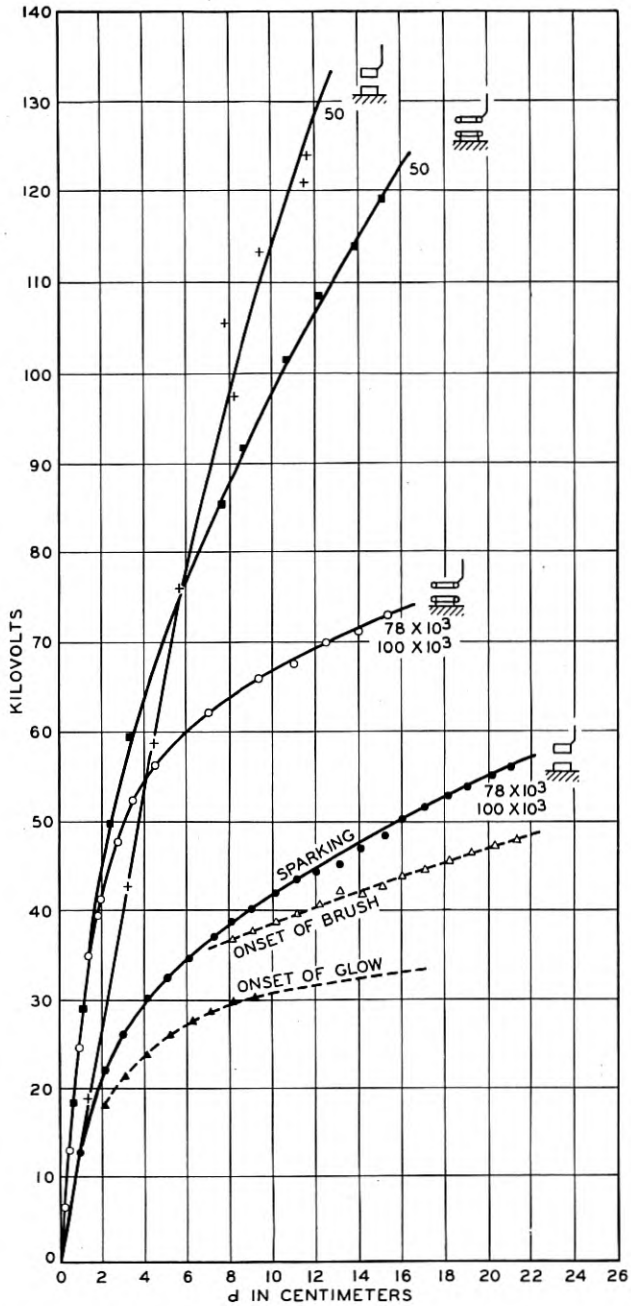


Fig. 15—Breakdown-potentials vs. gapwidth, in air, at the frequencies indicated, for the kinds of electrodes sketched in the figure and described in the text. (Kampschulte.)

axis of abscissæ. The common frequency is 73 or 107 kilocycles (Kampschulte seems to have found no difference between the behavior of the two), except for the curve marked "50 cycles" which as before may be regarded as the curve for steady voltage. The lowest of the curves pertains to electrodes sharpened at their ends to cones, with an angle of 30° at their points.

Fig. 15 is still more interesting, although the data were obtained with electrodes of a curious and inconvenient shape—collars or rings of metal, sometimes with sharp edges and sometimes with rounded edges, as the little sketches beside the curves suggest. Even for the blunt-edged electrodes the breakdown occurs at notably lower voltages for frequencies of the order of one hundred thousand than for 50-cycle or steady voltage, unless the gap-width is small. For the sharp-edged electrodes the difference is still more striking.

Most interesting of all is the triad of curves at the bottom of Fig. 15. If the gap-width between the sharp-edged electrodes is set (for instance) at 10 cm., and voltage of frequency 10^5 is applied and gradually increased, three transitions follow one after another: first, the establishment of a durable self-sustaining discharge of a certain aspect; then, its sudden transformation into another durable self-sustaining discharge of visibly different aspect; lastly, the advent of the spark. The first transition may be regarded as the breakdown of the initially undisturbed gas; the second, as the breakdown of the gas when ionized in the peculiar way prevailing in the first of the self-sustaining discharges; the third, correspondingly. With steady voltage likewise, sparkover is anticipated by the onset of a durable self-sustaining discharge if the ratio of gap-width to radius-of-curvature of one electrode is sufficiently high.[14] For the fifty-cycle A.C. applied to the sharp-edged electrodes, Kampschulte displays in Fig. 15 only the curve for sparkover; but he implies in the text that the other two discharges were observed to precede the spark.

Accurate explanation of these laws is lacking. The most that has been achieved is a rough test of a certain rough inference from theory.

Say that we establish a certain gap-width: determine the sparking-potential with steady or low-frequency A.C. voltage; and then apply

[14] "Electrical Phenomena in Gases," pp. 301–303, 443–445; note Fig. 64 on page 302 (taken from F. W. Peek, Jr., "Dielectric Phenomena in High-Voltage Engineering") which shows the critical potentials for the durable discharge or "corona" dropping below that for sparkover at a certain value of the ratio of gap-width to radius. The terms "*Glimm*" and "*Bürsten*" used by Kampschulte would be translated literally as "glow" and "brush," but usage in English is so uncertain that I have done so with hesitation.

to the electrodes potential-differences of successively higher and higher frequencies, with a certain constant peak value just inferior to the sparking-potential aforesaid. Electrons and positive ions will both oscillate in the field. For their oscillations, two sets of equations were written down in Part I: one for the extreme case of vacuum, the other for the opposite extreme case in which the collisions of the electrons with atoms are very numerous in a single cycle of the voltage. It is the latter extreme which fits more closely the present case of air at atmospheric pressure. I repeat from Part I the equation (there numbered 9) for the amplitude A of vibration of an ion of which the mobility is represented by μ:

$$A = \mu e/2\pi\nu. \tag{27}$$

The value is much greater for the electrons (because of their greater mobility) than for the positive ions.[15] At low frequencies this matters little, for both amplitudes are much greater than the gap-width and nearly all particles of both kinds are swept to the electrodes. As the frequency is raised, however, we eventually reach a point at which the amplitude of oscillation of the positive ions is depressed below the amount of the gap-width; many will then remain in the gas during cycle after cycle of the voltage while the electrons, as previously, will mostly be cleared out during the cycle in which they are formed. The effect of the positive ions in distorting the field by their space-charge will then be enhanced; the "rough inference" aforesaid is that on this account (or on some other) the breakdown-voltage will then be appreciably diminished.

To test the inference, one should measure the breakdown-voltage and compute the corresponding fieldstrength, at or near the "critical frequency" where the diminution begins; and into equation (27) one should insert the value μE for the drift-speed of the positive ions at the said fieldstrength, and for the amplitude A one should put the gap-width; and compare the resulting value of ν with the observed critical frequency. One is then baffled by the lack of measurements of drift-speed at such high fieldstrengths (the imminence of breakdown makes the customary methods of measurement difficult if not impossible). For this and other reasons, no more than an order-of-magnitude agreement is to be expected; and such a one is attained. Thus in

[15] This statement remains valid, despite the fact that (27) is probably not applicable to free electrons. It is based on the assumption that drift-speed is proportional to fieldstrength, *i.e.* that the mean kinetic energy of random motion of the electrons is independent of the field; this is certainly not true for electrons in a steady field, probably not in a high-frequency field. For positive ions it is true for low fieldstrengths, but should depart somewhat from exactness as the field is raised toward the value prevailing just before breakdown.

Lassen's experiments, the fieldstrength E at breakdown is about 30 kv./cm., for every gap-width between 0.3 and 2 cm.; if for the drift-speed of positive ions at this fieldstrength one puts 10^5, and for the amplitude of the oscillations puts the amount of the gap-width, one gets 10^5 for the critical frequency at gap-width 0.3 mm., and this —as Fig. 13 displays—is the proper order of magnitude. A like agreement is obtained with Reukema's data. But the values postulated for the drift speeds are scarcely more than guesses (in Lassen's case it is assumed that the mobility at 30 kv./cm. is two and a half times what it is at one volt per cm.); and plausible as the theory seems, the experiments help it but little.

On the other hand, observations have been made on the number of ions formed by an electron on its way across air at atmospheric pressure, at fieldstrengths of the order of those existing in these experiments.[16] This is an exponential function of E, and small variations of E thus make enormous differences in it. Lassen figures that just before breakdown at frequency $2.45 \cdot 10^6$, an electron crossing the gap (of any width between 0.2 and 2 cm.) produces 36 ion-pairs, while just before breakdown at constant voltage it would produce no fewer than ten million. This is a striking result.

Breakdown-Potentials in Gases at Low Pressures

Breakdown across a stratum of gas of low density—that is to say, having a pressure of a few millimeters of mercury, or a few tenths or a few hundredths of a millimeter—is normally followed by the establishment of a durable self-sustaining discharge, oftenest of the type called "glow." This rule, which for a gas at atmospheric pressure prevails only if one at least of the electrodes is so much rounded that its radius of curvature is decidedly smaller than the gap-width, is not thus limited at those lower densities. For an obvious reason, the rarefied gas is always confined within a tube, which in most of the experiments with high frequencies (those on the ring-discharge excepted) is a cylinder only a few centimeters wide; thus, to judge from experience with direct-current discharges, the presence of the wall must have a great influence on the phenomena. The electrodes are commonly either discs inside the tube near its ends, or belts of tinfoil wrapped around the outside of the tube; at high frequencies it often makes surprisingly little difference which, and yet such differences as have been reported are sometimes noteworthy. Breakdown-potentials are generally determined by raising the amplitude of the high-frequency

[16] M. Paavola, *Arch. f. Elektrotechnik*, **22**, 443–458 (1929); "Electrical Phenomena in Gases," p. 278.

voltage gradually till suddenly a visible discharge appears; the last previous reading of the voltage is then recorded. Some physicists have reported that the advent of the self-sustaining glow is difficult to observe, or capricious and unreproducible; others mention nothing of the sort.

There are now two independent variables, frequency and pressure, instead of the former only; this makes it harder to view the data. So long as the frequency is held constant, the curve of breakdown-potential versus pressure usually has the familiar form, characteristic of steady as well as alternating voltages: it is concave upward, with a single minimum, perhaps deep and striking, perhaps so flat as scarcely to be visible. There is consequently for each frequency an optimum pressure, P_{sm} say, for the onset of the glow; at pressures either lower or higher than P_{sm}, the critical potential is above its minimum value V_{sm}. In general terms, the reason is this: at high pressures, ionization is restricted by the fact that in their numerous collisions, the electrons lose energy so often that they seldom amass enough to ionize the molecules—at low pressures, it is restricted by the fact that there are few collisions—at the intermediate pressure P_{sm}, the best compromise prevails between the two disadvantages. There can be little doubt that if one were to vary the distance between the electrodes in lieu of the pressure, the effect would be the same, according to Paschen's law that breakdown-potentials depend on the product of pressure and distance.[17]

When one compares the breakdown-potential versus pressure or V_s-vs-p curves for various frequencies, the results are often far from simple, and different observations are sometimes hard to reconcile; even when one considers only undamped sinusoidal wavetrains, as I shall do.

Thus Hulburt, working with oxygen and hydrogen at pressures of 1 to 5 mm., in tubes with internal electrodes 5 to 30 mm. apart, experimented with steady voltages, with 50-cycle A.C., and with the high frequencies $0.86 \cdot 10^6$ and $5.3 \cdot 10^6$; and he detected *no* variation of the voltage for the onset of the glow over all this range. Likewise Rohde, working with a number of gases (oxygen, hydrogen, nitrogen, argon, neon, helium, mercury) in tubes with electrodes (usually internal) 19 or 38 mm. apart, applied frequencies ranging from 10^5 to $1.5 \cdot 10^8$. Up to about 10^6 the breakdown-voltage scarcely changes; thence-

[17] "Electrical Phenomena in Gases," pp. 304–308. Paschen's law in this form is valid only for broad plane-parallel electrodes; to make it hold for curved electrodes, their radii of curvature should be varied in the direct ratio of the distance or the inverse ratio of the pressure.

forward it declines.[18] In Fig. 16 I show three of his V_s-vs-p curves for oxygen, in a tube with electrodes 38 mm. apart; they correspond to wave-lengths 9.8, 5.03, 4.32 metres, therefore to frequencies $3.1 \cdot 10^7$, $6 \cdot 10^7$, $7 \cdot 10^7$. It is obvious that for any pressure in the range of these experiments, V_s diminishes as ν increases; also, that p_{sm} as well as V_{sm} diminish with increasing frequency.

From Townsend's school at Oxford, I will quote some observations of Hayman on helium and neon at pressures ranging from a few mm. to a few tenths of a mm., in cylindrical tubes with external collar electrodes. A curve of V_s-vs-p for frequency $3.75 \cdot 10^6$ displays a

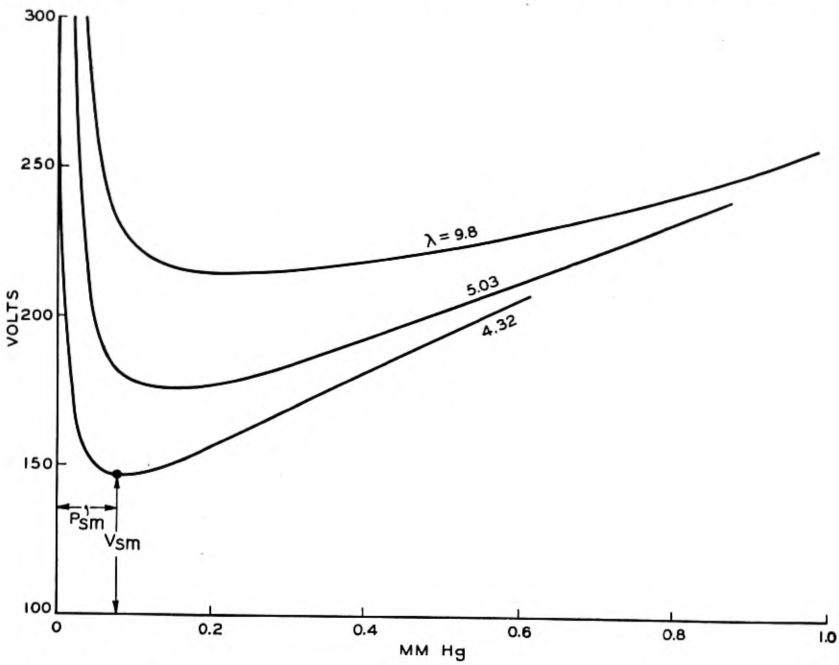


Fig. 16—Onset-potential vs. pressure, in rarefied oxygen, for self-sustaining glow-discharge at the indicated wave-lengths of the high-frequency voltage. (Rohde, *Ann. d. Phys.*)

minimum. Curves of V_s-vs-ν slope downwards toward higher frequencies over the range from $4.7 \cdot 10^5$ to $7.5 \cdot 10^6$, the slope being gentle at pressures above 2 mm. and very rapid at pressures rather lower (at 0.11 mm. there is a drop in a ratio greater than 6 : 1, as the frequency is raised from the bottom to the top of the aforesaid interval).

[18] Rohde devotes so much of his attention to the maintaining-potentials (see below) that his allusions to the onset-potential are scanty, and their degree of generality is hard to assess.

In a narrow tube, a greater voltage is required for breakdown than in a broad one—at pressure 1 mm., twice as great a voltage for a 1.5-cm. tube as for one of 3.9-cm. diameter. This last is an illustration of the effect of the walls; probably they influence the preliminaries to breakdown by capturing and retaining the electrons which approach them from the gas, so that the ionizing agencies at work in the gas must be strengthened to compensate that loss. Townsend and Nethercot also record a V_s-vs-p curve with a minimum, for frequency $7.5 \cdot 10^6$.

If one knew only of the foregoing papers, one would resume the situation as follows: for any value of pressure, breakdown-potential diminishes steadily with increase of frequency, but the diminution is

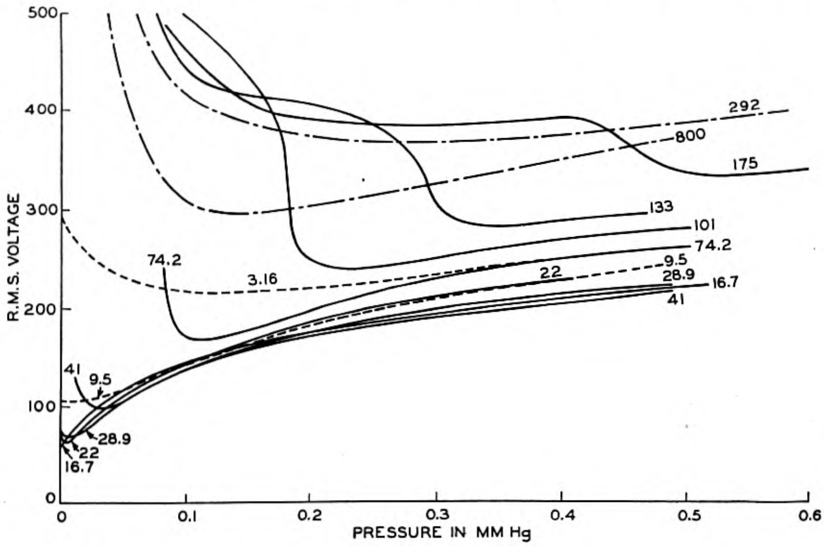


Fig. 17—Onset-potential vs. pressure, in rarefied hydrogen, for self-sustaining glow at the indicated wave-lengths. (H. Gutton, *Annales de Physique.*)

very small all the way from $\nu = 0$ to $\nu = 10^6$; for any value of frequency, the curve of V_s-vs-p has a single minimum; the coordinates p_{sm} and V_{sm} of that minimum decrease with increasing ν. There would be wide gaps in the range of frequency over which these statements had been tested, but nothing would suggest that there might be discrepancies within the gaps. However, the situation is not so simple. Mention must be made of remarkable and perplexing data obtained by C. and H. Gutton and collaborators of theirs, mainly with external-electrode tubes.

Fig. 17, relating to hydrogen, is taken from some of H. Gutton's most recent work: it is a set of V_s-vs-p curves for various frequencies of an extremely wide range (the wave-lengths in meters are marked beside the curves) obtained with a tube 10 cm. long closed at its ends by flat plates, covered outwardly by sheets of tinfoil serving as the electrodes. (Gutton never indicates the actual observations on his graphs.) It is superfluous to say that this family of curves is easy neither to envisage nor to describe. Most of them are of the type

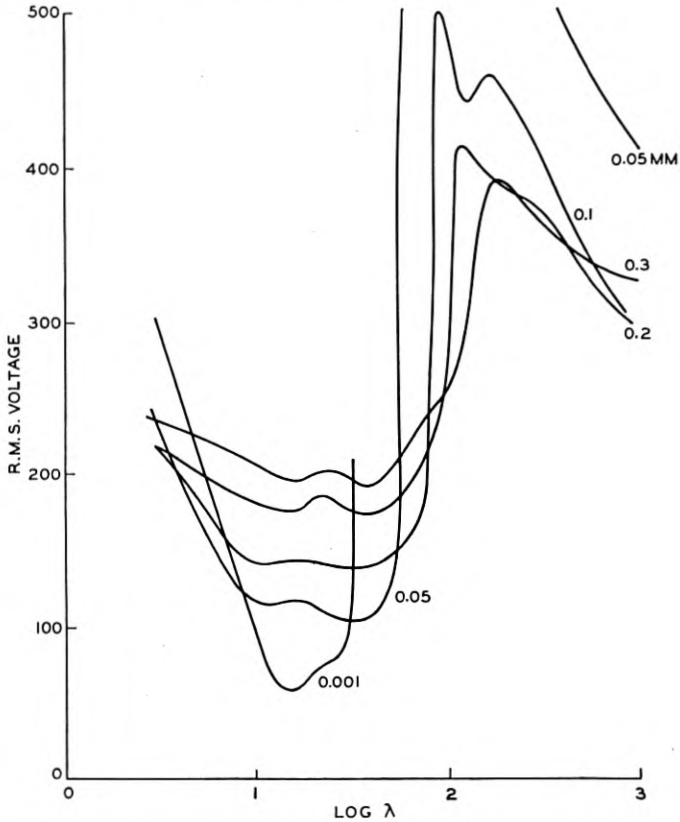


Fig. 18—Onset-potential vs. (logarithm of) wave-length, for self-sustaining glow at the indicated pressures. (H. Gutton.)

familiar from other researches, with a single flattish minimum; but some are very different, with no minimum at all in the range of experiment, but a couple of sharp bends with a linear segment between. The V_s-vs-ν curves for various pressures, exhibited in Fig. 18, form a set even more confusing.

Over the frequency-ranges where the curves of Fig. 17 have single minima, where accordingly we may define V_{sm} and p_{sm} as before, these do not always vary in the same sense with ν. As the frequency is raised from about $4 \cdot 10^5$, V_{sm} increases at first; then come the curves with curious shapes; when again the flattish minima return, at $4 \cdot 10^6$ cycles or thereabouts, V_{sm} is following the hitherto-familiar rule of decreasing with increase of frequency; but further along, beyond about $3 \cdot 10^7$, the trend again reverses, and V_{sm} rises once more. There is thus an "optimum frequency," at which (for a wide range of pres-

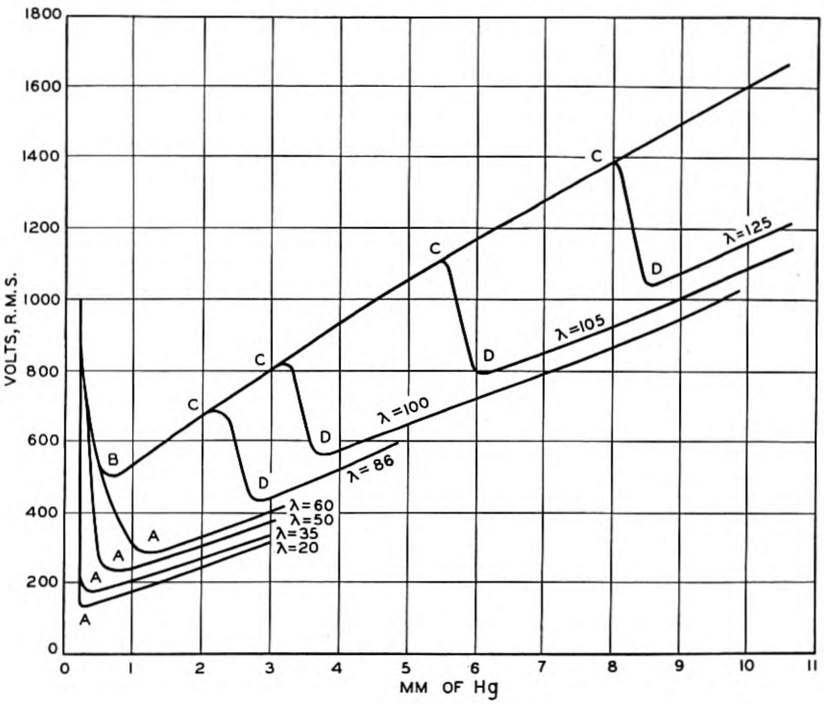


Fig. 19—Onset-potential vs. pressure, in rarefied air, for self-sustaining glow at the indicated wave-lengths, in tube described in context. (Gill & Donaldson, *Phil. Mag.*)

sure) breakdown occurs at a lower voltage than when ν is either lower or higher. This comes at about $3 \cdot 10^7$ cycles for tubes 10 to 20 cm. long, lower down for a 5-cm. tube.

These complexities far surpass what other observers report. The others, it is true, confined themselves to narrower ranges of frequency, and yet their ranges were often so located on the frequency-scale that they should have observed some of the striking reversals of trend and distortions of curves, had the conditions been the same; to seemingly minor differences in the conditions, then, the discrepancies must be ascribed. The complexities are not peculiar to hydrogen, for Gutton obtained a very similar set of curves with oxygen, and in much earlier work (1923) on rarefied air he found V_{sm} increasing with frequency up to about $7.5 \cdot 10^5$ cycles, and thenceforward diminishing all the way to the uppermost limit of his frequency-range, $2.14 \cdot 10^6$. This last-mentioned result was obtained with an external-electrode tube, the exterior tinfoil belts 24 mm. apart; on substituting an internal-electrode tube, he found V_{sm} increasing with ν over the entire frequency-range. But I must leave the reader to explore and collate Gutton's numerous curves for himself, and mention only in closing that in a tube of rarefied hydrogen with external electrodes 53 mm. apart, he got at frequency $2.5 \cdot 10^7$ a breakdown-potential of only 57 volts—an amazingly small value, far lower than anything ever obtained with direct current.

Gill and Donaldson produced V_s-vs-p curves with two minima apiece instead of one, by placing the long slender discharge-tube (20 cm. long, 3.3 cm. diameter) between two metal plates serving as the electrodes, with its axis parallel to their planes. These curves were obtained in rarefied air, with various frequencies between $3.5 \cdot 10^6$ and $2.3 \cdot 10^6$, corresponding to wave-lengths between 86 and 125 meters; one sees them in Fig. 19. (Below and to the left are curves for four other and higher frequencies, ranging from $5 \cdot 10^6$ to $1.5 \cdot 10^7$; these have the familiar single-minimum controur, and both V_{sm} and p_{sm} decrease as ν increases.) Thereupon, Gill and Donaldson re-oriented the tube so that its axis was perpendicular to the electrode-plates—owing to its length, it had to be passed through a pair of holes made specially in the plates—and repeated the observations. Now, of the two minima, the one to the right disappeared; for each of the several wave-lengths, the curve continued straight on past the point marked D in Fig. 8, to a single minimum lying far to the left.

Maintaining-Potentials of High-Frequency Glows in Rarefied Gases

When the high-frequency glow in a rarefied gas is established, the voltage between the electrodes—that is to say, the amplitude V of the oscillating voltage—is as a rule much smaller than the breakdown-potential. It would seem natural to begin the study of the glow by determining the curves of current versus voltage and current versus length (*i.e.* anode-to-cathode distance) for many values of pressure, as the custom is in dealing with direct-current discharges; but data of this sort are few. Further along I will speak of work of Townsend's school, in which over a limited range of conditions V was found to be almost independent of i (the amplitude of the oscillating current) and a linear function of the length l. Also Hayman speaks of observing a minimum in the curve of V versus i, occurring "at a value of current slightly greater than the least which gives a uniform glow in the tube." Often, however, the experimenters simply vary the strength of the oscillating current (usually by varying the filament-current of the vacuum-tube oscillator, which is coupled to the circuit containing the discharge-tube) and measure the voltage across the electrodes just before the glow disappears. This is called the "least maintaining-potential" or the "extinction-potential" or by some equivalent name. By analogy with direct-current discharges, it should depend on the constants of the circuit.

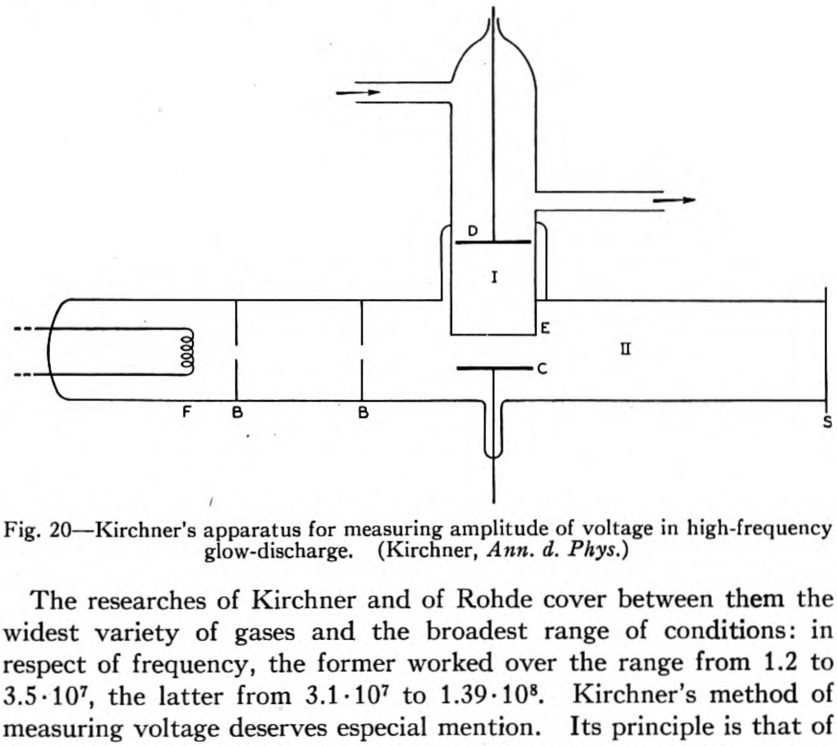


Fig. 20—Kirchner's apparatus for measuring amplitude of voltage in high-frequency glow-discharge. (Kirchner, *Ann. d. Phys.*)

The researches of Kirchner and of Rohde cover between them the widest variety of gases and the broadest range of conditions: in respect of frequency, the former worked over the range from 1.2 to $3.5 \cdot 10^7$, the latter from $3.1 \cdot 10^7$ to $1.39 \cdot 10^8$. Kirchner's method of measuring voltage deserves especial mention. Its principle is that of the cathode-ray oscillograph: a beam of fast electrons is deflected to and fro by the P.D. applied between two plates, one on either side of the beam. These could be the electrodes, but that the fast electrons might then perturb and be perturbed by the discharge, and there would be other disadvantages. Kirchner therefore designed three pieces of apparatus, of which one is figured in Fig. 20. The discharge is in the tube I, between the electrodes D and E, of which the latter is a sheet of metal separating I from the evacuated tube II; the beam of fast electrons, proceeding from the filament F and formed by the diaphragms B, passes between E and the lower plate C which is constantly at the same potential with D. The voltage between C and E is the same as that between the electrodes of the discharge; the measure of its amplitude is the length of the arc which the tip

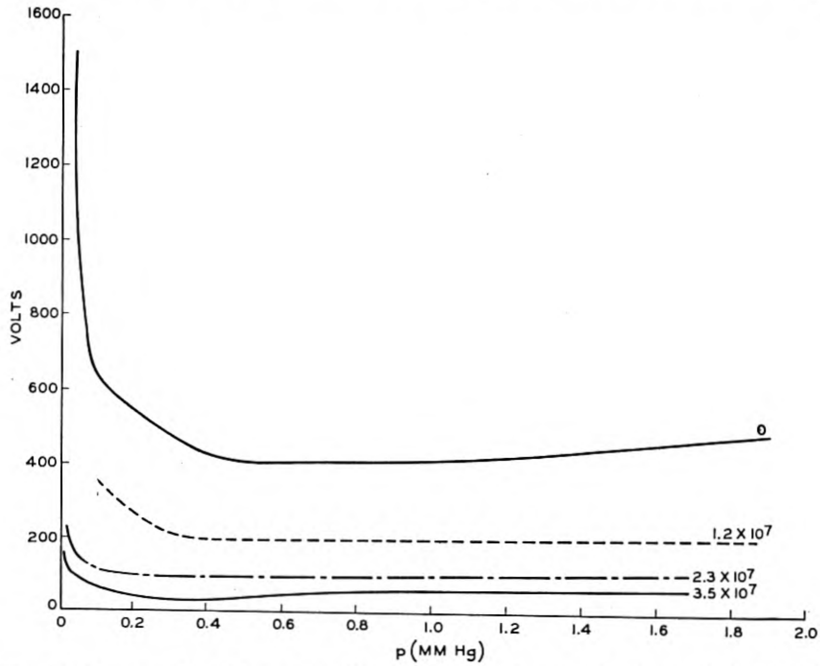


Fig. 21—Least maintaining-potential vs. pressure, in rarefied air, at the indicated frequencies. (Kirchner.)

of the beam describes, as it dashes back and forth over the surface of S, a fluorescent screen.

Mostly the curves of least maintaining-potential versus pressure, like those of onset-potential versus pressure, are concave-upward with single flattish minima. Fig. 21 shows four curves which Kirchner obtained with air. They are not very smooth nor are the minima clearly marked; I choose them for reproduction because they comprise a curve for direct-current discharge (marked 0) as well as three others for certain high frequencies marked beside them.

Let me denote by V_{mm} and p_{mm} the coordinates of the minimum of such a curve as those of Fig. 10, and call V_{mm} the "minimum of the least maintaining potential" or simply the "minimum maintaining potential" for the frequency in question (we must choose between lengthiness and lack of precision in our terms!). The value of p_{mm} and the value of V_{mm} both decrease with increase of ν, after the variation begins; consequently, a curve of V_{mm} vs ν, such as we will now consider, corresponds not to a single pressure but to as many different pressures as there are points.[19] Disregarding this complication, notice the curve of Fig. 22.

This is the curve of minimum maintaining-potential versus frequency for air in a tube of 24 mm. internal diameter, with electrodes 19 mm. apart. It is taken from Rohde, who says that the curves for oxygen,

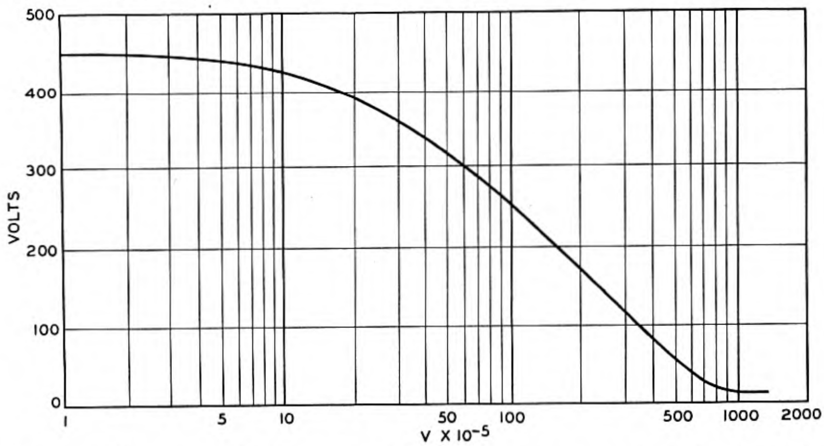


Fig. 22—Minimum maintaining-potential vs. frequency, in air, under conditions described in the context. (Rohde.)

nitrogen, hydrogen, helium, neon and argon are similar. The comparative constancy of V_{mm} at frequencies below about half a million, the rapid decline thenceforward as far almost as 10^8, are evident. Beyond, there is a definite hint that the voltage again becomes independent of ν, at a value far lower than its low-frequency or direct-current amount; for each of the aforesaid gases, hydrogen alone excepted, V_{mm} was sensibly the same at $1.39 \cdot 10^8$ (Rohde's highest frequency) as at $6.95 \cdot 10^7$.

One is struck by the resemblance to what Reukema and Lassen observed of the sparking-potential across atmospheric air: approximate constancy up to a certain critical frequency, ensuing decline, eventual

[19] The only extensive set of published curves of V_m-vs-ν is that of H. Gutton, which is more complex than one would expect (see below).

attainment of another and much lower constant value. There the critical frequency was found to depend on the gap-width and on the curvature of the electrodes; here, the data are too scanty to permit of a similar, or any conclusion. The behavior of the breakdown-potential V_{sm} in these tubes of rarefied air is of the same sort but (according to Rohde) the percentage-drop from its low-frequency value to its value at the highest frequency attained is much less striking than that of the minimum maintaining potential V_{mm}. The ratio of V_{mm} to V_{sm} therefore decreases with increase of ν, descending for argon to the value 0.1, for mercury to the fantastically low value 0.036.

The smallness of these lowest values of the maintaining potential is something extraordinary. They are, of course, much smaller than the minimum maintaining potential of the direct-current glow, which is the cathode-fall, and is of the order of hundreds of volts. Now, the office of the cathode-fall is to maintain the outflow of electrons from the cathode (this is proved by the fact, among others, that it becomes dispensable if the cathode is heated to such a degree that the outflow becomes spontaneous). The conclusion therefore is, that in the high-frequency discharge the demand for electrons from the electrodes is minimized if not abolished. Even so, the minuteness of the voltage-amplitudes remains astonishing. Taking Rohde's data for the frequency of 10^8, and going from the least toward the most striking case, we notice: air 14 volts, oxygen 12, nitrogen 12.5, hydrogen 15.5, helium 16, neon 11, argon 8, mercury 5 volts. I illustrate this by Rohde's curve (Fig. 23) of maintaining-potential versus pressure for neon, though in one respect the curve is quite untypical: no other gas exhibits so long a nearly horizontal arc (in a tube of 24 mm. diameter, and 19 mm. from one to the other of the electrodes).

More striking yet are some of the values obtained by C. and H. Gutton, whose flock of curves of V_m-vs-p for various frequencies and V_m-vs-ν for various pressures, obtained in long tubes with rarefied hydrogen within and metal electrodes outside, is almost as intricate and perplexing as the family of curves for the breakdown-potential of which I spoke above. Many indeed exhibit no minimum at all. However, with a tube 5.3 cm. long he maintained the discharge, at some $4 \cdot 10^7$ cycles, with a voltage of amplitude 5.7; and with a twenty-centimeter tube at $2 \cdot 10^7$ cycles he kept it alive with a voltage amplitude of 40, which considering the length is almost equally remarkable.[20]

[20] In consulting papers of the Guttons, remember that they give R.M.S. values of voltage and fieldstrength, not peak-values nor amplitudes.

One instinctively compares these values with the ionizing and resonance potentials of the gases, and finds them mostly lower. But actually there is no sense in making such a comparison, and indeed it is difficult to derive from theory anything with which they may profitably be compared. The most that one can do is to attempt to estimate the maximum kinetic energy which electrons should possess, not after having fallen through a constant potential-drop of the stated magnitude, but while they are under the influence of an oscillating fieldstrength of the corresponding amplitude.

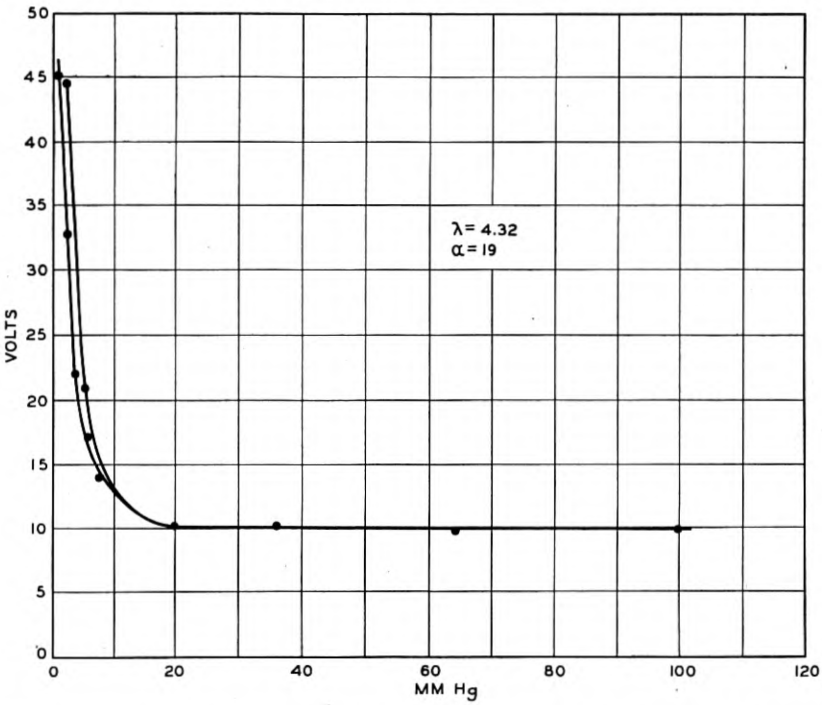


Fig. 23—Least maintaining-potential vs. pressure in rarefied neon at frequency $7 \cdot 10^7$. (Rohde.)

The formula required was given in equation (4) of Part I, but on examining it, one sees that it involves an unknowable quantity. Say that the fieldstrength is directed along the axis of x, and is given by the expression $eE \sin (2\pi\nu t)$, so that it is zero at $t = 0$ and positive immediately after; then this unknowable quantity, denoted by v_0, is the component along the x-axis of the velocity of the electron at $t = 0$. Indeed, there is a second unknowable, the component normal to the x-axis; call it v_n. For K_m, the maximum kinetic energy of the electron, we then have (repeating equation 4):

$$K_m = \frac{1}{2} m \left[\left(v_0 + \frac{1}{\pi \nu} \frac{eE}{m} \right)^2 + v_n{}^2 \right] \tag{28}$$

and one sees that it is idle to assign an exact maximum value to the energy of the free electrons roaming in a vacuum subjected to a high-frequency field, since we cannot possibly know the initial velocity-components v_0 and v_n of all these corpuscles at the instant $t = 0$.

If we do the easiest thing, and simply put $v_0 = 0$, we get for K_m the expression:

$$K_m = \frac{1}{2} m \left(\frac{1}{\pi \nu} \frac{eE}{m} \right)^2. \tag{29}$$

Here, of course, energy and fieldstrength are expressed in electrostatic units. Putting them in electron-volts and volts-per-cm. respectively, and denoting them by K_{mv} and E_v respectively to symbolize this choice of units, we obtain:

$$K_{mv} = \frac{10^8 e}{2m\pi^2 c} \left(\frac{E_v}{\nu} \right)^2 \tag{30}$$

$$= 8.95 \cdot 10^{13} \left(\frac{E_v}{\nu} \right)^2. \tag{31}$$

I recall from Part I that this choice of value for v_0, like every other except one, leads to the result that the electron oscillates not about a fixed but about a drifting centre. The solitary choice which results in the electron vibrating about a stationary centre,—to wit,

$$v_0 = - eE/2\pi\nu m \tag{32}$$

—produces for the maximum kinetic energy a value one-fourth as great as that given by equation (31).[21]

On inserting into equation (31) the various frequencies at which experiments have been made, and the amplitudes of the fieldstrength corresponding to the minimum maintaining potentials at these frequencies, one sometimes gets values of the order of magnitude of ionizing or resonance potentials, sometimes values much lower. Thus, Rohde's observations on helium give, at the frequency 10^8, a minimum maintaining-potential of 11 volts between electrodes 19 mm. apart, therefore an oscillating fieldstrength of amplitude 5.8 (if there is no

[21] If v_0 is negative and algebraically less than $- eE/2\pi\nu m$, the energy of the electron is always *less* than, or at most equal to $1/2m(v_0{}^2 + v_n{}^2)$: the initial vis viva is also the greatest.

distortion by space-charge, another dubious assumption!). By the equation we derive 0.3 electron-volts for the maximum energy of those electrons for which $v_0 = 0$—a value barely more than one per cent of the resonance-potential of helium, the least amount which a normal helium atom can absorb as the first stage toward ionization! If we apply the equation to some of H. Gutton's results, the conclusions are equally startling; thus, putting 2 for E_v and $2 \cdot 10^7$ for ν (values observed with hydrogen in a tube 20 cm. long), we find 0.9 electron-volts for the maximum energy.

Were the discrepancies between these values of K_{mv} and the resonance-potentials of the gases somewhat smaller—were they, say, of the order of fifty per cent—they could readily be excused. Occasional electrons, for instance, might make collisions with atoms in just such ways and at just such times as to increase their accumulation of energy; thus, an electron which had started from rest ($v_0 = 0$) and had been speeded up to its utmost during the first half-cycle of the field and was about to be slowed down again, might have its velocity reversed by an elastic impact just at the end at that first half-cycle, so that the second half-cycle would speed it up still more (Hiedemann's idea). Or, occasional electrons might acquire a fund of energy in other ways, and have a considerable value of kinetic energy $(1/2)m(v_0^2 + v_n^2)$ at the instant $t = 0$; the form of the right-hand member of equation (28) now shows that the high-frequency field would augment their vis viva, not merely by the amount which we have just computed, but by an extra amount proportional to v_0. But a discrepancy of two orders of magnitude seems too large to be explained in such a way; and although it is impossible to make any positive affirmation, I suspect that there must be a permanent distortion of the field by space-charge, the mean value of the potential in the middle of the gas differing by several volts from its mean value near the electrodes—being presumably more positive, owing to an accumulation of positive ions.[22]

We ought now to compute the distance D through which a free electron moves in the high-frequency field, while its energy is mounting from zero (or the minimum value, whatever that may be) to the greatest value which it attains. For, if it should turn out that this distance is not more than a small fraction of the electronic mean-free-

[22] This is the condition in the direct-current "low-voltage arc" ("Electrical Phenomena in Gases," pp. 383–386) where the P.D. between anode and cathode is less than the resonance-potential of the gas, but the P.D. between a certain region of the gas on the one hand and the cathode on the other is at least as great as the resonance-potential. In the low-voltage arc the electrons are expelled from the cathode by heat independently applied, so that there is no need of a cathode-fall.

path in gases under the conditions of the actual experiments, then the foregoing theory would be vitiated at the start; electrons would seldom or never acquire the maximum amount of energy for which we have derived the general formula and which we have computed in certain special cases.

The distance D is described during a half-cycle of the high-frequency field, but the phase at which that half-cycle must be supposed to begin depends on v_0, which makes the problem intricate. If we put $v_0 = 0$, the electron starts from rest at $t = 0$ and attains its maximum speed at $t = 1/(2\nu)$, after traversing the distance given by the first of the following formulæ. If we put for v_0 the particular value which corresponds to an electron describing oscillations about a fixed centre, the doubled amplitude of these oscillations is what we want; it is given by the second formula:

$$\left.\begin{aligned} D &= \frac{1}{2\pi}\frac{e}{m}\frac{E}{\nu^2} = 2.81\cdot 10^{14}\left(\frac{E_v}{\nu^2}\right) \\ D &= \frac{1}{2\pi^2}\frac{e}{m}\frac{E}{\nu^2} = 8.95\cdot 10^{13}\left(\frac{E_v}{\nu^2}\right) \end{aligned}\right\} \begin{aligned} &(v_0 = 0) \\ &(v_0 = -\,eE/2\pi\nu m), \end{aligned} \qquad (33)$$

E_v standing as before for the amplitude of the fieldstrength in volts per centimeter.

The most which we can infer from these formulæ is, that when we find recorded a value of E_v (amplitude of the fieldstrength in the self-sustaining high-frequency glow) we should evaluate the product $10^{14}E_v/\nu^2$, and compare it with the electronic mean-free-path in the gas in question at the pressure in question; if it is much smaller than the electronic mean-free-path the foregoing theory is worth whatever can be got out of it; if it is much larger than the electronic mean-free-path the theory is worthless. For the two special cases (from Rohde and Gutton) for which I have just computed the values of K_{mv}, those of the product $10^{14}E_v/\nu^2$ come out as 0.06 cm. and 0.50 cm. respectively. The pressure of the gases (helium and hydrogen respectively) amounted in the two experiments to 0.400 and 0.001 mm. Hg respectively. Now the measurements of electronic mean-free-path for electrons of these speeds are imprecise and uncertain, and the concept itself is vague. The values which it is probably best to take are those derived by Townsend and his school from measurements of the diffusion of free electrons in gases.[23] That for hydrogen at .001 mm. Hg is so high (of the order of 40 cm.) that the theory is justified by an ample margin; that for helium at 0.4 mm. Hg (of the order of 0.1 mm.) is

[23] "Electrical Phenomena in Gases," pp. 248–252.

high enough to make it probable that electrons would often acquire the stated energy. But this is not to be taken as universally true for all the values of fieldstrength which have been observed in high-frequency glow-discharges.[24]

Certain data were obtained by Brasefield in experiments on air over a frequency-range extending downward from Kirchner's, and contained in Gutton's: that is to say, from $2 \cdot 10^7$ down to $1.25 \cdot 10^6$. The electrodes—external belts of metal wrapped around a tube of 4.5 cm. diameter—were no less than 40 cm. apart; and instead of measuring the least maintaining potential, Brasefield measured at various pressures the amplitude V of the voltage existing between the electrodes when a current of amplitude 100 mils was passing. The resulting V-vs-p curves for diverse frequencies had the customary form, concave-upward with single minima. As the frequency was raised from $1.25 \cdot 10^6$ to $1.5 \cdot 10^7$, the value of the minimum voltage and that of the pressure at which it was attained both trended downward, though with peculiar brief rises. As the frequency was further raised from $1.5 \cdot 10^7$ to $2 \cdot 10^7$, there was a sudden tremendous upswing of the minimum voltage, and a rise of the corresponding pressure,—anomalies recalling the singularities of Gutton's curves. Under the conditions prevailing at the minima of the curves for these two highest frequencies, there was agreement (within the wide limits of uncertainty) between K_{mv} and the ionizing-potential of hydrogen, and between D from the first of equations (33) and the electronic mean free path.

In the direct-current glow-discharges in a cylinder of gas contained in a tube, under certain conditions, there is a region (the so-called "positive column") throughout which the fieldstrength is uniform and low, and either decreases slowly as the current or the current-density is increased, or else remains sensibly the same. This region is apparently uniform in color and brightness. (I am not taking account of cases where it is "striated," or cases in which it is visibly dimmer near the wall than near the axis.) In the high-frequency glow-discharge there is also, under certain conditions, a region of uniform color and brightness occupying all of the tube except small portions near the electrodes. Townsend and his school undertook to measure the (alternating) fieldstrength in this region, and to compare it with the values obtained in the direct-current glow.

[24] If D computed by equations (33) should turn out to be very many times as great as the electronic mean-free-path, the proper procedure would be to compute the maximum energy of the oscillating electrons by the conventional method from the general equation (equation 5 of Part I) for electrons moving in dense gases. I fear, however, that in most cases the ratio of D to the electronic mean-free path is not great enough to allow of passing to this limiting case.

In the experiments (for instance those of Townsend and Nethercot) the distance l between the electrodes was varied, the current maintained at some constant value, the voltage plotted as function of l. The resulting V-vs-l curves were rising straight lines over large (but not unlimited) ranges of conditions. In these experiments the gases were nitrogen, helium and neon; the electrodes were external collars surrounding the tube, one of which could be shifted. The same result was later obtained by other pupils of Townsend (Hayman, P. Johnson, F. L. Jones), who sometimes worked with internal-electrode tubes, displacing one of the electrode-discs by a magnetic device.

This result suggests that in the main part of the glowing gas there is an alternating potential-gradient of constant amplitude, independent of l. Denote its amplitude [25] by b, those of current and voltage by i and V: we have

$$V = a + bl.$$

Now a is to be interpreted as the sum of potential drops across regions near the electrodes, where conditions differ from those of the middle of the glow.

Plotting V against l for various values of i, Townsend and Nethercot found this important fact: the slope of the line, the potential-gradient b, is independent of current over wide ranges (for instance, over the range of i from 3 to 18 mils, in nitrogen contained in a tube of diameter 3.9 cm.). The difference $(V - bl)$, however, increases with the current; over a certain range of current-strengths, the increase is linear. The value of the gradient b is of the order of a few volts per cm. Townsend and Nethercot give for nitrogen in a tube of 3.1 cm. diameter the values 13.2 (volts/cm.) at the pressure 0.26 mm. and 19.3 at the pressure 0.53 mm. For helium at 1 mm. they give 5.1; for neon at 1.06 mm. the value 3.5. These were obtained at the aforesaid frequencies of 7.5 and 4 millions; and so we meet the question of the dependence of b on frequency.

The value of b was found to be nearly independent of frequency, so far as the rather scanty measurements go; in nitrogen, the same for the frequencies $4 \cdot 10^6$ and $7.5 \cdot 10^6$ (Townsend and Nethercot); in helium and neon, constant over the range from $4.7 \cdot 10^5$ to $7.5 \cdot 10^6$ cycles (Hayman); in neon, by further experiments, constant over the range from $2.5 \cdot 10^6$ to 10^7 cycles (Johnson). This brings us to the question: how does b, which is the amplitude of the alternating potential-gradient in the high-frequency glow, compare with the

[25] Townsend's school give root-mean-square instead of amplitude-values for sinusoidal quantities.

constant gradient in the positive column of the direct-current discharge? A discharge of the latter type was set up in tubes (equipped with internal electrodes, of course) which had been employed for the high-frequency glow; the gradient in its positive column, measured by Townsend and Nethercot with nitrogen and by Johnson with neon, agreed fairly well with the value of $2b/\pi$ which is the *mean* value of the gradient in the discharge taken over any half-cycle. As for the term $(V - bl)$, which has been interpreted as the sum of potential-drops localized near the electrodes, it seems to vary inversely as the frequency over the limited ranges aforesaid.

To anyone desirous of penetrating through phenomena to fundamental laws, the situation as presented in this article must seem deplorable. The laws of the high-frequency discharge are almost purely empirical, either unexplained altogether, or explained only in a vague and qualitative way. Even the data do not form a complete or coherent system. For the remaining type of high-frequency glow not treated here—the so-called electrodeless discharge, in which high-frequency magnetic as well as electric fields pervade the ionized and excited gas—the situation is yet more obscure. Still, if the reader will consult again the article which preceded this one, he will be reminded that considerable progress has been made already in interpreting by fundamental theory the events which happen, when high-frequency fields are applied to gas which is independently ionized by other agents; and this gives hope of future success in extending the theory to the phenomena which occur when the high-frequency fields are themselves the causes of the ionization.

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Three Superfluous Systems of Electromagnetic Units.[1] GEORGE A. CAMPBELL. At the present time the electromagnetic, electrostatic, Heaviside-Lorentz, practical and international systems of electric and magnetic units are used side by side in pure and applied electromagnetism. The question is here raised whether the use of this multiplicity of units should continue indefinitely into the future when

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the conversion tables for translating from any system to any other system show the essential equivalence of all five systems. It is recommended that but one system be legalized and used generally in place of the five systems, and that this universal system be the coherent meter-kilogram-second-ohm or definitive system. It is further recommended that the international ohm be used in this system. This unit is the one actually used in exploring the physical world because laboratory resistances for physics and test room resistances for engineering have been so calibrated. Of far greater importance is the fact that by retaining the international ohm it will be simpler, and completely feasible, to eliminate what Heaviside called "that unmitigated nuisance, the 4π factor of the present B.A. units" from our preferred system of units.

A Compensated Thermionic Electrometer.[2] K. G. COMPTON and H. E. HARING. A compensated single tube electrometer is described and the principles of its operation discussed. This apparatus has been found to compare favorably with "balanced tube" circuits both as regards stability and sensitivity and to be superior in many respects to the quadrant electrometers which usually have been used for the measurement of small currents, high resistance, or of voltage in circuits of high resistance and in those cases where only an infinitesimal current may be drawn from the source of the electromotive force. For most measurements the degree of compensation afforded has been found to be sufficient to make possible the use of dry cells or even properly controlled rectified alternating current as a power source.

Combined Reverberation Time of Electrically Coupled Rooms.[3] A. P. HILL. The importance of controlling the reverberation time of auditoriums, music rooms, etc., is well recognized, and curves showing the optimum reverberation times for buildings of different volumes have been drawn and have attained general acceptance. In the recording and reproduction of sound for talking motion pictures, however, the reverberation problem is somewhat more complex than is the case for rooms in which sound is originally produced, due to the fact that there are three factors to deal with: first, the reverberation time of the space in which the sound is recorded; second, that of the space in which it is reproduced; third, the resultant reverberation time produced by electrically coupling these two spaces together. This is, of course, done in actual practice. This paper deals with the

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