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The Practical Application of the Fourier Integral [1]

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ABSTRACT: The growing practical importance of transients and other non-periodic phenomena makes it desirable to simplify the application of the Fourier integral in particular problems of this kind and to extend the range of problems which can be solved in closed form by this method. Unless the physicist or technician is in a position to evaluate the definite integrals which occur, by mechanical means, he is usually entirely dependent upon the results obtained by the professional mathematician. To facilitate the use of the known closed form evaluations of Fourier integrals many of them have been compiled and tabulated in Table I. They are presented, however, not as definite integrals but as paired functions, one function being the coefficient for the cisoidal oscillation (or complex exponential) and the other function the reciprocally related coefficient for the unit impulse. This arrangement simplifies the table and promises to be most convenient in practical applications, since it is the coefficients of which immediate use is made, just as in the case of the Fourier series. Applications of the tabulated coefficient pairs to 85 transient problems are given, together with all necessary details, in Table II.

INTRODUCTION

THE Fourier integral and the Fourier series are alternative expressions of the Fourier theorem, the series being a limiting case of the integral and vice versa. Usually the theorem is approached from the side of the series, but there are also advantages in the approach from the integral side, which is the method followed in this paper. The generality and importance of the theorem is well expressed by Kelvin and Tait who said: ". . . Fourier's Theorem, which is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance." For any real understanding of the theorem it is necessary to appreciate why it is one of the most beautiful mathematical results and why it furnishes an indispensable instrument in physics.

The Fourier integral is a most beautiful mathematical result because of the economy of means employed in obtaining a most general result.

[1] Presented September 13, 1927, in preliminary form at the International Congress of Telegraphy and Telephony in Commemoration of Volta.

One form of integral is used both to analyze and to synthesize. In both cases it is the product of the arbitrary function and the elementary sinusoidal oscillation which is integrated. This achieves the mathematical counterpart of spectrum analysis and spectrum synthesis. The functions resulting from analysis and synthesis stand in a mutually reciprocal relation.[2] They are paired with each other. Either of these functions may be assigned with an astonishing degree of arbitrariness. Singular cases being excepted, the mate function is then determined uniquely and definitely by the integral. While the sine, cosine and complex exponential are most commonly used as the elementary expansion functions, an entire class of functions present the same fundamental relations and find applications in the more recondite problems.

The Fourier integral is an indispensable instrument in connection with physical systems in which cause and effect are linearly related (so that the principle of superposition holds) because it gives at once an explicit formal solution of general problems in terms of the solution for the sinusoidal case which is often readily found. This explicit general solution makes use of two Fourier integrals, one for the spectrum analysis of the arbitrary cause and the other for the spectrum synthesis of the component sinusoidal solutions. No further consideration of the actual physical system is necessary after the elementary sinusoidal solution has been obtained. This point of view has become a part of our general background of thought.

Unfortunately the actual evaluation of specific Fourier integrals in closed form presents formidable if not insuperable difficulties. Only a small number of distinct general integrals have been evaluated in closed form in the century and more which has elapsed since the Fourier integral discovery was announced. Additions to the list of evaluated Fourier integrals can ordinarily be made only by the professional mathematician. Unless the physicist or technician is in a position to evaluate Fourier integrals by mechanical means, or is satisfied to employ infinite series or other infinite processes in place of the definite integrals, he is usually entirely dependent upon the evaluations which the professional mathematician has made in the past or is able to make for his special use. On this account, it is often desirable to so formulate practical problems that only evaluated Fourier integrals will occur. It would be well for the physicist and technician to become familiar with the Fourier integral evaluations which the professional mathematician has achieved.

[2] The fundamental importance of the Fourier integral may be associated with an analogy which exists between the integral and the imaginary unit, both considered as operators. In both cases two iterations of the operation merely change a sign and four iterations completely restore the original function.

It is the purpose of this paper to take the first steps towards the preparation of two tables, one giving the evaluations of Fourier integrals and the other giving the sinusoidal solutions for physical systems. Together they would reduce the practical application of the Fourier integral to the selection of three results from these two tables. Thus by means of the first table the arbitrary cause could be resolved into a sum of sinusoidal causes; by means of the second table the solutions for these sinusoidal causes could be supplied; and, finally, by means of the first table again, the effect of superposing these sinusoidal solutions could be shown, and thus the answer to the original problem would be given.

The preparation of the tables calls primarily for a compilation of the results already obtained by pure analysis, after which new evaluations and new solutions should be added, in so far as is possible. No attempt has yet been made to completely cover the existing literature on the subject, which extends back over one hundred years and is extensive and widely scattered. But sufficient has been done to show that the forms of the tables which are proposed are most convenient for practical application.

Paired Coefficients—Terminology

The Fourier integral theorem has been expressed in several slightly different forms to better adapt it for particular applications. It has been recognized, almost from the start, however, that the form which best combines mathematical simplicity and complete generality makes use of the exponential oscillating function $e^{i2\pi ft}$. More recently the overwhelming advantage of using this oscillating function in the discussion of sinusoidal oscillatory systems has been generally recognized. It is, therefore, plain that this oscillating function should be adopted as the basic oscillation for both of the proposed tables. A name for this oscillation, associating it with sines and cosines, rather than with the real exponential function, seems desirable. The abbreviation cis x for ($\cos x + i \sin x$) suggests that we name this function a cis or a cisoidal oscillation. This term is tentatively employed in this paper. The notation $\text{cis}(2\pi ft)$ is also employed where it is desired to use an expression which is essentially one-valued, which avoids the use of exponentials, or which suggests periodic oscillations by its connection with cosine and sine.[3]

[3] Since the cisoidal oscillation is simply a uniform rotation at unit distance about the origin in the complex plane, it may be desirable to try some compact notation which directly suggests this rotation; for example, $\text{ru}(ft)$, 1^{ft}, i^{4ft} might be defined as the complex quantity obtained by rotating unity through ft complete turns or $4ft$ quadrants.

In a table of Fourier integrals, every integral expression would then contain, in addition to the arbitrary function $F(f)$, the same oscillating function $\text{cis}(2\pi ft)$, the same integral sign with limits $-\infty$, $+\infty$ and the same differential df. To repeat any such group of a dozen characters in each of several hundred entries seems quite unnecessary. It is, therefore, proposed merely to tabulate the arbitrary function $F(f)$ and the value $G(t)$ for the evaluated integral expressed as a function of the time. The table is thereby reduced to two parallel columns of associated functions, one of which is employed as the coefficient of the elementary cisoidal function while the other is a function of the independent time variable. The table would, however, be more symmetrical if both of the associated functions could be regarded as coefficients of an elementary function. This may be done by introducing the unit impulse as an elementary function, the impulse occurring at the epoch g at which instant it presents a unit area whereas its value is zero for all time before and all time after the epoch g. This is an essentially singular function and to recognize this fact it will be designated by $\mathbf{S}_0(t-g)$ which is intended to emphasize the singularity. The time function may now be replaced in the table by the same function of the parameter g, since the time function $G(t)$ is equal to the integral with respect to g between infinite limits of the product $G(g)\mathbf{S}_0(t-g)$.

The table of Fourier integrals has now become also a table of paired coefficient functions. This means that if the coefficient $F(f)$ is employed with the cisoid, and the coefficient $G(g)$ is employed with the unit impulse, and both products are summed for the entire infinite range of their parameters f and g, the same identical resulting time function is obtained.[4] Taken in connection with their respective elementary functions, the two associated coefficient functions are, therefore, equivalent, alternative ways of representing a particular time function. This is the fundamental geometrical or physical point of view which is needed in connection with the practical application of the Fourier integral theorem. For this reason the table has been headed a table of Paired Coefficients; as explained above, however, it may equally well be considered to be a table of Fourier Integrals.

There is another fundamental reason for placing both of the functions $F(f)$ and $G(g)$ on the same footing as coefficients. It is this:

[4] The use of frequency and epoch as the two parametric variables gives us many symmetrical formulas where, if the radian frequency were employed, an unsymmetrical 2π would occur. In practical applications the frequency of the coefficient pairs becomes the frequency which is ordinarily employed in acoustics, in music and in commercial alternating currents. The basic unit for frequency is the reciprocal second; the unit for epoch is the second.

Fourier's fundamental discovery was that the two functions may be transposed in the Fourier integral if the sign of one of the parameters is reversed. Thus, either one of the two functions constituting any coefficient pair may be taken as the coefficient of the cisoidal oscillation, provided only that the proper sign is given the epoch parameter occurring in the other function. For this reason also both functions are thus quite properly regarded as coefficients.

It is found convenient to call each coefficient of a coefficient pair the mate of the other coefficient, pair and mate being employed just as in the case of gloves. To find the mate of a glove, it is necessary to know all about the given glove including the fact as to whether it is the right or the left one of the pair. In the same way, to find the mate of a coefficient function, it is necessary to know not only the form of the function, but, in addition, whether its variable is the frequency or the epoch. The notation $\mathfrak{M}G(g)$, $\mathfrak{M}F(f)$ will be employed to indicate the mate of the particular coefficient $G(g)$, $F(f)$.

We have now defined and explained the proposed terminology for use in the practical application of the Fourier integral theorem. Before proceeding to practical applications, it is desirable to become familiar with these coefficient pairs considered in their own right. We may well begin by reminding the reader of the dissimilarity between the elementary oscillations.

The Two Elementary Functions Contrasted

The dissimilarity between the two elementary functions of the time, the cisoidal oscillation $\text{cis}(2\pi ft)$ and the unit impulse $\mathfrak{S}_0(t-g)$ is most striking. This is clearly shown by the wire models of Fig. 1 where each function is depicted for five values of its parameter. For the value zero the cisoidal oscillation degenerates into an infinite straight line parallel to the time axis and cutting the real axis at $x = 1$. For the same value zero of its parameter g, the unit impulse is zero everywhere except at the origin where it has a vanishingly narrow loop extending to $x = +\infty$.

For other values of the parameter, the cisoidal oscillation is always an infinite cylindrical helix, centered on the time axis, and passing through the point $x = 1$, while the infinite loop of the impulse function is displaced unchanged along the time axis to $t = g$. For positive values of the parameter f, the cisoidal oscillation is a right-handed helix, with pitch equal to f^{-1}, and thus decreasing as f increases. For negative values of f, the pitch is the same but the helix is left-handed.

Both functions have essential singularities, which are quite different both in character and in location. For the cisoidal oscillation the singularity is always located at $t = \infty$; for the impulse the singularity is at $t = g$.

The fundamental differences between the two elementary time functions adapt them for different uses. It is desirable to be in a position to employ first one and then the other, shifting from one to the other without any trouble or delay, so that at each step of a problem the elementary function best suited for use may be employed.

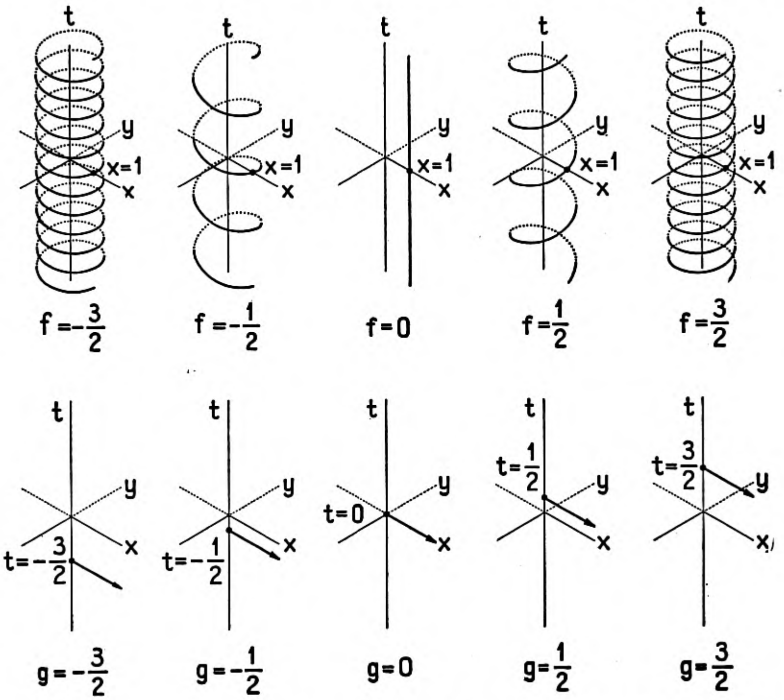

Fig. 1—Wire models of cisoidal oscillations cis $(2\pi ft)$ (above) and of unit impulses $\mathfrak{S}_0(t-g)$ (below) for the particular values 0, $\pm 1/2$, $\pm 3/2$, of the parameters f and g.

For this we require only an adequate table of pairs and a certain familiarity in the use of the pairs. It is desirable to acquire the habit of thinking of the coefficients of a pair as alternative representations of a curve.

The Use of Table I for Obtaining Coefficient Pairs[5]

The table is divided into nine parts. In Part 1 are given the general processes for deriving any coefficient mate; but such processes are to

[5] Five other closely related uses may be made of Table I as explained in the first footnote to that table. Operational expressions are brought within the scope of the table by substituting for the operator $p = d/dg$ the particular value $i2\pi f$, other possible interpretations of the operator, if any, being ignored.

be employed only when it is necessary to start from first principles. All mates which have once been determined may be taken from the latter sections of the table with a great saving of time and energy. Part 2 of the table shows the elementary transformations and combinations of pairs; these theorems may be employed either to extend a given table of coefficient pairs, or to cover a given group of coefficients with a shorter table of specific pairs. It is assumed that anyone desiring to make serious use of the table will first become familiar with these elementary combinations and transformations; even the simple addition, factor and transposition theorems (201), (205), (217) are most useful.

Part 3 of the table contains seven pairs, which are called key pairs because all specific pairs listed in the entire table may apparently be derived from them by specialization or by passing to a limit after any necessary use has been made of the elementary combinations and transformations of Part 2, amplified, as indicated, by the removal of certain unnecessary restrictions to real quantities. If an assigned coefficient is not included in Part 3 as thus generalized, then this coefficient cannot be found anywhere in Table I. Part 3, therefore, serves the useful purpose of giving a bird's-eye view of the entire table. The seven pairs are presumably redundant as they stand.

For applicational purposes it is most desirable to have a table which lists the precise pair required; many special cases which have been used in practical applications may be found in Parts 4–9 of Table I which constitute a short classified list of particular cases. It is important to remember that a given coefficient should be looked for on the other side of the table if it is not found on its own side since all pairs are transposable by (217) or (218). In the tables as they stand, some pairs have been transposed, but this is not true in the majority of cases.

Whenever an infinite process is to be employed, such as infinite series, integration or differentiation, the permissibility of the process is a question which must be answered for the particular case in hand; the formal result given in Table I may break down, for example, if either the original or the transformed pair is a singular pair. This general warning necessarily applies to every part of the subject of coefficient pairs just because it is a part of the general subject of mathematical analysis.

It is intended that the statement of each pair in the entire table shall eventually include every limitation and every warning which the mathematical sponsors for that pair would consider necessary to guarantee its safe use by anyone understanding the fundamental nature of coefficient pairs. A beginning has been made by specifying the branches of multiple-valued functions and the method of approaching limits. When this has been fully carried out, any pair may be taken from the table and used without the least concern as to the analytical methods by which the validity of the pairs has been established. Thus the finished table will make possible a complete separation of the analytical evaluation of all known Fourier integrals from their practical applications.

Having now explained, in a general way, the use of Table I, it will be useful to consider in detail a limited number of the pairs which are of special practical interest.

General Processes for Deriving the Mate

The table is naturally headed by the two fundamental Fourier integrals (101), (102) because of their intrinsic importance as explicit and implicit definitions of coefficient mates. The chief purpose of the table, however, is to make it possible for the technical man to make the fullest use of coefficient pairs without concerning himself at all as to the analytical work of evaluating either of these Fourier integrals. Pairs (101) and (102) are thus intended to serve mainly as definitions for the pairs which follow.

The statement has been made that essentially only one Fourier integral has been evaluated by determining the indefinite integral and substituting the integration limits. Whether or not this is precisely true, the statement does illustrate the fact that the formulation of the Fourier integral does not in itself suggest a practical finite analytical process for the actual evaluation of the definite integral. No such system of evaluating definite integrals is known. Writing down the Fourier integral amounts to little more than definitely formulating a question.

If the coefficient $F(f)$ is expanded as a finite or infinite series in powers of f (or p), the mate is given by pair (106*), and this involves a finite or infinite series of essentially singular functions which are further considered below in connection with Fig. 3. If a series expansion of $F(f)$ is made in terms of any functions of f for which the mates are known, there is a corresponding series for the mate. Some of these pairs are shown as (104*)–(112). The possibility of the formal infinite expansion does not necessarily imply the convergence of the series in the case of coefficient pairs any more than in other general developments.

The technical man is not ordinarily a master of infinite series, definite integrals or other infinite processes. It is, therefore, highly desirable to give him coefficient pairs which are in closed form, that is, involve only a finite number of operations with known functions. Accordingly, the portion of the table expressible in closed form has seemed to be the part which should be developed first. Specific pairs requiring infinite series for their expression have not been included in this preliminary draft of Table I. The omission of these series and of other infinite processes does not signify any failure to appreciate their importance. It is intended to include specific infinite series later.

The Elementary Transformations of Coefficient Pairs

The simple addition theorem (201) is of the greatest practical importance. The summation may include any number of pairs; they may be quite unrelated, or they may be the successive terms of power expansions as shown in (106*)–(111*). Next to the addition theorem we may place the multiplication theorem (202) or (203), special cases of which are of great practical importance. Among these special cases are (206)–(211) where any coefficient is multiplied or divided by its parameter or by a cisoidal oscillation of its parameter.

Any real linear substitution for the frequency and epoch parameters is made possible by the simple transformations (205)–(207), (214). The generalization of these transformations by the removal of the restriction to real numbers is allowable in important cases as is indicated by the parameters shown in square brackets with each pair of Part 3.

The differentiation and integration of coefficients with respect to the frequency, epoch or other parameter give the important transformations (208)–(213).

Some of the simple transformations continue to yield new results when they are repeated any number of times or when several transformations are combined in sequence. Pairs (216), (218)–(222) are examples of such combinations. All pairs in Parts 4–9 of this table may apparently be derived from the seven key pairs of Part 3 by means of these transformations employing complex parameters as indicated in Part 3, and passage to a limit in certain cases.

The resolution of pairs into the four types of i^n-multiple pairs, as shown by pairs (223)–(225), throws considerable light on the nature of coefficient pairs.

Some of the elementary properties of pairs are expressed in words as follows:

Elementary Properties of Pairs

(1) The sum or difference of pairs is a pair. Cf. pair (201).

(2) Any constant multiple of a pair is also a pair. Cf. pair (204).

(3) Any linear combination of pairs is also a pair. Cf. pairs (201), (204).

(4) The odd and even parts of every pair are also pairs.

(5) If both coefficients of a pair are real, both are even.

(6) If a pair has one real and one pure imaginary coefficient, both are odd.

(7) If a coefficient is even and real, its mate is also even and real.

(8) If a coefficient is odd and real, its mate is odd and pure imaginary, and vice versa.

(9) If a coefficient is real, its mate has conjugate values for opposite values of its parameter and conversely. Cf. pair (216).

(10) The conjugates of the coefficients of a pair are also a pair provided the sign of either frequency f or epoch g is reversed. Cf. pair (215).

(11) A pair with the signs of both frequency f and epoch g reversed is also a pair. Cf. pair (214).

(12) The parameter of either coefficient may be multiplied by a positive real constant provided the other parameter and coefficient are each divided by the same constant. Cf. pair (205).

(13) Coefficients of a pair may be interchanged if, when interchanging the parameters, the sign of one parameter, either f or g, is reversed. Cf. pair (217).

(14) Any pair may be resolved uniquely into the sum of four pairs by pairing together: the even, real parts; the even, imaginary parts; the odd, real part of each coefficient with the odd, imaginary part of the other coefficient.

(15) A pair may have the form $(F(f), \lambda F(g))$ where the multiplier λ is constant, if and only if λ has one of the four unit values $(1, i, -1, -i)$. Such a pair is called an i^n-multiple pair. Cf. pair (223).

(16) Any i^n-multiple pair has both coefficients odd or even according as n is odd or even.

(17) Any i^n-multiple pair with complex coefficients may be resolved into two i^n-multiple pairs with coefficients which are real or pure imaginary.

(18) The coefficients of any two i^n-multiple pairs are orthogonal if the i^n multipliers are different.

(19) The coefficients of any four i^n-multiple pairs with different i^n multipliers are linearly independent.

(20) Any pair may be resolved uniquely into the sum of four i^n-multiple pairs; i.e., pairs of the form $F_n(f)$, $i^n F_n(g)$. Cf. pair (224).

(21) Any pair may be resolved uniquely into the sum of eight i^n-multiple pairs where $F_n(f)$ is real or pure imaginary. Cf. pair (225)

Pairs Based on the Normal Error Law

The identical pair (703), $\exp(-\pi f^2)$, $\exp(-\pi g^2)$, is one of the simplest pairs and may well serve as the starting point in the consideration of specific coefficient pairs. Each coefficient is the broad impulse of the normal er ror law. It is remarkable that identical coefficients of this simple form should produce the same identical function when associated with either the cisoidal oscillation or the very different unit impulse.

If the differential transformation (222), taking the upper signs, is applied to the normal error law pair (703), the infinite series of ϕ_n pairs (702) is obtained. Of these derived pairs, the first eight are written out as pairs (704)–(711). The cisoidal coefficients are alternately even and odd functions which oscillate in the neighborhood of the origin, each successive coefficient having an added half oscillation. The ϕ_n pair has $(n + 1)$ half oscillations. Beyond these oscillations, every coefficient in the infinite sequence decreases rapidly and asymptotically to zero in both directions. The mates of these cisoidal coefficients are identically the same except for a constant coefficient which is i^n and thus goes cyclically through the four values, 1, i, -1, $-i$.

The $\phi_n(x)$ functions are shown by Fig. 2. They are essentially the parabolic cylinder functions of order n. These coefficients may be used for the expansion of every function which, with its first two derivatives, is continuous for all positive and negative values of the variable and for which a certain integral exists. This expansion is known as the Gram-Charlier series, which appears in pair (112).

Starting again with the normal law of error pair (703) in the form (701) and setting $\rho = \frac{1}{4}\beta/\pi$, and applying the differential transformation (208) repeatedly, we obtain the infinite sequence of pairs (713) of which the first five are listed as pairs (714)–(718). The cisoidal coefficients are the successive integral powers of p multiplied by the normal error exponential. The impulse coefficients are essentially the ϕ_n functions multiplied by the normal error exponentials. These pair, are plotted in Fig. 3 for the special case $\beta = a^2 = 1$.

Both of the infinite series of pairs derived from the error function and shown in Figs. 2 and 3 are regular throughout, are nowhere infinites and vanish at infinity.

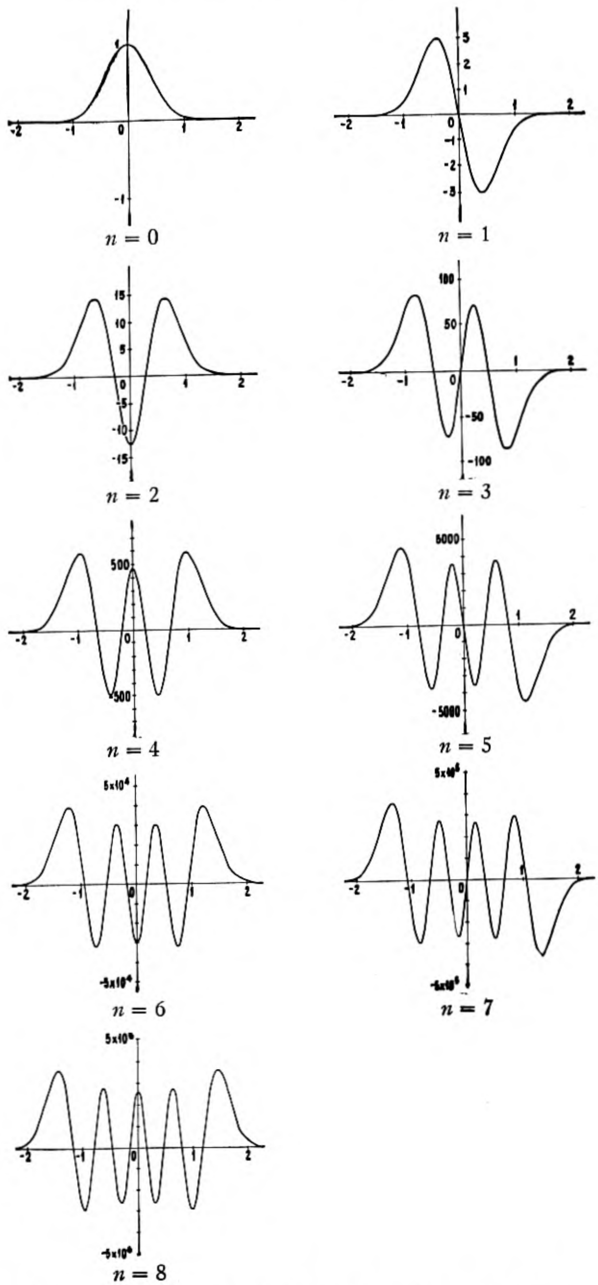

Fig. 2—Curves showing the $\phi_n(x)$ functions for $n = 0, 1, 2, \cdots, 8$
$\phi_n(x) = \exp(\pi x^2) D_x{}^n \exp(-2\pi x^2)$.

Essentially Singular Pairs for Integral Powers of the Parameter

If in Fig. 3, with the value of n held fixed, we allow a to approach the limit 0, the cisoidal coefficient becomes p^n and the impulse coefficient, which is compressed horizontally towards the origin and expanded vertically, with corresponding areas increasing as a^{-n}, ultimately vanishes everywhere except at the origin where it acquires an essential oscillating singular point. At the limit, then, a singular pair is obtained; it will be designated as p^n, $\mathfrak{S}_n(g)$. $\mathfrak{S}_n(g)$ is characterized by having all of its moments about the origin vanish except the nth moment, which is equal to $(-1)^n n!$ The dotted graphs on the left of Fig. 3 show p^n to the scales indicated. The curves on the right show $\mathfrak{S}_n(g)$ provided we assume that the horizontal scale is increased with a and the vertical scale increased inversely with a^{n+1} as a approaches the limit 0. Fig. 3 thus serves to picture the essentially singular function $\mathfrak{S}_n(g)$. That is, it is sufficient if the coefficient maintains this form while proceeding to the limit. This form is, however, not essential. It is necessary only that the method of approach to the limit give the same set of moments.

An alternative way of deriving the mate for the positive integral powers p^n is by means of a linear combination of $(n+1)$ pairs of the form of (603) with parameters equal to a, $2a$, $3a$, $\cdots$, $(n+1)a$, respectively, so that the first term in the power series expansion of the cisoidal coefficient is p^n. The corresponding impulse coefficient is a succession of $(n+1)$ bands, each of width a, the first band beginning at epoch zero, the heights of the successive bands being equal to the binomial coefficients for power n divided by a^{n+1} but alternately positive and negative. The mth moment of this impulse coefficient is 0 for $m < n$, equal to $(-1)^n n!$ for $m = n$, and proportional to a^{m-n} for $m > n$. Upon allowing a to approach zero, the cisoidal coefficient approaches p^n, and the impulse coefficient approaches $\mathfrak{S}_n(g)$, since in the limit the same set of moments is obtained as was found above to characterize the nth singularity function. This is pair (402*).

The special cases for $n = 0$, 1 are of most frequent occurrence. They are pairs (403*), (404*). $\mathfrak{S}_0$ is the unit impulse since its 0th moment equals unity; $\mathfrak{S}_1$ is the doublet with the moment -1 since its first moment is -1. $\mathfrak{S}_1$ and all higher order singular functions are included in the series coefficients of (104*), (106*).

Fig. 3 may be extended upward step by step from the normal error law pair by dividing by p on the left and integrating with respect to g on the right. At each step a constant of integration is introduced. The first two pairs thus obtained are pairs (725*) and (726*). Choos-

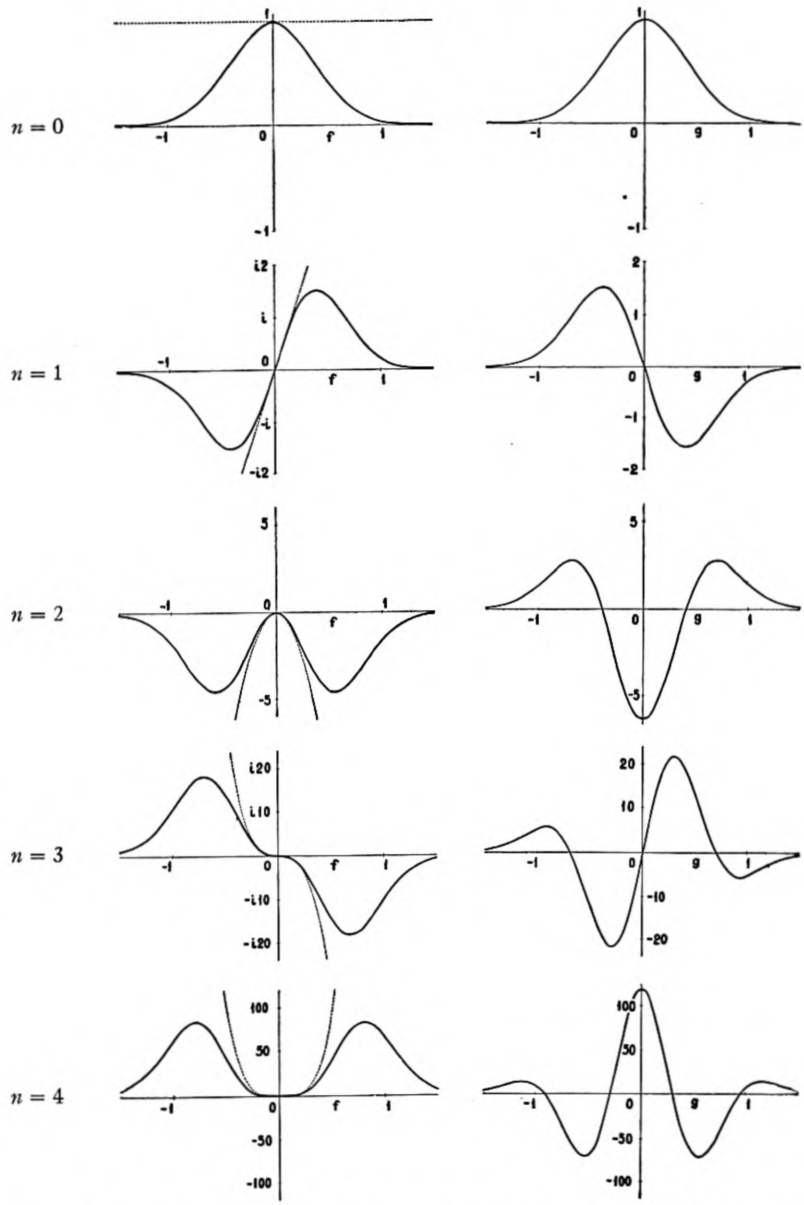

Fig. 3—Graphs for the family of pairs $p^n \exp(-\pi a^2 f^2)$ $a^{-1} D_g^n \exp(-\pi g^2/a^2)$. The heavy curves show the cases $a = 1$, $n = 0, 1, 2, 3, 4$; the dotted curves on the left are for the same values of n but for the limit $a \rightarrow 0$. On the right the curves apply for any value of a if the horizontal and vertical scales are multiplied by a and a^{-n-1} respectively.

ing the integration constants so as to make the impulse coefficients alternately odd and even, these two pairs are as shown in Fig. 4. If we now allow a to approach the limit zero, a new series of pairs is obtained of which the first two pairs are shown dotted in Fig. 4 for the particular choice of integration constants there made. The general limiting pair is designated as p^{-n}, $\mathfrak{S}_{-n}(g)$ and it is shown with its n arbitrary parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$ as pair (410*). In some ways it is simpler to derive the limiting pair for negative integral powers of p from rational functions of p, which may be accomplished as shown by pair (411*). Special cases are shown by pairs (408*), (409*), (415*), (416*).

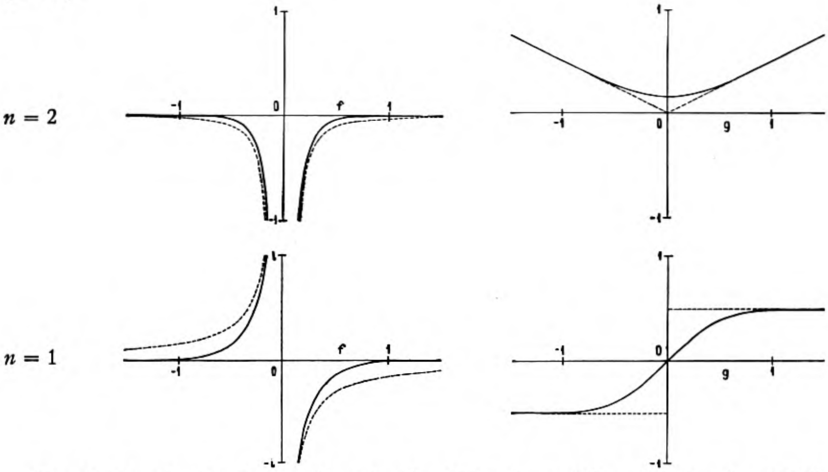

Fig. 4—Graphs for the family of pairs $p^{-n}\exp(-\pi a^2 f^2)$, $a^{-1}D_g^{-n}\exp(-\pi g^2/a^2)$, with the integration constants chosen so as to make the impulse coefficients alternately odd and even. The heavy curves show the cases $a = 1, n = 1, 2$; the dotted curves show the limit $a \rightarrow 0$, $n = 1, 2$.

The first of the series $\mathfrak{S}_{-1}(g)$ is a unit step at epoch 0 from a constant value $\lambda - \frac{1}{2}$ for all negative epochs to the constant value $\lambda + \frac{1}{2}$ for all positive epochs. The constant λ may have any value; this is a singular case marked by the failure of the general rule that the choice of the cisoidal coefficient uniquely determines the impulse coefficient. This means that in any well set problem some other condition determines the value of the constant λ. In some problems, for example, it is necessary that the epoch coefficient be an odd function, and then λ vanishes. In other problems where either the epoch function must be zero for all negative epochs or on the other hand the p occurring in the cisoidal coefficient is actually the limit of $p + a$ as a approaches zero through positive values, the constant λ equals $\frac{1}{2}$. This limiting condition may arise if we assume that resistance may be ignored, as a first approximation, in studying actual systems which necessarily involve at least a small amount of dissipation.

The mates of positive and negative integral powers of p, including the zero power, cannot be derived directly and definitely from the Fourier integral (101) without the specification of an additional passage to a limit. Such pairs therefore differ essentially from the great body of regular pairs where the choice of one coefficient completely determines the mate. In order to permanently ear-mark these limiting pairs, their serial numbers in Table I bear a star. These pairs may be thought of as lying on the periphery of the great domain which includes the totality of regular pairs.

Identical Mates and Other Simply Related Mates

Since one of the coefficients of a pair may be assigned quite arbitrarily, this choice allows us, if we so elect, to specify some relation between the two coefficients of a pair. We might specify that a linear combination $\lambda F_j(x) + \mu G_j(x)$ of the two coefficients of a pair both taken with the parameter x is to equal an arbitrary function $F(x)$. The pair (F_j, G_j) is then uniquely determined, unless $\lambda + i^n\mu = 0$, being equal to pair (224) after each F_n has been divided by $\lambda + i^n\mu$. Again if it is specified that one coefficient is to be the reciprocal of the other, a possible solution is pair (760).

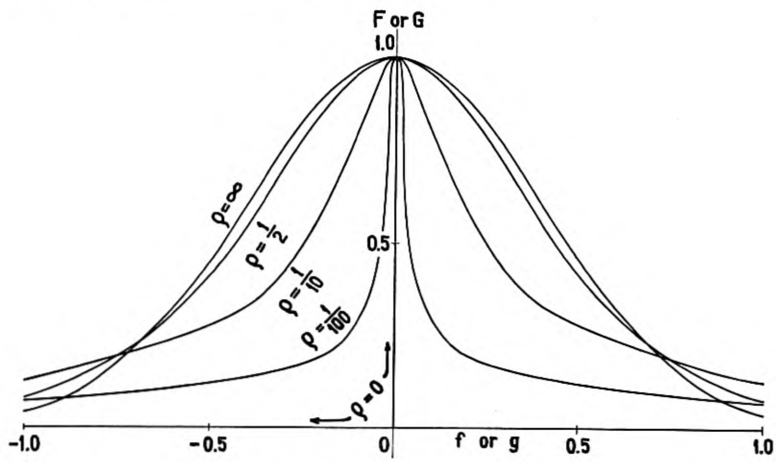

Fig. 5—Identical coefficient pairs of the form
$(1 + x^2/\rho^2)^{-\frac{1}{8}}K_{\frac{1}{4}}(2\pi\rho^2\sqrt{1 + x^2/\rho^2})/K_{\frac{1}{4}}(2\pi\rho^2)$, $x = f$ or g.

The condition that the mates shall be identically the same function of their parametric variables f and g is of special interest. In addition to the identical pairs shown on Fig. 2, $n = 0, 4, 8$, the table contains a number of identical pairs including (523), (625), (712), (761), (916).

The identical pair (916) divided by its value at the origin is shown in Fig. 5 for different real values of its parameter ρ. For $\rho = +\infty$, the curve is of the $\exp(-\pi x^2)$ or normal law of error form, and is identical pair (703). For $\rho = \frac{1}{2}$, the reciprocal hyperbolic cosine identical pair (625) is shown correctly within the width of the line, this being apparently a mere coincidence since pair (916) does not include it as a special case. Finally, for $\rho = 0$, the limiting curve coincides with the horizontal axis taken together with unit length of the positive vertical axis. This represents pair (523) divided by its value at the origin, which is infinite. The point to be especially noted is that the area under every curve of the family illustrated by Fig. 5 is the same and equal to unity. This must hold for the limit $\rho = 0$, when the curve encloses no area within a finite distance of the origin.

The identical pair $|f^{-\frac{1}{2}}|$, $|g^{-\frac{1}{2}}|$ is of great simplicity and it occupies a central position among algebraic pairs. Starting with the minus one-half power of the parameters in both coefficients, any increase in the power of one parameter requires an equal decrease in the power of the other parameter as is illustrated, for example, by pairs (502*), (516*), (524).

It is not permissible to specify any relation whatsoever between the two coefficients of a pair; for example, no pair exists for which one coefficient is twice the other. As stated above, the only multiples permissible are the four units 1, i, -1, $-i$. For each of these four cases there are an infinite number of solutions. These solutions satisfy the integral equations given in the foot-note to pair (223).

Practical Applications of Coefficient Pairs

Fourier gave the first comprehensive method of finding the solution for transients. His method involves three steps: viz.,

I. Spectrum analysis of the cause among all frequencies.
II. Solution for all frequencies.
III. Spectrum synthesis of the effects for all frequencies.

Fourier thus substituted three problems for one. With a table of Fourier coefficient pairs, these three steps may be made as follows:

I. Find the mate of the cause considered as an impulse coefficient.
II. Multiply this mate by the admittance for the system.
III. Find the mate of this product considered as a cisoidal coefficient.

These three steps define a perfectly definite result, since every arbitrarily chosen coefficient has a mate which is unique and determinate, or may be made so by the specification of some suitable passage to a limit.

The use of a table of pairs may also be stated in another and somewhat more general way as follows:

For any system where the principle of superposition holds, any cause $C(t)$, its effect $E(t)$ and the corresponding admittance $Y(f)$ are connected by a relation which may be written in any one of three ways which explicitly express each of the three quantities in terms of the remaining two, as follows:

$$E(g) = \mathfrak{M}[Y(f)\mathfrak{M}C(g)],$$

$$C(g) = \mathfrak{M}\left[\frac{\mathfrak{M}E(g)}{Y(f)}\right],$$

$$Y(f) = \frac{\mathfrak{M}E(g)}{\mathfrak{M}C(g)},$$

where $\mathfrak{M}$ is read "mate of."

The use of coefficient pairs may be most simply illustrated by reference to Figs. 3 and 4, in connection with the problem of finding transient currents through a perfect condenser of unit capacity due to impressed electromotive forces shown by each of the seven curves on the right considered as functions of the time. Any curve on the right being the cause, the next curve below it is the effect, considering Fig. 4 to be placed above Fig. 3. In the solution the first step is to find the mate of the curve on the right. This is the curve on the left. This mate is then to be multiplied by the admittance of the system which is p for a unit condenser. Reference to the titles of the figures shows that this product is given by the next lower curve on the left. To find the mate of this last curve is the third step in the solution and for this it is merely necessary to go to the curve on the right. The three steps then take us from any curve on the right to the next curve below it. Figs. 3 and 4, taken together, are a section of an infinite sequence of pairs which illustrate an infinite number of possible transients in a perfect condenser of unit capacity.

If, on the other hand, the system consisted of a perfect reactance coil of unit inductance and the impressed cause was again shown by any curve on the right, the effect would be shown by the next higher curve, assuming that the initial current at the beginning of time was that shown by the extreme left of the upper curve. Thus, when the cause is oscillating, there is one less half oscillation in the effect than in the cause. This is for an inductance. For a condenser, conditions are reversed; the effect has one more half oscillation than the cause.

The scales of Figs. 3 and 4 may be changed to correspond to any value of a, the parameter which appears in the coefficients of the pairs.

At the limit $a = 0$, the cause and effect would be the singular $\mathfrak{S}_n$ or $\mathfrak{S}_{-n}$ functions.

The curves on the right for $n = 0$ of Fig. 3 and $n = 1$ of Fig. 4 show that at the limit $a=0$ a unit step in the voltage produces a unit impulse in the current through a unit condenser; on the other hand, a unit impulse applied to a unit inductance gives a current which is a unit step.

The curves of Fig. 2 may be used to furnish another illustration of the use of coefficient pairs, in connection with the problem of finding networks in which assigned transient currents will be produced by assigned impressed electromotive forces. Any curve n being the assumed cause and the next curve $(n + 1)$ the assumed effect, the required admittance is $\phi_{n+1}(f)/[i\phi_n(f)]$. This admittance is presented by a ladder network of $(n + 1)$ elements: perfect inductance coils in the series arms, perfect condensers in the shunt arms, the ladder starting with a shunt condenser, the values of the shunt capacities being equal to 2, $2n(n - 1)^{-1}$, $2n(n - 1)^{-1}(n - 2)(n - 3)^{-1}$, etc., and the values of the series inductances being equal to $(2\pi n)^{-1}$, $(2\pi n)^{-1}(n - 1)(n - 2)^{-1}$, etc. In verifying the solution of this problem, it is to be noticed that the mates of the curves n and $(n + 1)$, regarded as impulse coefficients, are the same curves multiplied by i^{-n} and $i^{-(n+1)}$; the quotient of the latter mate divided by the former mate is the admittance of the network as given above.

On the other hand, any curve $(n + 1)$ being the cause, the curve n is the effect in the reciprocally related ladder network of $(n + 1)$ elements, starting with a series reactance coil, the values of the series inductances being equal to 2, $2n(n - 1)^{-1}$, $2n(n - 1)^{-1}(n - 2)(n - 3)^{-1}$, etc., and the values of the shunt capacities being equal to $(2\pi n)^{-1}$, $(2\pi n)^{-1}(n - 1)(n - 2)^{-1}$, etc.

Practical Applications of Coefficient Pairs in Table II.

In general, each of the three subsidiary problems employed by Fourier is unsolvable in closed form. In a strictly limited number of cases, however, all three problems have been solved and the final transient solution obtained. These solutions should be cherished and collected for ready reference. It is a needless waste of time to repeat the analytical work each time a solution is required. Except for a few special cases lying outside of the scope of the table, all practical applications of closed form coefficient pairs which were found in a preliminary search are included in the transient solutions of Table II. As it stands, the table is far from a complete list of closed form solutions, but it contains many important solutions and serves to illustrate the use of Table I. Table II contains 39 admittances, with references to 39 systems which serve to illustrate the occurrence of these admittances. In the third, fourth and fifth columns, 85 transient solutions are given of which 39 are for the unit impulse, 30 for the unit step, and 16 for the suddenly applied cisoid.

The causes producing the transients in Table II are but three in number: the unit impulse, the unit step, and the suddenly applied cisoid; and the mates for these causes are unity, p^{-1} and $(p - p_0)^{-1}$ as is shown by pairs (403*), (415*) and (440*). Multiplying these three mates by the admittances and taking the mates of the products, we have the effects, as is stated in the headings of the last three columns of the table.

To illustrate in detail the steps involved in finding a transient effect with the aid of Table I, consider system No. 14 of Table II with the cause equal to the unit step $\mathbf{S}_{-1}(t)$, $\lambda = \frac{1}{2}$. The mate of the unit step is p^{-1} by pair (415*). Multiplying this by $Y(f)$ as given in the second column of Table II, we have $up^{-\frac{1}{2}}(1 + \sqrt{p/\lambda})^{-1}$ for the cisoidal coefficient. By pair (551) the mate of this is $u\sqrt{\lambda}\,\exp(\lambda g)\,\mathrm{erfc}\,\sqrt{\lambda g}$, $0 < g$. Substituting for g the actual variable t, we have the transient solution as given in the fourth column and fourteenth row of Table II.

This simple example fully illustrates the three essential steps in finding any transient effect when the admittance and pairs are known. In this example the effect was considered to be the unknown. If either the cause or the admittance were the unknown, the same pairs would be involved but the two coefficients in a pair would be used in the reversed sequence in all but one instance.

There are still 32 squares of Table II left blank. It would be a simple matter to place series solutions or integral solutions in each of these squares. Thus if the impulse transient of column 3 is known, the other two transients are given at once in integral form by pairs (210) and (219); if the unit step transient of column 4 is known, the suddenly applied cisoidal transient is written immediately in integral form by the use of pair (220). The real problem is, however, either to find closed form solutions in terms of known functions or to show that this is impossible. When the failure of known functions has been established, we should next consider the choice of new functions so defined as to throw as much light as possible on the new solutions.

Table II may be regarded as another table of coefficient pairs. Column 2 contains cisoidal coefficients; column 3, the mates of these coefficients; column 4, the mates of these coefficients when multiplied by p^{-1}; and column 5, the mates of these coefficients when multiplied by $(p - p_0)^{-1}$. The corresponding pair in Table I is referred to in the lower left-hand corner of each square by its serial number. In a few cases, two or three pairs are referred to and there it is necessary to add the Table I pairs together or, in the case of systems 37–39, to apply the two pairs in sequence. In Table II, the customary physical notation is adhered to because it is often of long standing and this necessitates some change in notation when comparing pairs in the two tables.

Summary and Conclusions

Many practical applications of the Fourier integral have been simplified by the compilation of Tables I and II, which give coefficient pairs, admittances and transient solutions.

Minor changes in nomenclature and point of view have been introduced, all with the idea of simplifying the practical application of the Fourier integral, in the following ways:

(1) Using the cisoidal oscillation and the unit impulse side by side as alternative elementary expansion functions.

(2) Focusing attention upon coefficient pairs for these two elementary functions, both coefficients of a pair representing the resolution of the same arbitrary function.

(3) Using the frequency and epoch as the parametric variables, in place of the customary radian frequency and independent time variable.

(4) Employing as a coefficient any real or complex arbitrary function which may be practically useful by regarding it, where necessary, as a limit approached through coefficients which form regular pairs.

(5) Introducing the $\mathfrak{S}_n(g)$ functions having an essential oscillating singularity at the origin which mate with p^n, the positive integral powers of p.

(6) Using a notation which greatly reduces the number of occasions for employing the integral symbol in applications of the Fourier theorem.

Having established the inclusiveness and practical utility of the proposed coefficient pair method of applying the Fourier integral, we are now planning to critically verify the tables and make them as complete as is feasible. It is proposed to include eventually such references to the literature as may add to the interest of the tables. The contributions of integral equations and of the operational method to the present subject will also be incorporated in the tables. The preparation of similar tables for other elementary expansion functions, such as Bessel functions, is also a possibility. A comprehensive table might be made which would include in parallel columns the coefficient functions for a large number of elementary expansion functions, thus giving at once many alternative ways of representing particular time functions. This would make it possible to shift without trouble from any one expansion to any other expansion of the tabulation.

I am under great obligations to my colleagues for their contributions towards the preparation of this paper. I shall be grateful to any person who will call my attention to errors or omissions in any part of this paper.[6]

Notation

The following notation is employed in Table I; also in Table II, except as specifically restricted.

a, b, c	= positive reals.
br x	= branch x. For each multiple-valued function, branches are designated in one or more different ways. When no branch designation is given, branch zero is to be understood.
$C(z)$	$= \int_0 \cos(\frac{1}{2}\pi z^2)dz = -C(-z)$. $C(\pm\infty) = \pm\frac{1}{2}$.
$\text{cis}(z)$	$= \cos z + i \sin z = \exp(iz) = e^{iz}$ = cisoidal oscillation if $z = 2\pi ft$.
$D_\nu(z)$	= parabolic cylinder function of order ν. $D_n(z) = \exp(-\frac{1}{4}z^2)H_n(z)$. $D_{-\frac{1}{2}}(z) = (2\pi)^{-\frac{1}{2}}z^{\frac{1}{2}}K_{\frac{1}{4}}(\frac{1}{4}z^2)$. $D_{-1}(z) = (\frac{1}{2}\pi)^{\frac{1}{2}}\exp(\frac{1}{4}z^2)\,\text{erfc}(2^{-\frac{1}{2}}z)$.
$\text{erf}(z)$	$= \frac{2}{\sqrt{\pi}}\int_0^z \exp(-z^2)dz = -\text{erf}(-z)$. $\text{erf}(\pm\infty) = \pm 1$.
$\text{erfc}(z)$	$= \frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-z^2)dz = 1 - \text{erf}(z)$.
f	= frequency; parameter for the cisoidal oscillation. $-\infty < f < \infty$.
$F(f)$	= coefficient for cisoidal oscillation, parameter f.
$F_n(f)$	= coefficient of an i^n-multiple pair $(F_n(f), i^nF_n(g))$ in pairs (223)–(225).

[6] I am already much indebted to M. Paul Lévy for a number of suggestions including the expression of the general identical pair as the sum of any pair having even coefficients and its transposed pair.

g = epoch; parameter for the unit impulse. $-\infty < g < \infty$.

$g_0 < g < g_1$ restricts the given coefficient to the indicated limits; outside these limits the coefficient is zero.

$G(g)$ = coefficient for unit impulse, parameter g.

$H_n(z)$ $= z^n - \frac{n(n-1)}{2} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4} z^{n-4} - \cdots$

= Hermite polynomial of order n.

$H_\nu^{(1)}(z)$ $= \frac{2}{\pi} i^{-\nu-1} K_\nu(-iz)$ = Bessel function of the third kind.

$H_\nu^{(2)}(z)$ $= \frac{2}{\pi} i^{\nu+1} K_\nu(iz)$ = Bessel function of the third kind.

$I(z)$ = imaginary part of z. $z = R(z) + iI(z)$.

$I_\nu(z)$ $= i^{-\nu} J_\nu(iz)$.

j, k, l = integers greater than zero.

$J_\nu(z)$ $= i^\nu I_\nu(-iz)$ = Bessel function of the first kind.

$K_\nu(z)$ $= \frac{1}{2}\pi i^{\nu+1} H_\nu^{(1)}(iz)$.

m, n = positive integers, including zero.

$\mathcal{M}(\ )$ = mate of ( ).

p $= i2\pi f$, the imaginary radian frequency.

r, s = reals, positive or zero.

$R(z)$ = real part of z. $z = R(z) + iI(z)$.

$S(z)$ $= \int_0^z \sin(\frac{1}{2}\pi z^2) dz = -S(-z)$. $S(\pm\infty) = \pm\frac{1}{2}$.

$\mathfrak{S}_v(x)$ $= \lim_{a \to \infty} a D_x^v \exp(-\pi a^2 x^2)$ = vth singularity function.

$\mathfrak{S}_{-n}(x)$ $= \left(\lambda_1 \pm \frac{1}{2(n-1)!}\right) x^{n-1} + \lambda_2 x^{n-2} + \cdots + \lambda_n$, $0 < \pm x$, $0 < n$.

t = time. $-\infty < t < \infty$.

v, w = integers, positive, negative or zero.

x, y = reals, unrestricted.

Y = admittance of system for cisoidal oscillation.

$Y_\nu(z)$ $= \frac{1}{2} i [H_\nu^{(2)}(z) - H_\nu^{(1)}(z)]$ = Bessel function of the second kind.

z	= complex quantity, unrestricted.
$\bar{z}$	= conjugate of z.
z^μ, br x	$= \exp[\mu R(\log z) + i\mu \arg z]$, where $(2x-1)\pi < \arg z \leqq (2x+1)\pi$.
	$= e^{i2\pi\mu v} z^\mu$, br$(x - v)$. Branches $(x+v)$, $v = 0, \pm 1, \pm 2, \cdots$ form a complete set and without repetition unless μ is a rational real.
$\alpha, \beta, \gamma, \delta$	= complex quantities, real parts greater than zero.
θ	= principal argument. $-\pi < \theta \leqq \pi$.
λ, μ, ν	= complex quantities, unrestricted.
ρ, σ, τ	= complex quantities, real parts not less than zero.
$\phi_n(x)$	$= \exp(\pi x^2) D_x{}^n \exp(-2\pi x^2)$
	$= (-2\pi^{\frac{1}{2}})^n D_n(2\pi^{\frac{1}{2}} x)$ where D_n is the parabolic cylinder function of order n
	$= (-2\pi^{\frac{1}{2}})^n \exp(-\pi x^2) H_n(2\pi^{\frac{1}{2}} x)$ where H_n is the Hermite polynomial of order n.
$\psi(z)$	$= \Gamma'(z)/\Gamma(z)$ = logarithmic derivate of the gamma function. $-\psi(1)$ = Euler's constant $= 0.5772\cdots$.
*	marks a pair as being the limit approached by regular pairs.

	Not Restricted	Real Part	
		$\geqq 0$	> 0
Integers	v, w	m, n	j, k, l
Reals	f, g, t, x, y	r, s	a, b, c
Complex	z, λ, μ, ν	ρ, σ, τ	$\alpha, \beta, \gamma, \delta$

TABLE I

Paired Coefficients for the Cisoidal Oscillation and the Unit Impulse[1]

Part 1. General Processes for Deriving the Mate

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation $\text{cis}(2\pi ft) = \exp(pt)$	Coefficient $G(g)$ for the Unit Impulse $\mathfrak{S}_0(t-g) = \lim_{a\to 0} \frac{1}{a} e^{-\pi(t-g)^2/a^2}$
101.	F	$\int_{-\infty}^{\infty} F(f)\text{cis}(2\pi fg)df$
102.	$\int_{-\infty}^{\infty} G(g)\text{cis}(-2\pi fg)dg$	G
103.	F	$D_g \int_{-\infty}^{\infty} F(f)\text{cis}(2\pi fg)p^{-1}df$
104.*	$\lambda_1(p-p_0) + \lambda_2(p-p_0)^2 + \cdots,$ lim by 401*	$\text{cis}\ (2\pi f_0 g)[\lambda_1\mathfrak{S}_1(g) + \lambda_2\mathfrak{S}_2(g) + \cdots]$
105.*	$\lambda_1 \frac{1}{(p-p_0)} + \lambda_2 \frac{1}{(p-p_0)^2} + \cdots,$ lim by 408*	$\text{cis}\ (2\pi f_0 g)\left(\lambda_1 + \lambda_2 \frac{g}{1!} + \lambda_3 \frac{g^2}{2!} + \lambda_4 \frac{g^3}{3!} + \cdots\right),\quad 0 < g$
106.*	$\lambda_1 p + \lambda_2 p^2 + \lambda_3 p^3 + \cdots,$ lim by 401*	$\lambda_1\mathfrak{S}_1(g) + \lambda_2\mathfrak{S}_2(g) + \lambda_3\mathfrak{S}_3(g) + \cdots$
107.*	$\lambda_1 \frac{1}{p} + \lambda_2 \frac{1}{p^2} + \lambda_3 \frac{1}{p^3} + \cdots,$ lim by 408*	$\lambda_1 + \lambda_2 \frac{g}{1!} + \lambda_3 \frac{g^2}{2!} + \lambda_4 \frac{g^3}{3!} + \cdots,\quad 0<g$
108.*	$\frac{1}{p^a}\left(\lambda_0 + \lambda_1 \frac{1}{p} + \lambda_2 \frac{1}{p^2} + \cdots\right),$ $0 < a < 1$, lim by 516*	$\frac{1}{\Gamma(a)g^{1-a}}\left[\lambda_0 + \lambda_1 \frac{1}{a} g + \lambda_2 \frac{1}{a(a+1)} g^2 + \cdots\right],\quad 0 < g$

[1] The pair for the oscillation cis $(-2\pi ft)$ and the unit impulse is $F(f)$, $G(-g)$ which differs from the tabulated pair only in the sign of the epoch; similarly, for the oscillation cos $(2\pi ft)$ or sin $(2\pi ft)$ the only change is the substitution for $G(g)$ of the even part of $G(g)$ or of the odd part of $-iG(g)$, the pairs being $F(f)$, $\frac{1}{2}[G(g) + G(-g)]$, or $F(f)$, $-i\frac{1}{2}[G(g) - G(-g)]$, respectively. Every pair in Table I may be thrown into the form of an evaluated Fourier integral by equating the pair after writing either $\int_{-\infty}^{\infty} df$ cis $(2\pi fg)$ before the coefficient $F(f)$ or $\int_{-\infty}^{\infty} dg$ cis $(-2\pi fg)$ before the coefficient $G(g)$. Every pair in Table I may also be regarded as an operational expression $F(p/i2\pi)$ of the operator $p = i2\pi f = d/dg$ with $G(g)$ its explicit expression in g.

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
109.*	$\frac{1}{p^a}(\lambda_0 + \lambda_1 p + \lambda_2 p^2 + \cdots)$, $0 < a < 1$, lim by 501*	$\frac{1}{\Gamma(a)g^{1-a}}\left[\lambda_0 - \lambda_1(1-a)\frac{1}{g} + \lambda_2(1-a)(2-a)\frac{1}{g^2} + \cdots\right]$, $0 < g$
110.*	$\frac{1}{\sqrt{p}}\left(\lambda_0 + \lambda_1\frac{1}{p} + \lambda_2\frac{1}{p^2} + \lambda_3\frac{1}{p^3} + \cdots\right)$, lim by 518*	$\frac{1}{\sqrt{\pi g}}\left[\lambda_0 + \lambda_1\frac{2g}{1} + \lambda_2\frac{(2g)^2}{1\cdot 3} + \lambda_3\frac{(2g)^3}{1\cdot 3\cdot 5} + \cdots\right]$, $0 < g$
111.*	$\frac{1}{\sqrt{p}}(\lambda_0 + \lambda_1 p + \lambda_2 p^2 + \lambda_3 p^3 + \cdots)$, lim by 502*	$\frac{1}{\sqrt{\pi g}}\left[\lambda_0 - \lambda_1\frac{1}{2g} + \lambda_2\frac{1\cdot 3}{(2g)^2} - \lambda_3\frac{1\cdot 3\cdot 5}{(2g)^3} + \cdots\right]$, $0 < g$
112.	$\lambda_0\phi_0(f) + \lambda_1\phi_1(f) + \lambda_2\phi_2(f) + \cdots$	$\lambda_0\phi_0(g) + i\lambda_1\phi_1(g) + i^2\lambda_2\phi_2(g) + \cdots$

Part 2. Elementary Combinations and Transformations

201.	$F_1 \pm F_2$	$G_1 \pm G_2$
202.[2]	$F_1 F_2$	$\int_{-\infty}^{\infty} G_1(x)G_2(g - x)dx$
203.	$\int_{-\infty}^{\infty} F_1(-x)F_2(f + x)dx$	$G_1 G_2$
204.	λF	λG

[2] From (202) or (203), with g (or f) $= 0$, and (215) and (217) follow the important identities for the integrated product of two pairs of coefficients and for the integrated squared moduli of a pair of coefficients:

$$\int_{-\infty}^{\infty} F_1(f)F_2(\pm f)df = \int_{-\infty}^{\infty} G_1(g)G_2(\mp g)dg,$$

$$\int_{-\infty}^{\infty} |F^2|\,df = \int_{-\infty}^{\infty} |G^2|\,dg,$$

$$\int_{-\infty}^{\infty} F_1(x)G_2(x)dx = \int_{-\infty}^{\infty} G_1(x)F_2(x)dx.$$

The symmetry of these identities is to be noted; this would not be the case if the radian frequency $2\pi f$ were employed in place of the cyclic frequency f.

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
205.	$F(af)$	$\frac{1}{a}G\left(\frac{g}{a}\right)$
206.	$F(f-f_0) = F\left(\frac{p-p_0}{i2\pi}\right)$	$\text{cis}(2\pi f_0 g)G = e^{p_0 g}G$
207.	$\text{cis}(-2\pi f g_0)F = e^{-p g_0}F$	$G(g-g_0)$
208.	pF	$D_g G$
209.	$D_p F = \frac{1}{i2\pi} D_f F$	$-gG$
210.	$\frac{1}{p}F$	$\int_{-\infty}^{g} G dg = D_g^{-1}G$
211.	$\int_{-i\infty}^{p} F dp = i2\pi \int_{-\infty}^{f} F df = D_p^{-1}F$	$-\frac{1}{g}G$
212.	$D_\lambda F$	$D_\lambda G$
213.	$\int_{\lambda_0}^{\lambda} F d\lambda = D_\lambda^{-1}F$	$\int_{\lambda_0}^{\lambda} G d\lambda = D_\lambda^{-1}G$
214.	$F(-f)$	$G(-g)$
215.	$\bar{F}(\pm f)$	$\bar{G}(\mp g)$
216.	$F(f) \pm \bar{F}(f)$	$G(g) \pm \bar{G}(-g)$
217.	$G(\pm f)$	$F(\mp g)$
218.	$G(\pm ip)$	$\frac{1}{2\pi}F\left(\frac{\pm g}{2\pi}\right)$
219.	$\frac{F(f)}{p-p_0}$	$e^{p_0 g}\int_{-\infty}^{g} e^{-p_0 g}G(g)dg$
220.	$\frac{p}{p-p_0}F(f)$	$G(g) + p_0 e^{p_0 g}\int_{-\infty}^{g} e^{-p_0 g}G(g)dg$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
221.	$f^{\pm\frac{1}{2}}D_f^{\pm 1}(f^{\pm\frac{1}{2}}F) = (\frac{1}{2} + fD_f)^{\pm 1}F$	$-\, g^{\pm\frac{1}{2}}D_g^{\pm 1}(g^{\pm\frac{1}{2}}G) = -\, (\frac{1}{2} + gD_g)^{\pm 1}G$
222.	$e^{\pm\pi f^2}D_f^{v}(e^{\mp\pi f^2}F) = (\mp\, 2\pi f + D_f)^{v}F$	$i^{\pm v}e^{\pm\pi g^2}D_g^{v}(e^{\mp\pi g^2}G) = i^{\pm v}(\mp\, 2\pi g + D_g)^{v}G$
223.[3]	$F_n(f)$, $n = 0, 1, 2, \cdots, 11$ where $F_n(f) = \frac{1}{4}[F(f) + i^{2n}F(-f) + i^{-n}G(f) + i^{n}G(-f)]$, $F_{n+4} = R(F_n)$, $F_{n+8} = I(F_n)$, $n = 0, 1, 2, 3$	$i^{n}F_n(g)$
224.	$F_0(f) + F_1(f) + F_2(f) + F_3(f)$ where F_n is as for 223.	$F_0(g) + iF_1(g) - F_2(g) - iF_3(g)$
225.	$F_4(f) + F_6(f) + F_5(f) + F_7(f)$ $+\, i[F_8(f) + F_{10}(f) + F_9(f) + F_{11}(f)]$ where F_n is as for 223.	$F_4(g) - F_6(g) - F_9(g) + F_{11}(g)$ $+\, i[F_8(g) - F_{10}(g) + F_5(g) - F_7(g)]$.

Part 3. Key Pairs

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
301.	$\sec p$; $\left[\begin{array}{l}\gamma(p + \lambda) \text{ for } p, \text{ if} \\ -\, \frac{1}{2}\pi R(1/\gamma) < R(\lambda) < \frac{1}{2}\pi R(1/\gamma)\end{array}\right]$	$\frac{1}{2}\,\text{sech}(\frac{1}{2}\pi g)$
302.	$p^{\frac{1}{2}(\alpha-1)}K_{\alpha-1}(\sqrt{p})$; $\left[\begin{array}{l}\gamma(p + \rho) \text{ for } p, \text{ and } \nu \text{ for } \alpha \text{ with} \\ (p + \beta) \text{ for } p\end{array}\right]$	$(2g)^{-\alpha} \exp\left(-\dfrac{1}{4g}\right)$, $0 < g$
303.	$\dfrac{1}{p}\exp\left(-\dfrac{a^2 + 1}{p}\right) I_{\alpha-1}\left(\dfrac{2a}{p}\right)$; $\left[\begin{array}{l}\gamma(p + \beta) \text{ for } p, \text{ and } \sqrt{\gamma\delta} \text{ for } a \text{ with} \\ (p + \beta) \text{ for } p\end{array}\right]$	$J_{\alpha-1}(2a\sqrt{g})J_{\alpha-1}(2\sqrt{g})$, $0 < g$
304.	$\exp(\frac{1}{4}p^2)D_{-\alpha}(p)$; $[\sqrt{\gamma}(p + \rho) \text{ for } p]$	$\dfrac{1}{\Gamma(\alpha)}\, g^{\alpha-1} \exp(-\, \frac{1}{2}g^2)$, $0 < g$

The coefficients of the i^n-multiple pairs satisfy the following integral equations:

$$F_n(f) = (-\, 1)^{\frac{1}{2}}\; 2\int_0^{\infty} F_n(g)\cos(2\pi fg)dg, \quad n = 0, 2, 4, 6, 8, 10$$

$$F_n(f) = (-\, 1)^{\frac{1}{2}(n-1)}\, 2\int_0^{\infty} F_n(g)\sin(2\pi fg)dg, \quad n = 1, 3, 5, 7, 9, 11$$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
305.	$p^{-\frac{1}{2}\alpha}\exp\left(\frac{1}{4p}\right)D_{-\alpha}\left(\frac{1}{\sqrt{p}}\right)$; [$\gamma(p+\rho)$ for p]	$\frac{1}{\Gamma(\alpha)}(2g)^{\frac{1}{2}\alpha-1}\exp(-\sqrt{2g})$, $0 < g$
306.	$(p^2+b^2)^{\frac{1}{4}-\frac{1}{2}\alpha}K_{\alpha-\frac{1}{2}}(\sqrt{p^2+b^2})$, $0 < R(\alpha) < \frac{3}{2}$; [$(p+\rho)$ for p, and δ for b without the restriction on α with $(p+\beta)$ for p, if $-R(\beta) < R(i\delta) < R(\beta)$]	$\sqrt{\frac{\pi}{2}}\left(\frac{g^2-1}{b^2}\right)^{\frac{1}{2}\alpha-\frac{1}{4}}J_{\alpha-1}(b\sqrt{g^2-1})$, $1<g$
307.	$(\delta^2-p^2)^{\frac{1}{4}-\frac{1}{2}\alpha}J_{\alpha-\frac{1}{2}}(\sqrt{\delta^2-p^2})$; [$(p+\lambda)$ for p]	$\frac{1}{\sqrt{2\pi}}\left(\frac{1-g^2}{\delta^2}\right)^{\frac{1}{2}\alpha-\frac{1}{4}}J_{\alpha-1}(\delta\sqrt{1-g^2})$, $-1 < g < 1$

Part 4. Rational Algebraic Functions of f.

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
401.*	$p^n = \lim_{a\to 0} p^n e^{-\pi a^2 f^2}$	$\mathfrak{S}_n(g) = \lim_{a\to 0}\frac{1}{a}D_g{}^n e^{-\pi a^{-2}g^2}$
402.*	$p^n = \lim_{a\to 0}\sum_{k=1}^{n+1}\frac{(-1)^{k-1}(n+1)!(1-e^{-kap})}{k!(n-k+1)!\,a^{n+1}p}$	$\mathfrak{S}_n(g)$
403.*	$1 = \lim_{\beta\to 0} e^{-\beta\lvert p-p_0\rvert}$	$\mathfrak{S}_0(g)$, unit impulse at $g = 0$
404.*	$p = \lim_{\beta\to 0} p e^{-\pi\beta f^2}$	$\mathfrak{S}_1(g)$, negative unit doublet at $g = 0$
405.*	$\lvert p^{2n}\rvert = \lim_{\beta\to 0} D_\beta{}^{2n}e^{-\beta\lvert p\rvert}$	$(-1)^n\mathfrak{S}_{2n}(g)$
406.*	$\lvert p^{2n+1}\rvert = -\lim_{\beta\to 0} D_\beta{}^{2n+1}e^{-\beta\lvert p\rvert}$	$\frac{(-1)^{n+1}(2n+1)!}{\pi g^{2n+2}}$
407.*	$\lvert p\rvert$, lim by 406*	$-\frac{1}{\pi g^2}$
408.*	$p^{-n} = \lim_{\beta\to 0}(p+\beta)^{-n}$, $0 < n$	$\frac{g^{n-1}}{(n-1)!}$, $0 < g$
409.*	$p^{-n} = \lim_{\beta\to 0}(p-\beta)^{-n}$, $0 < n$	$\frac{-g^{n-1}}{(n-1)!}$, $g < 0$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
410.*	$p^{-n} = \lim_{a \to 0} p^{-n} e^{-\pi a^2 f^2}$, $0 < n$	$\mathfrak{S}_{-n}(g) = \lim_{a \to 0} \frac{1}{a} D_g^{-n} e^{-\pi a^{-2} g^2}$ $= \left(\lambda_1 \pm \frac{1}{2(n-1)!}\right) g^{n-1} + \lambda_2 g^{n-2} + \lambda_3 g^{n-3} + \cdots + \lambda_n$, $0 < \pm g$
411.*	$p^{-n} = \lim_{\beta \to 0} \left[\frac{\frac{1}{2}}{(p-\beta)^n} + \frac{\frac{1}{2}}{(p+\beta)^n} + \sum_{k=1}^{n} \left(\frac{\lambda_k (n-k)!}{(p+\beta)^{n-k+1}} - \frac{\lambda_k (n-k)!}{(p-\beta)^{n-k+1}} \right) \right]$, $0 < n$	$\mathfrak{S}_{-n}(g)$
415.*	$\frac{1}{p} = \lim_{\beta \to 0} \left(\frac{\frac{1}{2} - \lambda}{p - \beta} + \frac{\frac{1}{2} + \lambda}{p + \beta} \right)$	$\mathfrak{S}_{-1}(g) = \lambda \pm \frac{1}{2}$, $0 < \pm g$, unit step at $g = 0$
416.*	$\frac{1}{p^2} = \lim_{\beta \to 0} \left[\frac{\frac{1}{2} - \lambda}{(p-\beta)^2} + \frac{\frac{1}{2} + \lambda}{(p+\beta)^2} - \frac{2\beta\mu}{p^2 - \beta^2} \right]$	$\mathfrak{S}_{-2}(g) = \frac{1}{2}\|g\| + \lambda g + \mu$
421.*	$F(f) = \lim_{a \to 0} F_1(f)$, where F is any proper rational fraction in p with n distinct poles, the degree of pole z_j being n_j. All pure imaginary poles in F are the limits of corresponding poles in F_1 which have assigned real parts $\pm a$.	$\sum_{j=1}^{n} \sum_{k=1}^{n} \pm \lambda_{jk} e^{z_j g} g^{n_j - k}$, $0 < \pm g$, $\pm R(z_j) < 0$, where $\lambda_{jk} = \frac{\{D_p^{k-1}[F(f)(p - z_j)^{n_j}]\}_{p=z}}{(k-1)!(n_j - k)!}$ and the upper or lower signs for each term are employed according as the real part of z_j (either actual or vestigial) is less than or greater than zero.
431.	$\frac{1}{(p+\beta)^n}$, $0 < n$	$\frac{1}{(n-1)!} e^{-\beta g} g^{n-1}$, $0 < g$
432.	$\frac{1}{(p-\beta)^n}$, $0 < n$	$\frac{-1}{(n-1)!} e^{\beta g} g^{n-1}$, $g < 0$
433.	$\frac{1}{(p^2 - \beta^2)^n}$, $0 < n$	$\frac{(-1)^n}{(n-1)!} e^{-\beta\|g\|} \sum_{k=1}^{n} \frac{(n+k-2)!\,\|g^{n-k}\|}{(k-1)!(n-k)!(2\beta)^{n+k-1}}$ $= \frac{(-1)^n \|g^{n-\frac{1}{2}}\| K_{n-\frac{1}{2}}(\beta\|g\|)}{(n-1)!\,\pi^{\frac{1}{2}}(2\beta)^{n-\frac{1}{2}}}$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation		Coefficient $G(g)$ for the Unit Impulse	
438.	$\frac{1}{p+\beta}$		$e^{-\beta g}$,	$0 < g$
439.	$\frac{1}{p-\beta}$		$-e^{\beta g}$,	$g < 0$
440.*	$\frac{1}{p-p_0}$,	lim by 415*	$\text{cis}\,(2\pi f_0 g)\mathfrak{S}_{-1}(g)$	
441.*	$\frac{p}{p+\beta}$,	lim by 403*	$\mathfrak{S}_0(g) - \beta e^{-\beta g}$,	$0 < g$
442.	$\frac{1}{(p+\beta)^2}$		$ge^{-\beta g}$,	$0 < g$
443.	$\frac{1}{(p-\beta)^2}$		$-ge^{\beta g}$,	$g < 0$
444.	$\frac{1}{p^2-\beta^2}$		$-\frac{1}{2\beta}e^{-\beta\|g\|}$	
445.	$\frac{p}{p^2-\beta^2}$		$\pm\frac{1}{2}e^{\mp\beta g}$,	$0 < \pm g$
446.*	$\frac{a}{p^2+a^2}$,	lim by 415*	$\sin ag\,\mathfrak{S}_{-1}(g)$	
447.*	$\frac{p}{p^2+a^2}$,	lim by 415*	$\cos ag\,\mathfrak{S}_{-1}(g)$	
448.	$\frac{1}{(p+\alpha)(p+\beta)}$		$\frac{e^{-\beta g}-e^{-\alpha g}}{\alpha-\beta}$,	$0 < g$
449.	$\frac{p}{(p+\alpha)(p+\beta)}$		$\frac{\alpha e^{-\alpha g}-\beta e^{-\beta g}}{\alpha-\beta}$,	$0 < g$
450.	$\frac{1}{(p+\beta)^3}$		$\frac{1}{2}g^2e^{-\beta g}$,	$0 < g$
451.	$\frac{1}{(p-\beta)^3}$		$-\frac{1}{2}g^2e^{\beta g}$,	$g < 0$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
452.	$\dfrac{1}{(p+\alpha)(p+\beta)(p+\gamma)}$	$\dfrac{(\gamma-\beta)e^{-\alpha g}+(\alpha-\gamma)e^{-\beta g}+(\beta-\alpha)e^{-\gamma g}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$, $0 < g$
453.	$\dfrac{p}{(p+\alpha)(p+\beta)(p+\gamma)}$	$\dfrac{\alpha(\beta-\gamma)e^{-\alpha g}+\beta(\gamma-\alpha)e^{-\beta g}+\gamma(\alpha-\beta)e^{-\gamma g}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$, $0 < g$
454.	$\dfrac{\alpha\beta}{p(p+\alpha)(p+\beta)} - \dfrac{1}{p}$	$\dfrac{\beta e^{-\alpha g} - \alpha e^{-\beta g}}{\alpha - \beta}$, $0 < g$
455.*	$\dfrac{a^2}{p(p^2+a^2)}$, lim by 415*	$2\sin^2(\frac{1}{2}ag)\mathfrak{S}_{-1}(g)$
456.	$\dfrac{1}{(p+b+ia)(p+b-ia)}$	$\dfrac{1}{a}\sin ag\, e^{-bg}$, $0 < g$
457.	$\dfrac{1}{(p-b+ia)(p-b-ia)}$	$-\dfrac{1}{a}\sin ag\, e^{bg}$, $g < 0$
458.	$\dfrac{p}{(p+b+ia)(p+b-ia)}$	$\left(\cos ag - \dfrac{b}{a}\sin ag\right)e^{-bg}$, $0 < g$
459.	$\dfrac{p}{(p-b+ia)(p-b-ia)}$	$-\left(\cos ag + \dfrac{b}{a}\sin ag\right)e^{bg}$, $g < 0$

Part 5. Irrational Algebraic Functions of f.

Pair No.	Coefficient $F(f)$	Coefficient $G(g)$
501.*	$p^{n-\alpha} = \lim\limits_{\beta\to 0} p^{n-\alpha}e^{-\beta\|p\|}$, $0 < R(\alpha) < 1$	$\dfrac{1}{\Gamma(\alpha-n)}g^{\alpha-n-1}$, $0 < g$
502.*	$p^{n-\frac{1}{2}} = \lim\limits_{\beta\to 0} p^{n-\frac{1}{2}}e^{-\beta\|p\|}$, $0 < n$	$\dfrac{(-1)^n 1\cdot 3\cdot 5 \cdots (2n-1)}{(\frac{1}{2}\pi)^{\frac{1}{2}}}(2g)^{-n-\frac{1}{2}}$, $0 < g$
503.*	$p^{\frac{1}{2}}$, lim by 502*	$-\frac{1}{2}\pi^{-\frac{1}{2}}g^{-\frac{3}{2}}$, $0 < g$
504.*	$p^{\frac{3}{2}}$, lim by 502*	$\frac{3}{4}\pi^{-\frac{1}{2}}g^{-\frac{5}{2}}$, $0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation		Coefficient $G(g)$ for the Unit Impulse	
505.*	$\lvert f^{\frac{1}{2}} \rvert$,	lim by 503*	$\frac{-1}{4\pi} \lvert g^{-\frac{3}{2}} \rvert$	
506.*	$(p+\beta)^{\frac{1}{2}}$,	lim by 820	$-\frac{1}{2}\pi^{-\frac{1}{2}} e^{-\beta g} g^{-\frac{3}{2}}$,	$0 < g$
516.*	$p^{-\alpha} = \lim\limits_{\beta \to 0} (p+\beta)^{-\alpha}$,	br 0	$\frac{1}{\Gamma(\alpha)} g^{\alpha-1}$,	br 0, $0 < g$
517.*	$p^{-\alpha} = \lim\limits_{\beta \to 0} (p-\beta)^{-\alpha}$,	br $(-\frac{1}{2})$	$\frac{-1}{\Gamma(\alpha)} g^{\alpha-1}$,	br 0, $g < 0$
518.*	$p^{-n-\frac{1}{2}} = \lim\limits_{\beta \to 0} (p+\beta)^{-n-\frac{1}{2}}$,	br 0, $0 < n$	$\frac{(\frac{1}{2}\pi)^{-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \cdots (2n-1)} (2g)^{n-\frac{1}{2}}$,	br 0, $0 < g$
519.*	$p^{-n-\frac{1}{2}} = \lim\limits_{\beta \to 0} (p-\beta)^{-n-\frac{1}{2}}$,	br $\frac{1}{2}$, $0 < n$	$\frac{(\frac{1}{2}\pi)^{-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \cdots (2n-1)} (2g)^{n-\frac{1}{2}}$,	br 0, $g < 0$
520.*	$p^{-\frac{3}{2}}$,	lim by 518*	$2(g/\pi)^{\frac{1}{2}}$,	$0 < g$
521.	$p^{-\alpha}$,	$0 < R(\alpha) < 1$	$\frac{1}{\Gamma(\alpha)} g^{\alpha-1}$,	$0 < g$
522.	$p^{-\frac{1}{2}}$		$(\pi g)^{-\frac{1}{2}}$,	$0 < g$
523.	$\lvert f^{-\frac{1}{2}} \rvert$		$\lvert g^{-\frac{1}{2}} \rvert$	
524.	$(p+\beta)^{-\alpha}$,	br v	$\frac{1}{\Gamma(\alpha)} e^{-i2\pi\alpha(v+w)} e^{-\beta g} g^{\alpha-1}$,	br w, $0 < g$
525.	$(p-\beta)^{-\alpha}$,	br $(v-\frac{1}{2})$	$\frac{-1}{\Gamma(\alpha)} e^{-i2\pi\alpha(v+w)} e^{\beta g} g^{\alpha-1}$,	br w, $g < 0$
526.	$(p+\beta)^{-\frac{1}{2}}$,	br 0	$e^{-\beta g}(\pi g)^{-\frac{1}{2}}$,	br 0, $0 < g$
527.	$(p-\beta)^{-\frac{1}{2}}$,	br $\frac{1}{2}$	$e^{\beta g}(\pi g)^{-\frac{1}{2}}$,	br 0, $g < 0$
528.	$(p-\beta)^{-\frac{1}{2}}$,	br 0	$-e^{\beta g}(\pi g)^{-\frac{1}{2}}(\operatorname{erf}\sqrt{\beta g} - \frac{1}{2} \mp \frac{1}{2})$,	br 0, $0 < \pm g$
529.	$(p+\beta)^{-\frac{3}{2}}$,	br 0	$2e^{-\ g}(g/\pi)^{\frac{1}{2}}$,	br 0, $0 < g$
530.	$(p+\beta)^{1-\alpha} - (p+\gamma)^{1-\alpha}$		$\frac{1}{\Gamma(\alpha-1)} g^{\alpha-2}(e^{-\beta g} - e^{-\gamma g})$,	$0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
541.	$\dfrac{\sqrt{p}}{p+\gamma}$	$\dfrac{1}{\sqrt{\pi g}} + \sqrt{-\gamma}\, e^{-\gamma g} \operatorname{erf} \sqrt{-\gamma g}, \quad 0 < g$
542.	$\dfrac{1}{(p+\gamma)\sqrt{p}}$	$\dfrac{1}{\sqrt{-\gamma}} e^{-\gamma g} \operatorname{erf} \sqrt{-\gamma g}, \quad 0 < g$
543.	$\dfrac{1}{1+\sqrt{\beta p}}$	$\dfrac{1}{\sqrt{\pi \beta g}} - \dfrac{1}{\beta} \exp \dfrac{g}{\beta} \operatorname{erfc} \sqrt{\dfrac{g}{\beta}}, \quad 0 < g$
544.*	$\dfrac{p}{1+\sqrt{\beta p}}$	$-\dfrac{1}{\beta}\, \mathfrak{S}_0(g) + \dfrac{2g-\beta}{2\beta g\sqrt{\pi\beta g}} - \dfrac{1}{\beta^2} \exp \dfrac{g}{\beta} \operatorname{erfc} \sqrt{\dfrac{g}{\beta}}, \quad 0 < g$
545.	$\dfrac{p}{(p+\gamma)(1+\sqrt{\beta p})}$	$\dfrac{-\gamma e^{-\gamma g}}{1+\beta\gamma}(1 - \sqrt{-\beta\gamma} \operatorname{erf} \sqrt{-\gamma g}) + \dfrac{1}{\sqrt{\pi\beta g}} - \dfrac{\exp (g/\beta)}{\beta(1+\beta\gamma)} \operatorname{erfc} \sqrt{\dfrac{g}{\beta}}, \quad 0 < g$
546.	$\dfrac{1}{(p+\gamma)\sqrt{p+\beta}}$	$\dfrac{1}{\sqrt{\beta-\gamma}} e^{-\gamma g} \operatorname{erf} \sqrt{(\beta-\gamma)g}, \quad 0 < g$
547.	$\dfrac{\sqrt{\beta}}{p\sqrt{p+\beta}} - \dfrac{1}{p}$	$-\operatorname{erfc} \sqrt{\beta g}, \quad 0 < g$
548.	$\dfrac{1}{p}\sqrt{\dfrac{p}{\beta}+1} - \dfrac{1}{p}$	$\dfrac{1}{\sqrt{\pi\beta g}} e^{-\beta g} - \operatorname{erfc} \sqrt{\beta g}, \quad 0 < g$
549.	$\dfrac{\sqrt{p+\beta}}{p+\gamma}$	$\dfrac{1}{\sqrt{\pi g}} e^{-\beta g} + \sqrt{\beta-\gamma}\, e^{-\gamma g} \operatorname{erf} \sqrt{(\beta-\gamma)g}, \quad 0 < g$
550.*	$\dfrac{\sqrt{p}}{1+\sqrt{\beta p}}$	$\dfrac{1}{\sqrt{\beta}}\, \mathfrak{S}_0(g) - \dfrac{1}{\beta\sqrt{\pi g}} + \dfrac{1}{\beta\sqrt{\beta}} \exp \dfrac{g}{\beta} \operatorname{erfc} \sqrt{\dfrac{g}{\beta}}, \quad 0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
551.	$\dfrac{1}{\sqrt{p}(1+\sqrt{\beta p})}$	$\dfrac{1}{\sqrt{\beta}} \exp \dfrac{g}{\beta} \operatorname{erfc} \sqrt{\dfrac{g}{\beta}}, \quad 0 < g$
552.	$\dfrac{\sqrt{p}}{(p+\gamma)(1+\sqrt{\beta p})}$	$\dfrac{\sqrt{-\gamma}\, e^{-\gamma g}}{1+\beta\gamma}(\operatorname{erf} \sqrt{-\gamma g} - \sqrt{-\beta\gamma}) + \dfrac{\exp(g/\beta)}{\sqrt{\beta}(1+\beta\gamma)} \operatorname{erfc} \sqrt{\dfrac{g}{\beta}}, \quad 0 < g$
553.*	$\sqrt{\dfrac{p+\alpha}{p}}$	$\mathfrak{S}_0(g) + \dfrac{\alpha}{2} e^{-\frac{1}{2}\alpha g}[I_1(\frac{1}{2}\alpha g) + I_0(\frac{1}{2}\alpha g)], \quad 0 < g$
554.*	$\dfrac{1}{p}\sqrt{\dfrac{p+\alpha}{p}}$	$e^{-\frac{1}{2}\alpha g}[\alpha g I_1(\frac{1}{2}\alpha g) + (1+\alpha g) I_0(\frac{1}{2}\alpha g)], \quad 0 < g$
555.	$\dfrac{1}{\sqrt{(p+\alpha)(p+\beta)}}$	$e^{-\frac{1}{2}(\alpha+\beta)g} I_0[\frac{1}{2}(\alpha-\beta)g], \quad 0 < g$
556.*	$\sqrt{p^2+a^2}$	$\mathfrak{S}_1(g) + \dfrac{a}{g} J_1(ag), \quad 0 < g$
557.	$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(ag), \quad 0 < g$
558.	$\dfrac{1}{\sqrt{\beta^2-p^2}}$	$\dfrac{1}{\pi} K_0(\beta\|g\|)$
559.*	$\dfrac{\sqrt{p+\alpha}}{\sqrt{p}+\sqrt{p+\alpha}}$	$\frac{1}{2}\mathfrak{S}_0(g) + \dfrac{1}{2g} e^{-\frac{1}{2}\alpha g} I_1(\frac{1}{2}\alpha g), \quad 0 < g$
560.	$\dfrac{\sqrt{p+\alpha}}{p(\sqrt{p}+\sqrt{p+\alpha})} - \dfrac{1}{p}$	$-\frac{1}{2} e^{-\frac{1}{2}\alpha g}[I_1(\frac{1}{2}\alpha g) + I_0(\frac{1}{2}\alpha g)], \quad 0 < g$
571.	$(p^2+b^2)^{-\alpha}, \quad 0 < R(\alpha) < 1$	$\dfrac{\sqrt{\pi}}{\Gamma(\alpha)} \left(\dfrac{g}{2b}\right)^{\alpha-\frac{1}{2}} J_{\alpha-\frac{1}{2}}(bg), \quad 0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse	
572.	$[(p+\beta)^2+r^2]^{-\alpha}$	$\frac{\sqrt{\pi}}{\Gamma(\alpha)}\left(\frac{g}{2r}\right)^{\alpha-\frac{1}{2}} e^{-\beta g} J_{\alpha-\frac{1}{2}}(rg),$	$0 < g$
573.	$\sqrt{\frac{p}{p+\beta}}(\sqrt{p+\beta}+\sqrt{p})^{-2\alpha}$	$\frac{1}{4}\beta^{-\alpha+1}e^{-\frac{1}{2}\beta g}[I_{\alpha-1}(\frac{1}{2}\beta g) - 2I_\alpha(\frac{1}{2}\beta g) + I_{\alpha+1}(\frac{1}{2}\beta g)],$	$0 < g$
574.	$\frac{(\sqrt{p+\beta}+\sqrt{p})^{-2\sigma}}{\sqrt{p(p+\beta)}}$	$\beta^{-\sigma}e^{-\frac{1}{2}\beta g}I_\sigma(\frac{1}{2}\beta g),$	$0 < g$
575.	$\frac{[\sqrt{(p+\rho)^2+a^2}+(p+\rho)]^{-\alpha+1}}{\sqrt{(p+\rho)^2+a^2}}$	$a^{-\alpha+1}e^{-\rho g}J_{\alpha-1}(ag),$	$0 < g$
576.	$[\sqrt{(p+\rho)^2+a^2}+(p+\rho)]^{-\alpha}$	$\frac{\alpha e^{-\rho g}}{a^\alpha g} J_\alpha(ag),$	$0 < g$

Part 6. Exponential and Trigonometric Functions of f or f^{-1}.

601.*	e^{-rp}	$\mathfrak{S}_0(g-r)$	
602.*	$\frac{e^{-rp}}{p}$	$\mathfrak{S}_{-1}(g-r)$	
603.	$\frac{1}{p}(1-e^{-ap})$	1,	$0 < g < a$
604.	$\frac{e^{-rp}}{p+\beta}$	$e^{-\beta(g-r)},$	$r < g$
611.	$\frac{\alpha}{\sin \alpha p} - \frac{1}{p}$	$\frac{1}{2}\tanh\frac{\pi g}{2\alpha} \mp \frac{1}{2},$	$0 < \pm g$
612.*	$\tan \alpha p$	$\frac{-1}{2\alpha \sinh\frac{\pi g}{2\alpha}}$	
613.*	$\alpha \operatorname{ctn} \alpha p - \frac{1}{p}$	$\frac{1}{2}\operatorname{ctnh}\frac{\pi g}{2\alpha} \mp \frac{1}{2},$	$0 < \pm g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation		Coefficient $G(g)$ for the Unit Impulse	
614.	$\dfrac{p}{\sin \alpha p}$		$\dfrac{\pi}{4\alpha^2 \cosh^2 \dfrac{\pi g}{2\alpha}}$	
615.	$\dfrac{\sin \beta p}{\sin \alpha p},$	$R(\beta) < R(\alpha)$	$\dfrac{\dfrac{1}{2\alpha} \sin \dfrac{\pi\beta}{\alpha}}{\cosh \dfrac{\pi g}{\alpha} + \cos \dfrac{\pi\beta}{\alpha}}$	
616.	$\dfrac{\cos \beta p}{\cos \alpha p},$	$R(\beta) < R(\alpha)$	$\dfrac{\dfrac{1}{\alpha} \cos \dfrac{\pi\beta}{2\alpha} \cosh \dfrac{\pi g}{2\alpha}}{\cosh \dfrac{\pi g}{\alpha} + \cos \dfrac{\pi\beta}{\alpha}}$	
617.	$\dfrac{\alpha \sin \beta p}{\beta p \sin \alpha p} - \dfrac{1}{p},$	$R(\beta) < R(\alpha)$	$\dfrac{\alpha}{\pi\beta} \tan^{-1} \dfrac{\tanh \dfrac{\pi g}{2\alpha}}{\operatorname{ctn} \dfrac{\pi\beta}{2\alpha}} \mp \dfrac{1}{2},$	$0 < \pm g$
618.	$\dfrac{\cos \beta p}{p \cos \alpha p} - \dfrac{1}{p},$	$R(\beta) < R(\alpha)$	$-\dfrac{1}{\pi} \tan^{-1} \dfrac{\cos \dfrac{\pi\beta}{2\alpha}}{\sinh \dfrac{\pi g}{2\alpha}}$	
619.*	$\cosh(ap)$		$\frac{1}{2}[\mathfrak{S}_0(g + a) + \mathfrak{S}_0(g - a)]$	
620.	$\dfrac{\cosh(ap)}{p} - \dfrac{1}{p}$		$\mp \frac{1}{2},$	$0 < \pm g < a$
621.	$\dfrac{\cosh(ap)}{(p - p_0)\cosh(ap_0)} - \dfrac{1}{p - p_0}$		$\frac{1}{2} \operatorname{cis}(2\pi f_0 g)[\tanh(ap_0) \mp 1],$	$0 < \pm g < a$
622.	$\dfrac{\sinh(ap)}{p}$		$\frac{1}{2},$	$-a < g < a$
623.	$\dfrac{\sinh(ap)}{ap^2} - \dfrac{1}{p}$		$\dfrac{g}{2a} \mp \frac{1}{2},$	$0 < \pm g < a$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
624.	$\dfrac{p_0 \sinh(ap)}{p(p-p_0)\sinh(ap_0)} - \dfrac{1}{p-p_0}$	$\frac{1}{2}\{\text{cis}(2\pi f_0 g)[\text{ctnh}(ap_0) \mp 1] - \text{csch}(ap_0)\}$, $0 < \pm g < a$
625.	$\text{sech}\ \pi f$	$\text{sech}\ \pi g$
631.	$p^{\alpha-1}e^{-\beta\|p\|}$	$\dfrac{\Gamma(\alpha)}{i2\pi}[(-g-i\beta)^{-\alpha} - (-g+i\beta)^{-\alpha}]$
632.	$e^{-\beta\|p\|}$	$\dfrac{1}{\pi} \cdot \dfrac{\beta}{\beta^2+g^2}$
633.	$\dfrac{1}{p}e^{-\beta\|p\|} - \dfrac{1}{p}$	$-\dfrac{1}{\pi}\tan^{-1}\dfrac{\beta}{g}$
634.	$\dfrac{e^{-\beta\|p\|}}{\sqrt{\|p\|}}$	$\sqrt{\dfrac{\beta+\sqrt{\beta^2+g^2}}{2\pi(\beta^2+g^2)}}$
635.	$\|p\|e^{-\beta\|p\|}$	$\dfrac{\beta^2-g^2}{\pi(\beta^2+g^2)^2}$
645.*	$\sin(a\|p\|)$	$\dfrac{a}{\pi(a^2-g^2)}$
651.	$\dfrac{1}{\sqrt{p+\beta}}\exp\dfrac{\rho}{p+\beta}$	$\dfrac{e^{-\beta g}}{\sqrt{\pi g}}\cosh(2\sqrt{\rho g})$, $0 < g$
652.*	$\dfrac{1}{\sqrt{p}}\exp\dfrac{1}{4p}$	$\dfrac{1}{\sqrt{\pi g}}\cosh\sqrt{g}$, $0 < g$
653.	$(p+\beta)^{-\frac{3}{2}}\exp\dfrac{\rho}{p+\beta}$	$\dfrac{e^{-\beta g}}{\sqrt{\pi\rho}}\sinh(2\sqrt{\rho g})$, $0 < g$
654.*	$\exp\left(-\dfrac{1}{p}\right)$	$\mathfrak{S}_0(g) - \dfrac{1}{\sqrt{g}}J_1(2\sqrt{g})$, $0 < g$
655.	$\dfrac{1}{p}\exp\left(-\dfrac{1}{p}\right)$	$J_0(2\sqrt{g})$, $0 < g$
656.*	$\dfrac{1}{p^2}\exp\left(-\dfrac{1}{p}\right)$	$\sqrt{g}\,J_1(2\sqrt{g})$, $0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Part 7. Exponential and Trigonometric Functions of f^2.

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
701.	$e^{\rho p^2}$, $\quad 0 < \lvert\rho\rvert$	$\dfrac{1}{2\sqrt{\pi\rho}}\, e^{-\frac{1}{4}g^2/\rho}$
702.	$\phi_n(f) = e^{\pi f^2} D_f{}^n e^{-2\pi f^2}$	$i^n \phi_n(g)$
703.	$\phi_0(f) = e^{-\frac{1}{2}x^2} = e^{-\pi f^2}$, $\quad x = f\sqrt{4\pi}$ in 703–711	$\phi_0(g) = e^{-\pi g^2}$
704.	$\phi_1(f) = -\, e^{-\frac{1}{2}x^2}(4\pi)^{\frac{1}{2}}x$	$i\phi_1(g)$
705.	$\phi_2(f) = e^{-\frac{1}{2}x^2}(4\pi)(x^2 - 1)$	$-\, \phi_2(g)$
706.	$\phi_3(f) = -\, e^{-\frac{1}{2}x^2}(4\pi)^{\frac{3}{2}}(x^3 - 3x)$	$-\, i\phi_3(g)$
707.	$\phi_4(f) = e^{-\frac{1}{2}x^2}(4\pi)^2(x^4 - 6x^2 + 3)$	$\phi_4(g)$
708.	$\phi_5(f) = -\, e^{-\frac{1}{2}x^2}(4\pi)^{\frac{5}{2}}(x^5 - 10x^3 + 15x)$	$i\phi_5(g)$
709.	$\phi_6(f) = e^{-\frac{1}{2}x^2}(4\pi)^3(x^6 - 15x^4 + 45x^2 - 15)$	$-\, \phi_6(g)$
710.	$\phi_7(f) = -\, e^{-\frac{1}{2}x^2}(4\pi)^{\frac{7}{2}}(x^7 - 21x^5 + 105x^3 - 105x)$	$-\, i\phi_7(g)$
711.	$\phi_8(f) = e^{-\frac{1}{2}x^2}(4\pi)^4(x^8 - 28x^6 + 210x^4 - 420x^2 + 105)$	$\phi_8(g)$
712.	$e^{-\pi f^2}(4\pi f^2 - 3)^2$	$e^{-\pi g^2}(4\pi g^2 - 3)^2$
713.	$p^n e^{-\pi\beta f^2}$	$\dfrac{1}{\sqrt{\beta}} D_g{}^n e^{-\pi g^2/\beta} = \dfrac{1}{\sqrt{\beta}(\sqrt{2\beta})^n} \times e^{-\frac{1}{2}\pi g^2/\beta}\phi_n\left(\dfrac{g}{\sqrt{2\beta}}\right)$
714.	$e^{-\pi\beta f^2}$	$\dfrac{1}{\sqrt{\beta}}\, e^{-\pi g^2/\beta}$
715.	$pe^{-\pi\beta f^2}$	$-\dfrac{2\pi g}{\beta\sqrt{\beta}}\, e^{-\pi g^2/\beta}$
716.	$p^2 e^{-\pi\beta f^2}$	$\dfrac{2\pi}{\beta^2\sqrt{\beta}}\, e^{-\pi g^2/\beta}(2\pi g^2 - \beta)$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
717.	$p^3e^{-\pi\beta f_2}$	$-\frac{4\pi^2 g}{\beta^3\sqrt{\beta}}e^{-\pi g^2/\beta}(2\pi g^2 - 3\beta)$
718.	$p^4e^{-\pi\beta f^2}$	$\frac{4\pi^2}{\beta^4\sqrt{\beta}}e^{-\pi g^2/\beta}(4\pi^2 g^4 - 12\pi\beta g^2 + 3\beta^2)$
719.	$p^n e^{-\frac{1}{2}\pi f^2}$	$\sqrt{2}D_g{}^n e^{-2\pi g^2} = \sqrt{2}e^{-\pi g^2}\phi_n(g)$
720.	$e^{-\frac{1}{2}\pi f^2}$	$\sqrt{2}e^{-2\pi g^2}$
721.	$pe^{-\frac{1}{2}\pi f^2}$	$-4\pi g\sqrt{2}e^{-2\pi g^2}$
722.	$p^2e^{-\frac{1}{2}\pi f^2}$	$4\pi\sqrt{2}e^{-2\pi g^2}(4\pi g^2 - 1)$
723.	$p^3e^{-\frac{1}{2}\pi f^2}$	$-16\pi^2 g\sqrt{2}e^{-2\pi g^2}(4\pi g^2 - 3)$
724.	$p^4e^{-\frac{1}{2}\pi f^2}$	$16\pi^2\sqrt{2}e^{-2\pi g^2}(16\pi^2 g^4 - 24\pi g^2 + 3)$
725.*	$\frac{1}{p}e^{-\pi\beta f^2}$	$\mathfrak{S}_{-1}(g) + \frac{1}{2}\operatorname{erf}(g\sqrt{\pi/\beta}) \mp \frac{1}{2}, \quad 0 < \pm g$
726.*	$\frac{1}{p^2}e^{-\pi\beta f^2}$	$\mathfrak{S}_{-2}(g) + \frac{1}{2}g\operatorname{erf}(g\sqrt{\pi/\beta}) + \frac{\sqrt{\beta}}{2\pi}e^{-\pi g^2/\beta} \mp \frac{1}{2}g, \quad 0 < \pm g$
727.	$\frac{1}{p}\exp(\alpha p^2) - \frac{1}{p}$	$\mp\frac{1}{2}\operatorname{erfc}\frac{\lvert g\rvert}{2\sqrt{\alpha}}, \quad 0 < \pm g$
728.	$\frac{1}{p - p_0}\exp[\alpha(p^2 - p_0{}^2)] - \frac{1}{p - p_0}$	$\frac{1}{2}\operatorname{cis}(2\pi f_0 g)\left(\operatorname{erf}\frac{g + 2\alpha p_0}{2\sqrt{\alpha}} \mp 1\right), \quad 0 < \pm g$
729.	$\exp(\rho p^2 + \sigma p), \quad 0 < \lvert\rho\rvert$	$\frac{1}{2\sqrt{\pi\rho}}\exp\left[-\frac{(g+\sigma)^2}{4\rho}\right]$
751.	$\sin(ap^2)$	$\frac{1}{2\sqrt{\pi a}}\sin\left(\frac{g^2}{4a} - \frac{\pi}{4}\right)$
752.	$\cos(ap^2)$	$\frac{1}{2\sqrt{\pi a}}\sin\left(\frac{g^2}{4a} + \frac{\pi}{4}\right)$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
753.	$\frac{\sin(ap^2)}{p}$	$\frac{1}{2}\left[S\left(\frac{g}{\sqrt{2\pi a}}\right) - C\left(\frac{g}{\sqrt{2\pi a}}\right)\right]$
754.	$\frac{\cos(ap^2)}{p} - \frac{1}{p}$	$\frac{1}{2}\left[S\left(\frac{g}{\sqrt{2\pi a}}\right) + C\left(\frac{g}{\sqrt{2\pi a}}\right) \mp 1\right]$, $0 < \pm g$
755.	$\frac{\cos(ap^2)}{(p - p_0)\cos(ap_0^2)} - \frac{1}{p - p_0}$	$\frac{\text{cis}(2\pi f_0 g)}{4\cos(ap_0^2)}\left[\exp(iap_0^2)\,\text{erf}\left(\frac{g}{2\sqrt{ia}} + p_0\sqrt{ia}\right) + \exp(-iap_0^2)\,\text{erf}\left(\frac{g}{2\sqrt{-ia}} + p_0\sqrt{-ia}\right) \mp 2\cos(ap_0^2)\right]$, $0 < \pm g$
756.	$\frac{\sin(ap^2)}{p^2}$	$\sqrt{\frac{a}{\pi}}\sin\left(\frac{g^2}{4a} + \frac{\pi}{4}\right) + \frac{g}{2}\left[S\left(\frac{g}{\sqrt{2\pi a}}\right) - C\left(\frac{g}{\sqrt{2\pi a}}\right)\right]$
757.	$\frac{\sin(ap^2)}{p^3} - \frac{a}{p}$	$\frac{1}{2}\left[\left(a + \frac{g^2}{2}\right)S\left(\frac{g}{\sqrt{2\pi a}}\right) + \left(a - \frac{g^2}{2}\right)C\left(\frac{g}{\sqrt{2\pi a}}\right) + g\sqrt{\frac{a}{\pi}}\sin\left(\frac{g^2}{4a} + \frac{\pi}{4}\right) \mp a\right]$, $0 < \pm g$
758.	$\sin(ap^2 + \lambda)$	$\frac{1}{2\sqrt{\pi a}}\sin\left(\frac{g^2}{4a} + \lambda - \frac{\pi}{4}\right)$
759.	$\cos(ap^2 + \lambda)$	$\frac{1}{2\sqrt{\pi a}}\sin\left(\frac{g^2}{4a} + \lambda + \frac{\pi}{4}\right)$
760.	$\text{cis}\,[\pm\pi(f^2 - \frac{1}{8})]$	$\text{cis}\,[\mp\pi(g^2 - \frac{1}{8})]$
761.	$\cos\,[\pi(f^2 - \frac{1}{8})]$	$\cos\,[\pi(g^2 - \frac{1}{8})]$
762.	$\sin\,[\pi(f^2 - \frac{1}{8})]$	$-\sin\,[\pi(g^2 - \frac{1}{8})]$

TABLE I (Continued)

Part 8. Other Elementary Transcendental Functions of f.

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
801.	$\exp(-\alpha\sqrt{p})$	$\frac{\alpha}{2g\sqrt{\pi g}}\exp\left(-\frac{\alpha^2}{4g}\right)$, $0 < g$
802.	$p\exp(-\alpha\sqrt{p})$	$\left(\frac{\alpha^2}{2g}-3\right)\frac{\alpha}{4g^2\sqrt{\pi g}}\exp\left(-\frac{\alpha^2}{4g}\right)$, $0 < g$
803.	$\frac{1}{p}[1-\exp(-\alpha\sqrt{p})]$	$\operatorname{erf}\frac{\alpha}{2\sqrt{g}}$, $0 < g$
804.*	$\frac{1}{p^2}[1-\exp(-\alpha\sqrt{p})]+\frac{\alpha^2}{2p}$	$\left(g+\frac{\alpha^2}{2}\right)\operatorname{erf}\frac{\alpha}{2\sqrt{g}} + \alpha\sqrt{\frac{g}{\pi}}\exp\left(-\frac{\alpha^2}{4g}\right)$, $0 < g$
805.	$\frac{1-\exp(-\alpha\sqrt{p})}{p(p+\gamma)}$	$\frac{e^{-\gamma g}}{2\gamma}\left[\exp(-\alpha\sqrt{-\gamma})\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{g}} - \sqrt{-\gamma g}\right) + \exp(\alpha\sqrt{-\gamma})\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{g}} + \sqrt{-\gamma g}\right)\right] + \frac{1}{\gamma}\operatorname{erf}\frac{\alpha}{2\sqrt{g}} - \frac{1}{\gamma}e^{-\gamma g}$, $0 < g$
806.	$\sqrt{p}\exp(-\alpha\sqrt{p})$	$\left(\frac{\alpha^2}{2g}-1\right)\frac{1}{2g\sqrt{\pi g}}\exp\left(-\frac{\alpha^2}{4g}\right)$, $0 < g$
807.	$\frac{1}{\sqrt{p}}\exp(-\alpha\sqrt{p})$	$\frac{1}{\sqrt{\pi g}}\exp\left(-\frac{\alpha^2}{4g}\right)$, $0 < g$
808.*	$\frac{1}{\alpha p\sqrt{p}}\exp(-\alpha\sqrt{p})+\frac{1}{p}$	$\frac{2}{\alpha}\sqrt{\frac{g}{\pi}}\exp\left(-\frac{\alpha^2}{4g}\right)+\operatorname{erf}\frac{\alpha}{2\sqrt{g}}$, $0 < g$
809.	$\frac{\exp(-\alpha\sqrt{p})}{1+\sqrt{\beta p}}$	$\frac{1}{\sqrt{\pi\beta g}}\exp\left(-\frac{\alpha^2}{4g}\right)-\frac{1}{\beta}\exp\frac{\alpha\sqrt{\beta}+g}{\beta} \times \operatorname{erfc}\frac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}}$, $0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
810.	$\dfrac{p\exp(-\alpha\sqrt{p})}{1+\sqrt{\beta p}}$	$\dfrac{\alpha^2\beta-2(\beta+\alpha\sqrt{\beta})g+4g^2}{4\beta g^2\sqrt{\pi\beta g}}\exp\left(-\dfrac{\alpha^2}{4g}\right) - \dfrac{1}{\beta^2}\exp\dfrac{\alpha\sqrt{\beta}+g}{\beta}\operatorname{erfc}\dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}},$ $0<g$
811.	$\dfrac{\exp(-\alpha\sqrt{p})}{p(1+\sqrt{\beta p})}-\dfrac{1}{p}$	$-\operatorname{erf}\dfrac{\alpha}{2\sqrt{g}}-\exp\dfrac{\alpha\sqrt{\beta}+g}{\beta} \times\operatorname{erfc}\dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}},\quad 0<g$
812.	$\dfrac{\exp(-\alpha\sqrt{p})}{(p+\gamma)(1+\sqrt{\beta p})}$	$\dfrac{\exp(-\alpha\sqrt{-\gamma}-\gamma g)}{2(1+\sqrt{-\beta\gamma})}\operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}}-\sqrt{-\gamma g}\right)+\dfrac{\exp(\alpha\sqrt{-\gamma}-\gamma g)}{2(1-\sqrt{-\beta\gamma})} \times\operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}}+\sqrt{-\gamma g}\right) -\dfrac{1}{1+\beta\gamma}\exp\dfrac{\alpha\sqrt{\beta}+g}{\beta} \times\operatorname{erfc}\dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}},\quad 0<g$
813.	$\dfrac{p\exp(-\alpha\sqrt{p})}{(p+\gamma)(1+\sqrt{\beta p})}$	$\dfrac{-\gamma\exp(-\alpha\sqrt{-\gamma}-\gamma g)}{2(1+\sqrt{-\beta\gamma})} \times\operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}}-\sqrt{-\gamma g}\right) +\dfrac{-\gamma\exp(\alpha\sqrt{-\gamma}-\gamma g)}{2(1-\sqrt{-\beta\gamma})} \times\operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}}+\sqrt{-\gamma g}\right) -\dfrac{1}{\beta(1+\beta\gamma)}\exp\dfrac{\alpha\sqrt{\beta}+g}{\beta} \times\operatorname{erfc}\dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}}+\dfrac{1}{\sqrt{\pi\beta g}}\exp\left(-\dfrac{\alpha^2}{4g}\right),$ $0<g$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
814.	$\dfrac{\sqrt{p}\exp(-\alpha\sqrt{p})}{1+\sqrt{\beta p}}$	$\dfrac{\alpha\sqrt{\beta}-2g}{2\beta g\sqrt{\pi g}}\exp\left(-\dfrac{\alpha^2}{4g}\right) + \dfrac{1}{\beta\sqrt{\beta}}\exp\dfrac{\alpha\sqrt{\beta}+g}{\beta}\operatorname{erfc}\dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}},\quad 0<g$
815.	$\dfrac{\exp(-\alpha\sqrt{p})}{\sqrt{p}(1+\sqrt{\beta p})}$	$\dfrac{1}{\sqrt{\beta}}\exp\dfrac{\alpha\sqrt{\beta}+g}{\beta}\operatorname{erfc}\dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}},\quad 0<g$
816.	$\dfrac{\sqrt{p}\exp(-\alpha\sqrt{p})}{(p+\gamma)(1+\sqrt{\beta p})}$	$\dfrac{\sqrt{-\gamma}\exp(-\alpha\sqrt{-\gamma}-\gamma g)}{2(1+\sqrt{-\beta\gamma})}\times\operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}}-\sqrt{-\gamma g}\right) - \dfrac{\sqrt{-\gamma}\exp(\alpha\sqrt{-\gamma}-\gamma g)}{2(1-\sqrt{-\beta\gamma})}\times\operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}}+\sqrt{-\gamma g}\right) + \dfrac{1}{\sqrt{\beta}(1+\beta\gamma)}\exp\dfrac{\alpha\sqrt{\beta}+g}{\beta}\times\operatorname{erfc}\dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}},\quad 0<g$
817.	$\exp(-\alpha\sqrt{p+\beta})$	$\dfrac{\alpha}{2g\sqrt{\pi g}}\exp\left(-\beta g-\dfrac{\alpha^2}{4g}\right),\quad 0<g$
818.	$\dfrac{1}{p}\exp(-\alpha\sqrt{p+\beta}+\alpha\sqrt{\beta})-\dfrac{1}{p}$	$-\dfrac{1}{2}\left[\operatorname{erfc}\left(\sqrt{\beta g}-\dfrac{\alpha}{2\sqrt{g}}\right) - \exp(2\alpha\sqrt{\beta})\operatorname{erfc}\left(\sqrt{\beta g}+\dfrac{\alpha}{2\sqrt{g}}\right)\right],\quad 0<g$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
819.	$\dfrac{\exp(-\alpha\sqrt{p+\beta})}{p+\gamma}$	$\dfrac{1}{2}\Big[\exp(-\alpha\sqrt{\beta-\gamma}-\gamma g)\,\mathrm{erfc}\Big(\dfrac{\alpha}{2\sqrt{g}} - \sqrt{(\beta-\gamma)g}\Big) + \exp(\alpha\sqrt{\beta-\gamma}-\gamma g) \times \mathrm{erfc}\Big(\dfrac{\alpha}{2\sqrt{g}} + \sqrt{(\beta-\gamma)g}\Big)\Big],\quad 0 < g$
820.	$\sqrt{p+\beta}\exp(-\alpha\sqrt{p+\beta})$	$\Big(\dfrac{\alpha^2}{2g}-1\Big)\dfrac{1}{2g\sqrt{\pi g}}\exp\Big(-\dfrac{\alpha^2}{4g}-\beta g\Big),\quad 0 < g$
821.	$\dfrac{1}{p}\sqrt{\dfrac{p}{\beta}+1}\,\exp(-\alpha\sqrt{p+\beta}+\alpha\sqrt{\beta})-\dfrac{1}{p}$	$\dfrac{1}{\sqrt{\pi\beta g}}\exp\Big(-\dfrac{\alpha^2}{4g}-\beta g+\alpha\sqrt{\beta}\Big) - \dfrac{1}{2}\Big[\mathrm{erfc}\Big(\sqrt{\beta g}-\dfrac{\alpha}{2\sqrt{g}}\Big) + \exp(2\alpha\sqrt{\beta})\,\mathrm{erfc}\Big(\sqrt{\beta g}+\dfrac{\alpha}{2\sqrt{g}}\Big)\Big],\quad 0 < g$
822.	$\dfrac{\sqrt{p+\beta}\exp(-\alpha\sqrt{p+\beta})}{p+\gamma}$	$\dfrac{\sqrt{\beta-\gamma}}{2}\Big[\exp(-\alpha\sqrt{\beta-\gamma}-\gamma g) \times \mathrm{erfc}\Big(\dfrac{\alpha}{2\sqrt{g}}-\sqrt{(\beta-\gamma)g}\Big) - \exp(\alpha\sqrt{\beta-\gamma}-\gamma g) \times \mathrm{erfc}\Big(\dfrac{\alpha}{2\sqrt{g}}+\sqrt{(\beta-\gamma)g}\Big)\Big] + \dfrac{1}{\sqrt{\pi g}}\exp\Big(-\dfrac{\alpha^2}{4g}-\beta g\Big),\quad 0 < g$
823.	$\dfrac{\exp(-\alpha\sqrt{p+\beta})}{\sqrt{p+\beta}}$	$\dfrac{1}{\sqrt{\pi g}}\exp\Big(-\beta g-\dfrac{\alpha^2}{4g}\Big),\quad 0 < g$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
824.	$\dfrac{\exp(-\alpha\sqrt{p+\beta}+\alpha\sqrt{\beta})}{p\sqrt{\dfrac{p}{\beta}+1}}-\dfrac{1}{p}$	$-\frac{1}{2}\left[\operatorname{erfc}\left(\sqrt{\beta g}-\frac{\alpha}{2\sqrt{g}}\right)+\exp(2\alpha\sqrt{\beta})\operatorname{erfc}\left(\sqrt{\beta g}+\frac{\alpha}{2\sqrt{g}}\right)\right],\quad 0<g$
825.	$\dfrac{\exp(-\alpha\sqrt{p+\beta})}{(p+\gamma)\sqrt{p+\beta}}$	$\frac{1}{2\sqrt{\beta-\gamma}}\left[\exp(-\alpha\sqrt{\beta-\gamma}-\gamma g)\times\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{g}}-\sqrt{(\beta-\gamma)g}\right)-\exp(\alpha\sqrt{\beta-\gamma}-\gamma g)\times\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{g}}+\sqrt{(\beta-\gamma)g}\right)\right],\quad 0<g$
841.	$\dfrac{1}{\sqrt{\|p\|}}\exp(-\rho\sqrt{\|p\|})$	$\sqrt{\frac{2}{\pi\|g\|}}\left[\sin\frac{\rho^2}{4\|g\|}C\left(\frac{\rho}{\sqrt{2\pi\|g\|}}\right)-\cos\frac{\rho^2}{4\|g\|}S\left(\frac{\rho}{\sqrt{2\pi\|g\|}}\right)\right]+\frac{1}{\sqrt{\pi\|g\|}}\cos\left(\frac{\rho^2}{4\|g\|}+\frac{\pi}{4}\right)$
842.	$\dfrac{1}{\sqrt{\|p\|}}\exp(-\rho\sqrt{\|p\|}-\sigma\|p\|)$	$\frac{1}{2\sqrt{\pi(\sigma+ig)}}\exp\frac{\rho^2}{4(\sigma+ig)}\operatorname{erfc}\frac{\rho}{2\sqrt{\sigma+ig}}+\frac{1}{2\sqrt{\pi(\sigma-ig)}}\exp\frac{\rho^2}{4(\sigma-ig)}\times\operatorname{erfc}\frac{\rho}{2\sqrt{\sigma-ig}}$
843.*	$\sqrt{\|p\|}\sin(a\sqrt{\|p\|})$	$-\frac{a}{2\pi g^2}-\frac{1}{\|g\|\sqrt{2\pi\|g\|}}\left[\left(\cos\frac{a^2}{4\|g\|}-\frac{a^2}{2\|g\|}\sin\frac{a^2}{4\|g\|}\right)C\left(\frac{a}{\sqrt{2\pi\|g\|}}\right)+\left(\sin\frac{a^2}{4\|g\|}+\frac{a^2}{2\|g\|}\cos\frac{a^2}{4\|g\|}\right)\times S\left(\frac{a}{\sqrt{2\pi\|g\|}}\right)\right]$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
844.	$\dfrac{\sqrt{\lvert p\rvert}\sin(a\sqrt{\lvert p\rvert})}{p}$	$\pm\sqrt{\dfrac{2}{\pi\lvert g\rvert}}\left[\sin\dfrac{a^2}{4\lvert g\rvert}S\left(\dfrac{a}{\sqrt{2\pi\lvert g\rvert}}\right) + \cos\dfrac{a^2}{4\lvert g\rvert}C\left(\dfrac{a}{\sqrt{2\pi\lvert g\rvert}}\right)\right], \quad 0 < \pm g$
845.	$\cos(\alpha\sqrt{\lvert p\rvert})\,e^{-\beta\lvert p\rvert}$	$\dfrac{\beta}{\pi(\beta^2+g^2)} + \dfrac{i\alpha}{4(\beta+ig)\sqrt{\pi(\beta+ig)}} \times \exp\left[-\dfrac{\alpha^2}{4(\beta+ig)}\right]\operatorname{erf}\dfrac{i\alpha}{2\sqrt{\beta+ig}} + \dfrac{i\alpha}{4(\beta-ig)\sqrt{\pi(\beta-ig)}} \times \exp\left[-\dfrac{\alpha^2}{4(\beta-ig)}\right]\operatorname{erf}\dfrac{i\alpha}{2\sqrt{\beta-ig}}$
846.	$\dfrac{\sin(\alpha\sqrt{\lvert p\rvert})\,e^{-\beta\lvert p\rvert}}{\sqrt{\lvert p\rvert}}$	$\dfrac{1}{2i\sqrt{\pi(\beta+ig)}}\exp\left[-\dfrac{\alpha^2}{4(\beta+ig)}\right] \times \operatorname{erf}\dfrac{i\alpha}{2\sqrt{\beta+ig}} + \dfrac{1}{2i\sqrt{\pi(\beta-ig)}} \times \exp\left[-\dfrac{\alpha^2}{4(\beta-ig)}\right]\operatorname{erf}\dfrac{i\alpha}{2\sqrt{\beta-ig}}$
861.	$\dfrac{\exp[-c\sqrt{p(p+\alpha)}]}{\sqrt{p(p+\alpha)}}$	$e^{-\frac{1}{2}\alpha g}I_0\left(\dfrac{\alpha}{2}\sqrt{g^2-c^2}\right), \quad c < g$
862.*	$\sqrt{\dfrac{p}{p+\alpha}}\exp[-c\sqrt{p(p+\alpha)}]$	$e^{-\frac{1}{2}\alpha c}\mathfrak{S}_0(g-c) + \dfrac{\alpha}{2}e^{-\frac{1}{2}\alpha g}\left[\dfrac{g}{\sqrt{g^2-c^2}}I_1\left(\dfrac{\alpha}{2}\sqrt{g^2-c^2}\right) - I_0\left(\dfrac{\alpha}{2}\sqrt{g^2-c^2}\right)\right], \quad c < g$
863.*	$\exp[-c\sqrt{(p+\alpha)(p+\beta)}]$	$e^{-\frac{1}{2}(\alpha+\beta)c}\mathfrak{S}_0(g-c) + \dfrac{c(\alpha-\beta)}{2\sqrt{g^2-c^2}}e^{-\frac{1}{2}(\alpha+\beta)g}I_1\left(\dfrac{\alpha-\beta}{2} \times \sqrt{g^2-c^2}\right), \quad c < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
864.*	$\sqrt{\frac{p+\alpha}{p+\beta}} \exp\left[-c\sqrt{(p+\alpha)(p+\beta)}\right]$	$e^{-\frac{1}{2}(\alpha+\beta)c}\mathbf{S}_0(g-c) + \frac{\alpha-\beta}{2} e^{-\frac{1}{2}(\alpha+\beta)g}\left[\frac{g}{\sqrt{g^2-c^2}} I_1\left(\frac{\alpha-\beta}{2} \times \sqrt{g^2-c^2}\right) + I_0\left(\frac{\alpha-\beta}{2}\sqrt{g^2-c^2}\right)\right]$, $c < g$
865.*	$\exp(-c\sqrt{p^2+a^2})$	$\mathbf{S}_0(g-c) - \frac{ac}{\sqrt{g^2-c^2}} J_1(a\sqrt{g^2-c^2})$, $c < g$
866.	$\frac{\exp(-c\sqrt{p^2+a^2})}{\sqrt{p^2+a^2}}$	$J_0(a\sqrt{g^2-c^2})$, $c < g$
867.	$\exp(-\alpha\sqrt{\beta^2-p^2})$	$\frac{\alpha\beta K_1(\beta\sqrt{\alpha^2+g^2})}{\pi\sqrt{\alpha^2+g^2}}$
868.	$\frac{\exp(-\alpha\sqrt{\beta^2-p^2})}{\sqrt{\beta^2-p^2}}$	$\frac{1}{\pi} K_0(\beta\sqrt{\alpha^2+g^2})$
869.	$\frac{(\sqrt{p^2+a^2}+p)^{-\nu}}{\sqrt{p^2+a^2}} \exp(-r\sqrt{p^2+a^2})$	$a^{-\nu}\left(\frac{g-r}{g+r}\right)^{\frac{1}{2}\nu} J_\nu(a\sqrt{g^2-r^2})$, $r < g$
871.*	$\cos(a\sqrt{\beta^2-p^2})$	$\frac{1}{2}\mathbf{S}_0(g+a) + \frac{1}{2}\mathbf{S}_0(g-a) - \frac{a\beta J_1(\beta\sqrt{a^2-g^2})}{2\sqrt{a^2-g^2}}$, $-a < g < a$
872.	$\frac{\sin(a\sqrt{\beta^2-p^2})}{\sqrt{\beta^2-p^2}}$	$\frac{1}{2} J_0(\beta\sqrt{a^2-g^2})$, $-a < g < a$
881.	$\tan^{-1}\frac{r}{p+\rho}$	$\frac{1}{g} e^{-\rho g}\sin rg$, $0 < g$
891.	$(p+\beta)^{-\alpha}\log(p+\beta)$	$\frac{1}{\Gamma(\alpha)} g^{\alpha-1}e^{-\beta g}[\psi(\alpha) - \log g]$, $0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation		Coefficient $G(g)$ for the Unit Impulse	
892.	$p^{-\alpha} \log p,$	$0 < R(\alpha) < 1$	$\frac{\psi(\alpha) - \log g}{\Gamma(\alpha)} g^{\alpha-1},$	$0 < g$
893.*	$\frac{\log p}{p} = \lim_{\beta \to 0} \frac{\log (p + \beta)}{p + \beta}$		$\psi(1) - \log g,$	$0 < g$
894.	$\log \frac{p + \gamma}{p + \beta}$		$\frac{1}{g} (e^{-\beta g} - e^{-\gamma g}),$	$0 < g$
	Part 9. Other Transcendental Functions of f.			
901.	$\exp p^2 \operatorname{erfc} p$		$\frac{1}{\sqrt{\pi}} \exp (- \frac{1}{4} g^2),$	$0 < g$
902.	$\frac{1}{\sqrt{p}} \exp \left(\frac{1}{p} \right) \operatorname{erfc} \left(\frac{1}{\sqrt{p}} \right)$		$\frac{1}{\sqrt{\pi g}} \exp (- 2\sqrt{g}),$	$0 < g$
911.	$p^{\frac{1}{2}-\alpha} K_{\frac{1}{2}-\alpha}(ap),$	$0 < R(\alpha) < 1$	$\frac{\sqrt{\pi}}{\Gamma(\alpha)} (2a)^{\frac{1}{2}-\alpha} (g^2 - a^2)^{\alpha-1},$	$a < g$
912.	$K_0(ap)$		$\frac{1}{\sqrt{g^2 - a^2}},$	$a < g$
913.*	$\frac{1}{p} K_0(ap) = \lim_{\beta \to 0} \frac{K_0[a(p + \beta)]}{p + \beta}$		$\cosh^{-1} \frac{g}{a},$	$a < g$
914.	$p^{\frac{1}{2}-\alpha} I_{\alpha-\frac{1}{2}}(ap)$		$\frac{1}{\sqrt{\pi}\Gamma(\alpha)} (2a)^{\frac{1}{2}-\alpha} (a^2 - g^2)^{\alpha-1},$	$-a < g < a$
915.	$(\beta^2 - p^2)^{\frac{1}{2}\nu} K_\nu(\sqrt{\beta^2 - p^2})$		$\frac{1}{\sqrt{2\pi}} \left(\frac{\beta^2}{g^2 + 1} \right)^{\frac{1}{2}\nu+\frac{1}{4}} K_{\nu+\frac{1}{2}}(\beta\sqrt{g^2 + 1})$	
916.	$(\rho^2 + f^2)^{-\frac{1}{8}} K_{\frac{1}{4}}(2\pi\rho\sqrt{\rho^2 + f^2})$		$(\rho^2 + g^2)^{-\frac{1}{8}} K_{\frac{1}{4}}(2\pi\rho\sqrt{\rho^2 + g^2})$	
917.	$K_0(\beta\sqrt{\rho^2 - p^2})$		$\frac{\exp (- \rho\sqrt{g^2 + \beta^2})}{2(g^2 + \beta^2)^{\frac{1}{2}}}$	

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse	
918.	$K_0(\alpha \lvert p \rvert)$	$\dfrac{1}{2\sqrt{\alpha^2 + g^2}}$	
919.	$\lvert p \rvert K_1(\alpha \lvert p \rvert)$	$\dfrac{\alpha}{2\sqrt{(\alpha^2 + g^2)^3}}$	
920.	$K_0(\alpha\sqrt{p})$	$\dfrac{1}{2g} \exp\left(-\dfrac{\alpha^2}{4g}\right)$,	$0 < g$
921.	$\sqrt{p}\, K_1(\alpha\sqrt{p})$	$\dfrac{\alpha}{4g^2} \exp\left(-\dfrac{\alpha^2}{4g}\right)$,	$0 < g$
922.	$\dfrac{\alpha}{\sqrt{p}} K_1(\alpha\sqrt{p}) - \dfrac{1}{p}$	$\exp\left(-\dfrac{\alpha^2}{4g}\right) - 1$,	$0 < g$
923.	$\sqrt{p} \exp (p^2) K_{\frac{1}{4}}(p^2)$	$\dfrac{1}{\sqrt{2g}} \exp\left(-\tfrac{1}{8}g^2\right)$,	$0 < g$
924.	$\dfrac{1}{\sqrt{p}} \exp\left(\dfrac{1}{p}\right) K_{\frac{1}{4}}\left(\dfrac{1}{p}\right)$	$(2g)^{-\frac{3}{4}} \exp\left(-2\sqrt{2g}\right)$,	$0 < g$
925.	$\dfrac{1}{\sqrt{p}} \exp\left(-\dfrac{1}{p}\right) I_{\alpha-\frac{1}{2}}\left(\dfrac{1}{p}\right)$	$\dfrac{1}{\sqrt{\pi g}} J_{2\alpha-1}(2\sqrt{2g})$,	$0 < g$
931.	$\sqrt{p} K_{\nu+\frac{1}{4}}(p) K_{\nu-\frac{1}{4}}(p), \quad -\tfrac{3}{4} < R(\nu) < \tfrac{3}{4}$	$\dfrac{\sqrt{\pi}(y^{2\nu} + y^{-2\nu})}{\sqrt{2g(g^2 - 4)}}, \quad y = \tfrac{1}{2}(g + \sqrt{g^2 - 4})$,	$2 < g$
932.	$\sqrt{p}\left[I_{\nu-x}(p) I_{-\nu-x}(p) \right]_{x=-\frac{1}{4}}^{x=\frac{1}{4}}$	$\dfrac{\sqrt{2}(y^{2\nu} + y^{-2\nu})}{\pi\sqrt{\pi g(4 - g^2)}}, \quad y = \tfrac{1}{2}(g + \sqrt{g^2 - 4})$,	$0 < g < 2$
933.	$\sqrt{p} I_v(p) K_{v+\frac{1}{2}}(p)$	$\dfrac{(-1)^v (x^{2v+\frac{1}{2}} + x^{-2v-\frac{1}{2}})}{\sqrt{2\pi g(4 - g^2)}}$, $x = \tfrac{1}{2}(g + \sqrt{g^2 - 4})$,	$0 < g < 2$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
934.	$\sqrt{p}\left[J_{-\frac{3}{4}+x+\alpha}(p) Y_{-\frac{1}{4}-x+\alpha}(p) \right]_{x=-\frac{1}{4}}^{x=\frac{1}{4}}$	$\dfrac{2^{2\alpha}(g+\sqrt{g^2+4})^{\frac{3}{2}-2\alpha}}{\pi\sqrt{\pi g(g^2+4)}}, \quad 0 < g$
935.	$J_{\alpha-\frac{1}{2}}(\sqrt{\delta^2-p^2}+ip)J_{\alpha-\frac{1}{2}}(\sqrt{\delta^2-p^2}-ip)$	$\dfrac{1}{\pi}(4-g^2)^{-\frac{1}{2}}J_{2\alpha-1}(\delta\sqrt{4-g^2}), \quad -2 < g < 2$
936.	$I_{\alpha-\frac{1}{2}}(\sqrt{p^2+b^2}-p)K_{\alpha-\frac{1}{2}}(\sqrt{p^2+b^2}+p)$	$(g^2-4)^{-\frac{1}{2}}J_{2\alpha-1}(b\sqrt{g^2-4}), \quad 2 < g$
981.*	$\mathfrak{S}_0(f) = \dfrac{1}{\pi}\lim_{\beta\to 0}\dfrac{\beta}{\beta^2+f^2}$	1
982.*	$\mathfrak{S}_0(f-f_0)$, lim by 981*	$\text{cis}(2\pi f_0 g)$
983.*	$\mathfrak{S}_0(f-f_0)+\mathfrak{S}_0(f+f_0)$, lim by 981*	$2\cos(2\pi f_0 g)$
984.*	$\mathfrak{S}_0(f-f_0)-\mathfrak{S}_0(f+f_0)$, lim by 981*	$i2\sin(2\pi f_0 g)$
985.*	$\mathfrak{S}_1(f) = \lim_{\beta\to 0}\left(-\dfrac{2\pi f}{\beta\sqrt{\beta}}e^{-\pi f^2/\beta}\right)$	$-i2\pi g$
986.*	$\mathfrak{S}_{-1}(f) = \lim_{\beta\to 0}\left(\lambda \pm \dfrac{1}{2}\right)e^{-\beta\|f\|}, \quad 0 < \pm f$	$-\dfrac{1}{i2\pi g}+\lambda\mathfrak{S}_0(g)$
987.*	$\mathfrak{S}_{-2}(f) = \lim_{\beta\to 0}\left(\dfrac{1}{2}\|f\| + \lambda f + \mu\right)e^{-\beta\|f\|}$	$-\dfrac{1}{4\pi^2 g^2}+\dfrac{\lambda}{i2\pi}\mathfrak{S}_1(g)+\mu\mathfrak{S}_0(g)$

* A star marks a pair as being the limit approached by regular pairs.

TABLE II—ADMITTANCES OF

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse $= \mathfrak{S}_0(t)$ Effect: $\mathfrak{M}Y(p)$ 403*
1	$Y(p) = \dfrac{Cp + G}{LC(p - p_1)(p - p_2)}$ Inductance L and resistance R in series with parallel combination of capacity C and conductance G. Cause: Voltage across terminals. Effect: Current through network. $\left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} = \dfrac{-(RC + LG) \pm \Delta}{2LC}$, $\Delta = \sqrt{(RC - LG)^2 - 4LC}$.	$\dfrac{1}{\Delta}[(Cp_1 + G)e^{p_1 t} - (Cp_2 + G)e^{p_2 t}]$, $0 < t$ 448, 449
2	$Y(p) = Cp + G$ Same as 1, except $R = 0$, $L = 0$.	$C\mathfrak{S}_1(t) + G\mathfrak{S}_0(t)$ 403*, 404*
3	$Y(p) = \exp\left[-\dfrac{x}{v}\sqrt{(p + \rho)^2 - \sigma^2} \right]$ Semi-infinite smooth line (resistance R, inductance L, conductance G, and capacity C per unit length). Cause: Initial voltage. Effect: Voltage at distance x from end. $v = (LC)^{-\frac{1}{2}}$, $k = (L/C)^{\frac{1}{2}}$, $\alpha = R/(2L)$, $\beta = G/(2C)$, $\rho = \alpha + \beta$, $\sigma = \alpha - \beta$, $z = \sqrt{t^2 - (x/v)^2}$.	$\exp\left(-\dfrac{\rho x}{v} \right) \mathfrak{S}_0\left(t - \dfrac{x}{v} \right) + \dfrac{\sigma x}{vz} e^{-\rho t} I_1(\sigma z)$, $\dfrac{x}{v} < t$ 863*
4	$Y(p) = \dfrac{1}{k}\sqrt{\dfrac{p + \rho - \sigma}{p + \rho + \sigma}} \times \exp\left[-\dfrac{x}{v}\sqrt{(p + \rho)^2 - \sigma^2} \right]$ Same as 3, except Effect: Current at distance x from end.	$\dfrac{1}{k}\exp\left(-\dfrac{\rho x}{v} \right) \mathfrak{S}_0\left(t - \dfrac{x}{v} \right) + \dfrac{1}{k} e^{-\rho t}\left[\dfrac{\sigma t}{z} I_1(\sigma z) - \sigma I_0(\sigma z) \right]$, $\dfrac{x}{v} < t$ 864*

AND TRANSIENTS IN PHYSICAL SYSTEMS

Cause: Unit Step (0, 1) $= \mathfrak{S}_{-1}(t),\ \lambda = \frac{1}{2}$ Effect: $\mathfrak{M}[Y(p)/p]$ 415*	Cause: Unit Cisoid × Unit Step (0, 1) $= e^{p_0 t}\mathfrak{S}_{-1}(t),\ \lambda = \frac{1}{2}$ Effect: $\mathfrak{M}[Y(p)/(p - p_0)]$ 440*
$\frac{1}{R + G^{-1}} + \frac{Cp_1 + G}{\Delta p_1} e^{p_1 t} - \frac{Cp_2 + G}{\Delta p_2} e^{p_2 t}, \quad 0 < t$ 448, 454, 415*	$\frac{(Cp_0 + G)e^{p_0 t}}{LC(p_0 - p_1)(p_0 - p_2)} + \frac{(Cp_1 + G)e^{p_1 t}}{\Delta(p_1 - p_0)} - \frac{(Cp_2 + G)e^{p_2 t}}{\Delta(p_2 - p_0)}, \quad 0 < t$ 452, 453
$C\mathfrak{S}_0(t) + G, \quad 0 < t$ 403*, 415*	$C\mathfrak{S}_0(t) + (Cp_0 + G)e^{p_0 t}, \quad 0 < t$ 438, 441*

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathcal{M}\, Y(p)$
5	$Y(p) = \exp\,(-\,y\sqrt{p + 2\beta})$ Same as 3, except $L = 0$. $y = x\sqrt{RC}$.	$\dfrac{y}{2t\sqrt{\pi t}} \exp\left(-\dfrac{y^2}{4t} - 2\beta t\right), \qquad 0 < t$ 817
6	$Y(p) = u\sqrt{p + 2\beta}\exp\,(-\,y\sqrt{p + 2\beta})$ Same as 4, except $L = 0$. $y = x\sqrt{RC}, \qquad u = \sqrt{C/R}$.	$\dfrac{u(y^2 - 2t)}{4t^2\sqrt{\pi t}} \exp\left(-\dfrac{y^2}{4t} - 2\beta t\right), 0 < t$ 820
7	$Y(p) = \dfrac{\exp\,(-\,y\sqrt{p + 2\beta})}{u\sqrt{p + 2\beta}}$ Same as 3, except $L = 0$ and Cause: Initial current. $y = x\sqrt{RC}, \qquad u = \sqrt{C/R}$.	$\dfrac{1}{u\sqrt{\pi t}} \exp\left(-\dfrac{y^2}{4t} - 2\beta t\right), \qquad 0 < t$ 823
8	$Y(p) = \dfrac{1}{k}\sqrt{\dfrac{p}{p + 2\alpha}}$ $\times \exp\left[-\dfrac{x}{v}\sqrt{p(p + 2\alpha)}\right]$ Same as 4, except $G = 0$.	$\dfrac{1}{k}\exp\left(-\dfrac{\alpha x}{v}\right)\mathfrak{S}_0\left(t - \dfrac{x}{v}\right)$ $+\dfrac{1}{k}e^{-\alpha t}\left[\dfrac{\alpha t}{z} I_1(\alpha z) - \alpha I_0(\alpha z)\right], \dfrac{x}{v} < t$ 862*
9	$Y(p) = k\sqrt{\dfrac{p + 2\alpha}{p}}$ Same as 3, except $G = 0$, $x = 0$, and Cause: Initial current.	$k\mathfrak{S}_0(t) + k\alpha e^{-\alpha t}[I_1(\alpha t) + I_0(\alpha t)],$ $0 < t$ 553*

Continued

Cause: Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid $\times$ Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/(p - p_0)]$
$\frac{1}{2}\left[\exp(-y\sqrt{2\beta})\operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{2\beta t}\right) + \exp(y\sqrt{2\beta})\operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{2\beta t}\right)\right], \quad 0 < t$ 818, 415*	$\frac{1}{2}e^{p_0 t}\left[\exp(-y\sqrt{2\beta + p_0}) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{(2\beta + p_0)t}\right) + \exp(y\sqrt{2\beta + p_0}) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{(2\beta + p_0)t}\right)\right], \quad 0 < t$ 819
$\frac{u}{\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t} - 2\beta t\right) + \frac{u\sqrt{2\beta}}{2} \times \left[\exp(-y\sqrt{2\beta})\operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{2\beta t}\right) - \exp(y\sqrt{2\beta})\operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{2\beta t}\right)\right], \quad 0 < t$ 821, 415*	$\frac{u}{\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t} - 2\beta t\right) + \frac{u\sqrt{2\beta + p_0}}{2}e^{p_0 t}\left[\exp(-y\sqrt{2\beta + p_0}) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{(2\beta + p_0)t}\right) - \exp(y\sqrt{2\beta + p_0}) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{(2\beta + p_0)t}\right)\right], \quad 0 < t$ 822
$\frac{1}{2u\sqrt{2\beta}}\left[\exp(-y\sqrt{2\beta}) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{2\beta t}\right) - \exp(y\sqrt{2\beta}) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{2\beta t}\right)\right], \quad 0 < t$ 824, 415*	$\frac{e^{p_0 t}}{2u\sqrt{2\beta + p_0}}\left[\exp(-y\sqrt{2\beta + p_0}) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{(2\beta + p_0)t}\right) - \exp(y\sqrt{2\beta + p_0}) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{(2\beta + p_0)t}\right)\right], \quad 0 < t$ 825
$\frac{1}{k}e^{-\alpha t}I_0(\alpha z), \quad \frac{x}{v} < t$ 861	
$ke^{-\alpha t}[2\alpha t I_1(\alpha t) + (1 + 2\alpha t)I_0(\alpha t)], \quad 0 < t$ **554***	

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{M}Y(p)$
10	$Y(p) = \exp\left(-\frac{\rho x}{v} - \frac{x}{v}p\right)$ Same as 3, except $R/L = G/C$.	$\exp\left(-\frac{\rho x}{v}\right)\mathfrak{S}_0\left(t - \frac{x}{v}\right)$ 601*
11	$Y(p) = \frac{\sqrt{p + 2\alpha}}{\sqrt{p} + \sqrt{p + 2\alpha}}$ Semi-infinite smooth line (resistance R, inductance L, and capacity C per unit length). Cause: Voltage applied through resistance $R_0 = \sqrt{L/C}$. Effect: Voltage at end of line. $\alpha = R/(2L)$.	$\frac{1}{2}\mathfrak{S}_0(t) + \frac{1}{2t}e^{-\alpha t}I_1(\alpha t), \quad 0 < t$ 559*
12	$Y(p) = \frac{\exp(-y\sqrt{p})}{1 + \sqrt{p/\lambda}}$ Semi-infinite smooth line (resistance R and capacity C per unit length). Cause: Voltage applied through resistance R_0. Effect: Voltage at distance x from end. $y = x\sqrt{RC}, \quad \lambda = R/(CR_0^2)$.	$\sqrt{\frac{\lambda}{\pi t}}\exp\left(-\frac{y^2}{4t}\right) - \lambda\exp(y\sqrt{\lambda} + \lambda t)$ $\times\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{\lambda t}\right), \quad 0 < t$ 809
13	$Y(p) = \frac{u\sqrt{p}\exp(-y\sqrt{p})}{1 + \sqrt{p/\lambda}}$ Same as 12, except Effect: Current at distance x from end. $u = \sqrt{C/R}$.	$\frac{u(y - 2t\sqrt{\lambda})}{2t}\sqrt{\frac{\lambda}{\pi t}}\exp\left(-\frac{y^2}{4t}\right)$ $+ u\lambda\sqrt{\lambda}\exp(y\sqrt{\lambda} + \lambda t)$ $\times\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{\lambda t}\right), \quad 0 < t$ 814
14	$Y(p) = \frac{u\sqrt{p}}{1 + \sqrt{p/\lambda}}$ Same as 13, except $x = 0$. $u = \sqrt{C/R}$.	$u\sqrt{\lambda}\left[\mathfrak{S}_0(t) - \sqrt{\frac{\lambda}{\pi t}} + \lambda e^{\lambda t}\,\mathrm{erfc}\,\sqrt{\lambda t}\right],$ $0 < t$ 550*

Continued

Cause: Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid × Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/(p - p_0)]$
$\exp\left(-\frac{\rho x}{v}\right), \quad \frac{x}{v} < t$ 602*	$\exp\left[-\frac{x}{v}(\rho + p_0) + p_0 t\right], \quad \frac{x}{v} < t$ 604
$1 - \frac{1}{2}e^{-\alpha t}[I_0(\alpha t) + I_1(\alpha t)], \quad 0 < t$ 560, 415*	
$\operatorname{erfc}\frac{y}{2\sqrt{t}} - \exp(y\sqrt{\lambda} + \lambda t) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{\lambda t}\right), \quad 0 < t$ 811, 415*	$\frac{1}{2}e^{p_0 t}\left[\frac{\exp(-y\sqrt{p_0})}{1 + \sqrt{p_0/\lambda}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{p_0 t}\right) + \frac{\exp(y\sqrt{p_0})}{1 - \sqrt{p_0/\lambda}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{p_0 t}\right)\right] - \frac{1}{1 - p_0/\lambda}\exp(y\sqrt{\lambda} + \lambda t) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{\lambda t}\right), \quad 0 < t$ 812
$u\sqrt{\lambda}\exp(y\sqrt{\lambda} + \lambda t)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{\lambda t}\right), \quad 0 < t$ 815	$\frac{u\sqrt{p_0}}{2}e^{p_0 t}\left[\frac{\exp(-y\sqrt{p_0})}{1 + \sqrt{p_0/\lambda}} \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{p_0 t}\right) - \frac{\exp(y\sqrt{p_0})}{1 - \sqrt{p_0/\lambda}} \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{p_0 t}\right)\right] + \frac{u\sqrt{\lambda}}{1 - p_0/\lambda} \times \exp(y\sqrt{\lambda} + \lambda t)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{\lambda t}\right), \quad 0 < t$ 816
$u\sqrt{\lambda}e^{\lambda t}\operatorname{erfc}\sqrt{\lambda t}, \quad 0 < t$ 551	$\frac{u\sqrt{\lambda}}{1 - p_0/\lambda}\left[\sqrt{\frac{p_0}{\lambda}}e^{p_0 t}\operatorname{erf}\sqrt{p_0 t} - \frac{p_0}{\lambda}e^{p_0 t} + e^{\lambda t}\operatorname{erfc}\sqrt{\lambda t}\right], \quad 0 < t$ 552

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\partial\mathcal{M}Y(p)$
15	$Y(p) = \dfrac{C_0 p \exp(-y\sqrt{p})}{1 + \sqrt{p/\mu}}$ Semi-infinite smooth line (resistance R and capacity C per unit length). Cause: Voltage applied through capacity C_0. Effect: Current at distance x from end. $y = x\sqrt{RC}, \quad \mu = C/(RC_0^2)$.	$\dfrac{C_0(y^2 - 2yt\sqrt{\mu} - 2t + 4\mu t^2)}{4t^2}\sqrt{\dfrac{\mu}{\pi t}}$ $\times \exp\left(-\dfrac{y^2}{4t}\right)$ $- C_0\mu^2 \exp(y\sqrt{\mu} + \mu t)$ $\times \operatorname{erfc}\left(\dfrac{y}{2\sqrt{t}} + \sqrt{\mu t}\right), \quad 0 < t$ 810
16	$Y(p) = \dfrac{C_0 p}{1 + \sqrt{p/\mu}}$ Same as 15, except $x = 0$.	$- C_0\mu \mathbf{S}_0(t) + \dfrac{C_0(2\mu t - 1)}{2t}\sqrt{\dfrac{\mu}{\pi t}}$ $- C_0\mu^2 e^{\mu t} \operatorname{erfc}\sqrt{\mu t}, \quad 0 < t$ 544*
17	$Y(p) = \dfrac{w^{2n+1}[\sqrt{(p+\lambda)^2 + w^2} + (p+\lambda)]^{-2n}}{k\sqrt{(p+\lambda)^2 + w^2}}$ Semi-infinite artificial line (series element: resistance R and inductance L; shunt element: conductance G and capacity C; $R/L = G/C$; mid-series termination). Cause: Applied voltage. Effect: Current in nth section. $k = (L/C)^{\frac{1}{2}}$, $\lambda = R/L = G/C$, $w = 2(LC)^{-\frac{1}{2}}$.	$\dfrac{2}{L} e^{-\lambda t} J_{2n}(wt), \quad 0 < t$ 575
18	$Y(p) = \dfrac{2(2\alpha)^n}{R}\sqrt{\dfrac{p}{p + 2\alpha}} \times (\sqrt{p + 2\alpha} + \sqrt{p})^{-2n}$ Semi-infinite artificial line (series element: resistance R; shunt element: capacity C; mid-series termination). Cause: Applied voltage. Effect: Current in nth section. $\alpha = 2/(RC)$.	$\dfrac{\alpha}{R} e^{-\alpha t}[I_{n-1}(\alpha t) - 2I_n(\alpha t) + I_{n+1}(\alpha t)], \quad 0 < t$ 573

Continued

Cause: Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid $\times$ Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/(p-p_0)]$
$C_0\sqrt{\frac{\mu}{\pi t}}\exp\left(-\frac{y^2}{4t}\right) - C_0\mu\exp(y\sqrt{\mu}+\mu t)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{\mu t}\right),\quad 0<t$ 809	$\frac{1}{2}C_0p_0\, e^{p_0t}\left[\frac{\exp(-y\sqrt{p_0})}{1+\sqrt{p_0/\mu}}\times\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}-\sqrt{p_0t}\right)+\frac{\exp(y\sqrt{p_0})}{1-\sqrt{p_0/\mu}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{p_0t}\right)\right] + C_0\sqrt{\frac{\mu}{\pi t}}\exp\left(-\frac{y^2}{4t}\right)-\frac{C_0\mu}{1-p_0/\mu}\times\exp(y\sqrt{\mu}+\mu t)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{\mu t}\right),\quad 0<t$ 813
$C_0\sqrt{\frac{\mu}{\pi t}} - C_0\mu e^{\mu t}\operatorname{erfc}\sqrt{\mu t},\quad 0<t$ 543	$C_0\sqrt{\frac{\mu}{\pi t}}+\frac{C_0\mu}{1-p_0/\mu}\left[\frac{p_0}{\mu}e^{p_0t}-\frac{p_0}{\mu}\sqrt{\frac{p_0}{\mu}}\,e^{p_0t}\times\operatorname{erf}\sqrt{p_0t}-e^{\mu t}\operatorname{erfc}\sqrt{\mu t}\right],\quad 0<t$ 545
$\frac{2}{R}e^{-\alpha t}I_n(\alpha t),\quad 0<t$ 574	

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{M}\, Y(p)$
19	$Y(p) = \exp\left(x - \lvert x\rvert \sqrt{p^2 + 1}\right)$ Vertical atmospheric waves, axis of x vertically upwards, velocity = 1, height of homogeneous atmosphere = $\frac{1}{2}$. Cause: Vertical displacement at $x = 0$. Effect: Vertical displacement at time t of particle whose undisturbed position is x.	$e^x \mathfrak{S}_0(t - \lvert x\rvert) - \dfrac{\lvert x\rvert e^x}{\sqrt{t^2 - x^2}} \times J_1(\sqrt{t^2 - x^2}), \quad \lvert x\rvert < t$ 865*
20	$Y(p) = \dfrac{\exp\left(x - \lvert x\rvert \sqrt{p^2 + 1}\right)}{2\sqrt{p^2 + 1}}$ Same as 19, except Cause: Vertical force at $x = 0$.	$\dfrac{e^x}{2} J_0(\sqrt{t^2 - x^2}), \quad \lvert x\rvert < t$ 866
21	$Y(p) = \dfrac{\lvert y\rvert}{\pi r} \sqrt{p}\, K_1(r\sqrt{p})$ Flow of heat in infinite plane. Cause: Temperature impulse at origin, temperature maintained zero along x-axis, except at origin. Effect: Temperature of point with coördinates (x, y) at time t. $r = \sqrt{x^2 + y^2}$.	$\dfrac{\lvert y\rvert}{4\pi t^2} \exp\left(-\dfrac{r^2}{4t}\right), \quad 0 < t$ 921
22	$Y(p) = \dfrac{1}{p}\left[1 - \exp\left(-y\sqrt{\dfrac{p}{\nu}}\right)\right]$ Horizontal oscillations of deep viscous fluid, axis of y vertical, bottom plane $y = 0$, kinematic coefficient of viscosity $= \nu$. Cause: Applied horizontal force. Effect: Displacement of particle at y at time t, y assumed small.	$\operatorname{erf} \dfrac{y}{2\sqrt{\nu t}}, \quad 0 < t$ 803
23	$Y(p) = \dfrac{1}{2\pi} K_0\left(\dfrac{rp}{c}\right)$ Water waves radiating from center in an unlimited sheet of uniform depth h, gravity constant $= g$. Cause: Pressure at the origin. Effect: Velocity potential at distance r, time t. $c^2 = gh$.	$\dfrac{1}{2\pi} \cdot \dfrac{1}{\sqrt{t^2 - \dfrac{r^2}{c^2}}}, \quad \dfrac{r}{c} < t$ 912

Continued

Cause: Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid × Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/(p-p_0)]$
$\frac{\lvert y\rvert}{\pi r^2}\exp\left(-\frac{r^2}{4t}\right)$, $0 < t$ 922, 415*	
$y\sqrt{\frac{t}{\nu\pi}}\exp\left(-\frac{y^2}{4\nu t}\right)-\frac{y^2}{2\nu}+\left(\frac{y^2}{2\nu}+t\right)\operatorname{erf}\frac{y}{2\sqrt{\nu t}}$, $0 < t$ 804*, 415*	$\frac{1}{p_0}e^{p_0t}-\frac{1}{p_0}\operatorname{erf}\frac{y}{2\sqrt{\nu t}}-\frac{e^{p_0t}}{2p_0}\left[\exp\left(-y\sqrt{\frac{p_0}{\nu}}\right)\times\operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}-\sqrt{p_0t}\right)+\exp\left(y\sqrt{\frac{p_0}{\nu}}\right)\operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}+\sqrt{p_0t}\right)\right]$, 805 $0 < t$
$\frac{1}{2\pi}\cosh^{-1}\frac{ct}{r}$, $\frac{r}{c} < t$ 913*	

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{N}Y(p)$
24	$Y(p) = \dfrac{\sin [(\pi - y)p]}{\sin \pi p}$ Flow of electricity in thin plane infinite strip, axis of x along lower edge of strip, axis of y across, width of strip $= \pi$, upper edge ($y = \pi$) maintained at zero potential. Cause: Potential along x axis. Effect: Potential at point (x, y).	$\dfrac{1}{2\pi} \cdot \dfrac{\sin y}{\cosh x - \cos y}$ 615
25	$Y(p) = \dfrac{\cos [(\pi - y)p]}{\cos \pi p}$ Same as 24, except upper edge ($y = \pi$) is insulated.	$\dfrac{1}{\pi} \cdot \dfrac{\sin \frac{1}{2}y \cosh \frac{1}{2}x}{\cosh x - \cos y}$ 616
26	$Y(p) = \exp (\kappa t p^2)$ Linear flow of heat in infinite solid, diffusivity κ, axis of x in direction of flow. Cause: Initial temperature. Effect: Temperature at time t at point x.	$\dfrac{1}{2\sqrt{\pi \kappa t}} \exp \left(- \dfrac{x^2}{4\kappa t} \right)$ 701
27	$Y(p) = \cos (tp^2)$ Transverse oscillations of infinite elastic plate; x and y axes in the plate, but all points with same y coördinate have same displacement. Cause: Initial displacement. Effect: Displacement perpendicular to plate at time t of point whose coördinate is x.	$\dfrac{1}{2\sqrt{\pi t}} \sin \left(\dfrac{x^2}{4t} + \dfrac{\pi}{4} \right)$ 752

Continued

Cause: Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid × Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/(p-p_0)]$
$\frac{1}{\pi}\tan^{-1}\frac{\tanh\frac{1}{2}x}{\tan\frac{1}{2}y}$ 617, 415*	
$\frac{1}{\pi}\tan^{-1}\frac{\sinh\frac{1}{2}x}{\sin\frac{1}{2}y}$ 618, 415*	
$\frac{1}{2}\operatorname{erf}\frac{x}{2\sqrt{\kappa t}}$ 727, 415*	$\frac{1}{2}\exp(\kappa t p_0^2 + p_0 x)\operatorname{erf}\left(\frac{x}{2\sqrt{\kappa t}} + p_0\sqrt{\kappa t}\right)$ 728, 440*
$\frac{1}{2}\left[S\left(\frac{x}{\sqrt{2\pi t}}\right) + C\left(\frac{x}{\sqrt{2\pi t}}\right)\right]$ 754, 415*	$\frac{1}{4}\exp(p_0 x + itp_0^2)\operatorname{erf}\left(\frac{x}{2\sqrt{it}} + p_0\sqrt{it}\right) + \frac{1}{4}\exp(p_0 x - itp_0^2) \times \operatorname{erf}\left(\frac{x}{2\sqrt{-it}} + p_0\sqrt{-it}\right)$ 755, 440*

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{M}\,Y(p)$
28	$Y(p) = \frac{\sin(tp^2)}{p^2}$ Same as 27, except Cause: Initial velocity.	$\sqrt{\frac{t}{\pi}}\sin\left(\frac{x^2}{4t}+\frac{\pi}{4}\right) + \frac{x}{2}\left[S\left(\frac{x}{\sqrt{2\pi t}}\right) - C\left(\frac{x}{\sqrt{2\pi t}}\right)\right]$ 756
29	$Y(p) = \cos(t\sqrt{1-p^2})$ Same as 19, except Cause: Initial displacement multiplied by e^{-x}. Effect: Vertical displacement multiplied by e^{-x} at time t of particle whose undisturbed position is x.	$\frac{1}{2}[\mathfrak{S}_0(x-t) + \mathfrak{S}_0(x+t)] - \frac{tJ_1(\sqrt{t^2-x^2})}{2\sqrt{t^2-x^2}}, \quad -t < x < t$ 871*
30	$Y(p) = \frac{\sin(t\sqrt{1-p^2})}{\sqrt{1-p^2}}$ Same as 29, except Cause: Initial velocity multiplied by e^{-x}.	$\frac{1}{2}J_0(\sqrt{t^2-x^2}), \quad -t < x < t$ 872
31	$Y(p) = e^{-\lvert yp\rvert}$ Flow of electricity in infinite thin plane, x and y axes in the plane. Cause: Potential along x axis. Effect: Potential at point (x, y).	$\frac{1}{\pi}\cdot\frac{\lvert y\rvert}{x^2+y^2}$ 632
32	$Y(p) = \cosh(atp)$ Transverse motion of infinite stretched elastic string, axis of x along equilibrium position of string, velocity of propagation along string $= a$. Cause: Initial displacement. Effect: Normal displacement of particle at x at time t.	$\frac{1}{2}[\mathfrak{S}_0(x-at) + \mathfrak{S}_0(x+at)]$ 619*

Continued

Cause: Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid × Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/(p-p_0)]$
$\frac{1}{2}\left[\left(t+\frac{x^2}{2}\right)S\left(\frac{x}{\sqrt{2\pi t}}\right)+\left(t-\frac{x^2}{2}\right)\times C\left(\frac{x}{\sqrt{2\pi t}}\right)+x\sqrt{\frac{t}{\pi}}\sin\left(\frac{x^2}{4t}+\frac{\pi}{4}\right)\right]$ 757, 415*	
$\frac{1}{\pi}\tan^{-1}\frac{x}{\lvert y\rvert}$ 633, 415*	
$\pm\frac{1}{2}, \quad at < \pm x$ 620, 415*	$\begin{cases}\pm\frac{1}{2}e^{p_0x}\cosh(atp_0), & at < \pm x\\ \frac{1}{2}e^{p_0x}\sinh(atp_0), & -at < x < at\end{cases}$ 621, 440*

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{M}Y(p)$
33	$Y(p) = \frac{\sinh(atp)}{ap}$ Same as 32, except Cause: Initial velocity.	$\frac{1}{2a}, \qquad -at < x < at$ 622
34	$Y(p) = \frac{1}{\rho}\cos(\alpha\sqrt{\lvert p\rvert})e^{y\lvert p\rvert}$ Waves on deep water, axis of y vertically upwards, axis of x in the surface, density $= \rho$, gravity constant $= g$, $y \leqq 0$. Cause: Initial surface-impulse along x axis. Effect: Velocity potential at time t at point (x, y). $h = \frac{\alpha}{\sqrt{2\pi\lvert x\rvert}}, \qquad \alpha = t\sqrt{g}$.	$\frac{-y}{\pi\rho(x^2+y^2)} + \frac{i\alpha}{4\rho\sqrt{\pi}(-y+ix)^{3/2}} \times \exp\left[-\frac{\alpha^2}{4(-y+ix)}\right] \times \operatorname{erf}\frac{i\alpha}{2\sqrt{-y+ix}} + \frac{i\alpha}{4\rho\sqrt{\pi}(-y-ix)^{3/2}} \times \exp\left[-\frac{\alpha^2}{4(-y-ix)}\right] \times \operatorname{erf}\frac{i\alpha}{2\sqrt{-y-ix}}$ 845
35	$Y(p) = -\frac{\sqrt{\lvert p\rvert}}{\rho\sqrt{g}}\sin(\alpha\sqrt{\lvert p\rvert})$ Same as 34, except Effect: Surface elevation at time t at point x.	$\frac{\alpha}{2\pi\rho x^2\sqrt{g}} + \frac{1}{\rho\lvert x\rvert\sqrt{2\pi g\lvert x\rvert}} \times \{[\cos(\frac{1}{2}\pi h^2) - \pi h^2 \sin(\frac{1}{2}\pi h^2)]C(h) + [\sin(\frac{1}{2}\pi h^2) + \pi h^2\cos(\frac{1}{2}\pi h^2)]S(h)\}$ 843*
36	$Y(p) = \sqrt{g}\,\frac{\sin(\alpha\sqrt{\lvert p\rvert})}{\sqrt{\lvert p\rvert}}\,e^{y\lvert p\rvert}$ Same as 34, except Cause: Initial surface elevation.	$\frac{\sqrt{g}}{2i\sqrt{\pi(-y+ix)}} \times \exp\left[-\frac{\alpha^2}{4(-y+ix)}\right] \times \operatorname{erf}\frac{i\alpha}{2\sqrt{-y+ix}} + \frac{\sqrt{g}}{2i\sqrt{\pi(-y-ix)}} \times \exp\left[-\frac{\alpha^2}{4(-y-ix)}\right] \times \operatorname{erf}\frac{i\alpha}{2\sqrt{-y-ix}}$ 846

Continued

Cause: Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid × Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/(p-p_0)]$
$\begin{cases} \pm\frac{1}{2}t, & at < \pm x \\ \frac{x}{2a}, & -at < x < at \end{cases}$ 623, 415*	$\begin{cases} \pm\frac{1}{2ap_0}e^{p_0x}\sinh(atp_0), & at < \pm x \\ \frac{1}{2ap_0}[e^{p_0x}\cosh(atp_0) - 1], & -at < x < at \end{cases}$ 624, 440*
$\mp\frac{1}{\rho}\sqrt{\frac{2}{\pi g\lvert x\rvert}}[\sin(\frac{1}{2}\pi h^2)S(h) + \cos(\frac{1}{2}\pi h^2)C(h)], \quad 0 < \pm x$ 844	

TABLE II

No.	Admittance $Y(p_1, p_2)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{M}_1\mathfrak{M}_2 Y(p_1, p_2)$
37	$Y(p_1, p_2) = \cos[t(p_1^2 + p_2^2)]$ Transverse oscillations of infinite elastic plate, x and y axes in the plate. Cause: Initial displacement. Effect: Displacement perpendicular to plate at time t of point whose coördinates are x and y.	$\frac{1}{4\pi t} \sin \frac{x^2 + y^2}{4t}$ 759, 758
38	$Y(p_1, p_2) = \exp(-z\sqrt{\lvert p_1^2 + p_2^2 \rvert})$ Velocity potential function in semi-infinite incompressible fluid, x and y axes in surface of fluid, z extending down, $z \geqq 0$. Cause: Velocity potential at surface, $z = 0$. Effect: Velocity potential at point (x, y, z).	$\frac{1}{2\pi} \cdot \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$ 867, 919
39	$Y(p_1, p_2) = \frac{\exp(-z\sqrt{\lvert p_1^2 + p_2^2 \rvert})}{\sqrt{\lvert p_1^2 + p_2^2 \rvert}}$ Newtonian potential function in semi-infinite solid, x and y axes in face of solid, z extending into solid, $z \geqq 0$. Cause: Normal potential derivative at surface, $z = 0$. Effect: Potential at point (x, y, z).	$\frac{1}{2\pi} \cdot \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ 868, 918

Continued

Cause: Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}_1\mathfrak{M}_2[Y(p_1, p_2)/(p_1p_2)]$	Cause: Unit Cisoid $\times$ Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}_1\mathfrak{M}_2 \dfrac{Y(p_1, p_2)}{(p_1-p_0)(p_2-p_0)}$
$\frac{1}{2}\left[S\left(\frac{x}{\sqrt{2\pi t}}\right)C\left(\frac{y}{\sqrt{2\pi t}}\right) + C\left(\frac{x}{\sqrt{2\pi t}}\right)S\left(\frac{y}{\sqrt{2\pi t}}\right)\right]$ 753; 754, 415*	
$\frac{1}{2\pi}\tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}$ †	
$\frac{x}{4\pi}\log\frac{\sqrt{x^2+y^2+z^2}+y}{\sqrt{x^2+y^2+z^2}-y} + \frac{y}{4\pi}\log\frac{\sqrt{x^2+y^2+z^2}+x}{\sqrt{x^2+y^2+z^2}-x} - \frac{z}{2\pi}\tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}$ †	

† This solution was obtained by double integration of the unit impulse solution, not by the operation indicated at the head of the column. The two pairs required for this operation nave not yet been found in closed form.

Automatic Machine Gaging

By C. W. ROBBINS

NOTE: This paper discusses the advantages to be gained in certain types of large scale production by the substitution of automatic machine gaging for hand testing. For testing carbon protector blocks, a machine has been developed which accepts all blocks in case a certain dimension lies between 0.0024″ and 0.0032″ and rejects those when the dimension is 0.0023″ or less or 0.0033″ or more. This machine will effect a saving of $8000 per year over the cost of hand gaging on an output of 4,500,000 blocks. The saving effected by another recently developed machine replacing a manual test is approximately $1200 a year on a production of 2,500,000 pieces, but a far more important consideration than this money saving is the elimination of an operation so monotonous that it was difficult to keep any operator on it for more than a brief period. The author points out that in some instances automatic machine gaging of the entire product will cost less than a sampling inspection in which there must be included in the direct cost of inspection the cost of some additional supervision and control.

THE cost of testing and gaging parts manufactured in large quantities frequently warrants the construction of special machinery for this work which may be more or less automatic in operation. At the Hawthorne Works of the Western Electric Company considerable study has been given to the problem during the past ten or twelve years and several such machines have been developed. The work has recently assumed more important proportions and many important developments have materialized in the last two or three years.

Some of these machines perform a single operation while others perform several operations successively. Some are automatically fed from a hopper; others are fed by an operator, who at the same time performs some visual operation. Usually each type of piece to be gaged forms a distinct problem, and a single paper to be most useful can be suggestive only as to the procedure to be followed and methods that may be used. To this end it seems best to describe with considerable detail some of the machines that are in successful operation.

SINGLE TEST MACHINE, AUTOMATIC FEED

A single purpose gaging machine with automatic feed is shown in Figs. 1 and 2. The part to be tested, shown in Fig. 2 (*a*), is used in the construction of switchboard plugs. It consists of a piece of 5/32 in. brass tubing, $1\frac{1}{2}$ in. long, having a sleeve soldered at one end and a plug soldered in the other. The machine applies a 50 pound test to the two soldered joints simultaneously.

Within the hopper *H*, Fig. 1, a shaft having three slotted arms is revolved slowly through the ratchet *R*. The slotted arms pick up the parts suspended in the slot by the sleeve and deliver them into the chute *A*, Figs. 1 and 2. An intermittently revolving turret *B*, Fig. 2,

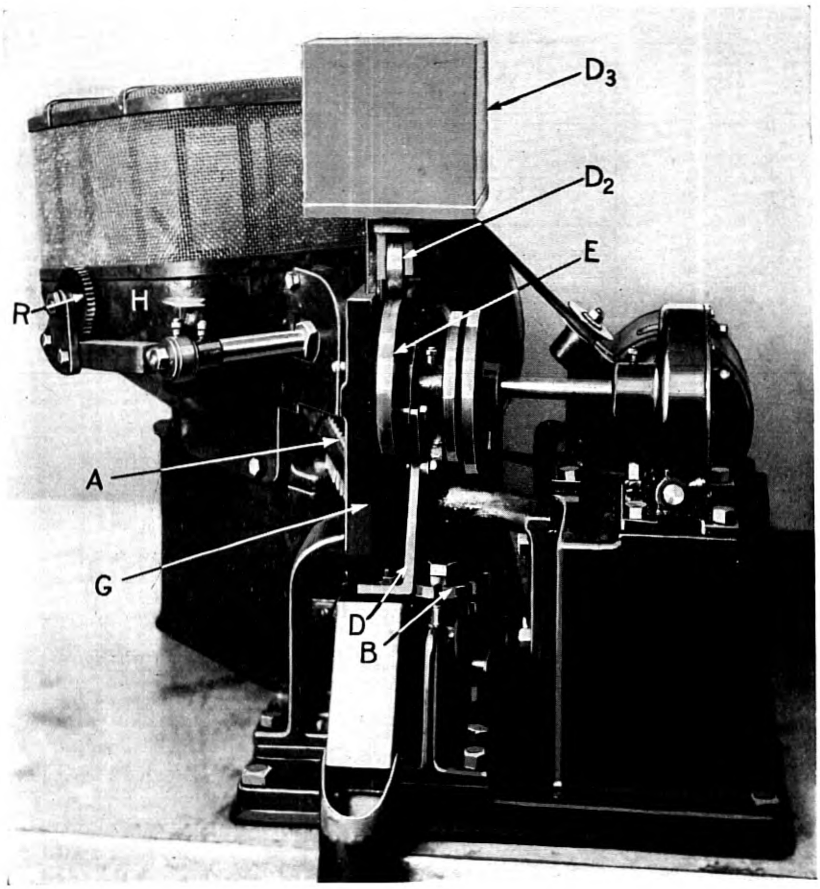

Fig. 1

having four notches or chucks, receives the parts from the chute and carries them under the plunger *D*, attached to the slide D_1, which carries the 50 pound weight D_3. The slide with the weight is raised and lowered at the proper time by the cam *E* acting on the roller D_2 attached to the slide.

After the turret has carried the part to the testing position and stopped, the plunger *D* enters the tube, gradually applying the load (furnished by the weight D_3) to the soldered joints *X* and *Y*, Fig. 2.

If either joint is defective, the weight and slide will not be supported but will drop to the limit allowed by the cam at E_1. This permits the wedge-shaped piece D_4, mounted on the slide D_1, to push the far side of the lever F (pivoted to the base at C_3) to the right. The near side of F

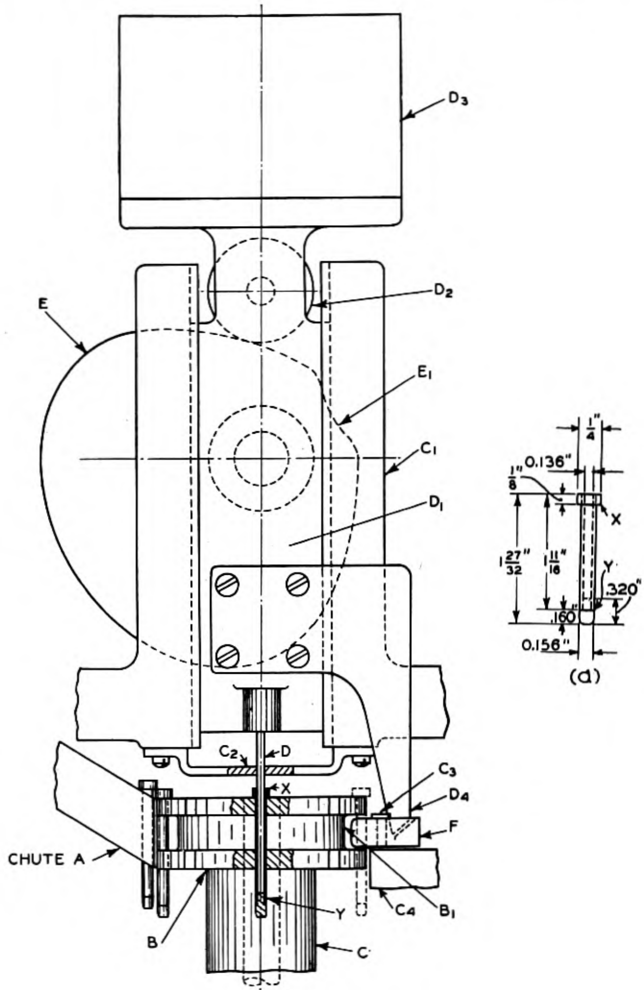

Fig. 2

is pushed into the groove B, and when the turret turns the defective piece is ejected. The lever F is reset by a cam after each operation. If the soldering is good, the load is supported by the piece for a short time while the roller D_2 is passing across the depression E_1, after which the cam raises the weight, the turret turns and the piece is discharged into the O.K. chute after passing the lever *F*.

The saving effected by the automatic machine replacing the manual test is approximately $1,200 per year on a production of 2,500,000

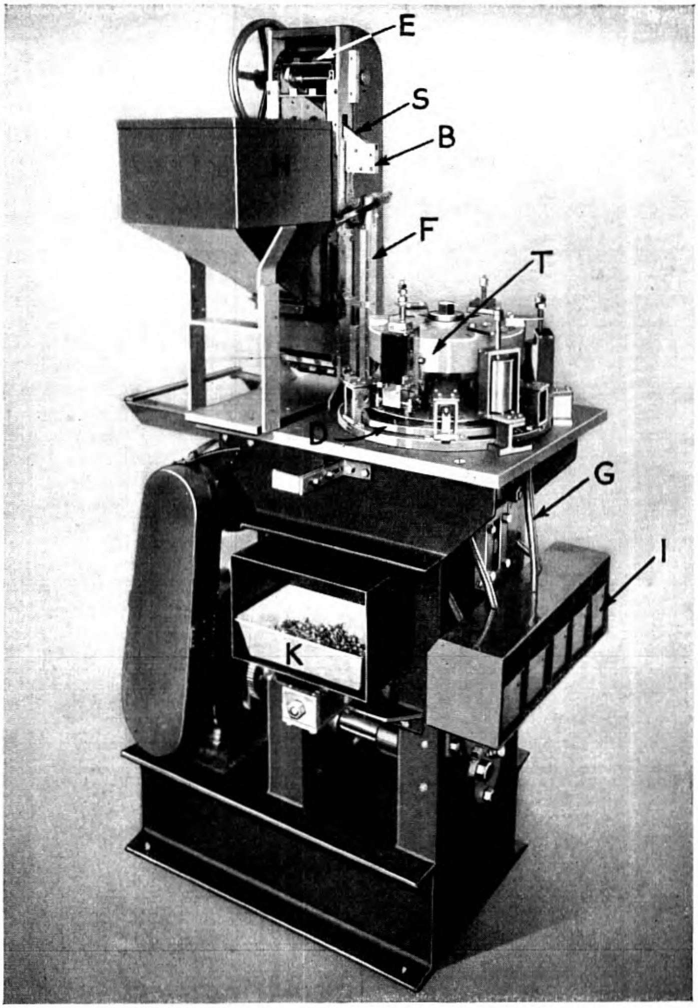

Fig. 3

pieces. However, by far the more important consideration is the elimination of an operation that was so monotonous and tiresome as to make it very difficult to keep any operator on it for more than a brief period.

Multi-Test Machine, Automatic Feed

An automatic gaging machine for applying four tests to a piece of telephone apparatus is shown in Fig. 3.

The part tested, shown in Fig. 4 (*a*), is a heat coil used to protect telephone exchange equipment against excessive electrical currents that may accidentally come in over the line wires. It consists of a tiny coil wound around a copper sleeve, into the end of which sleeve is

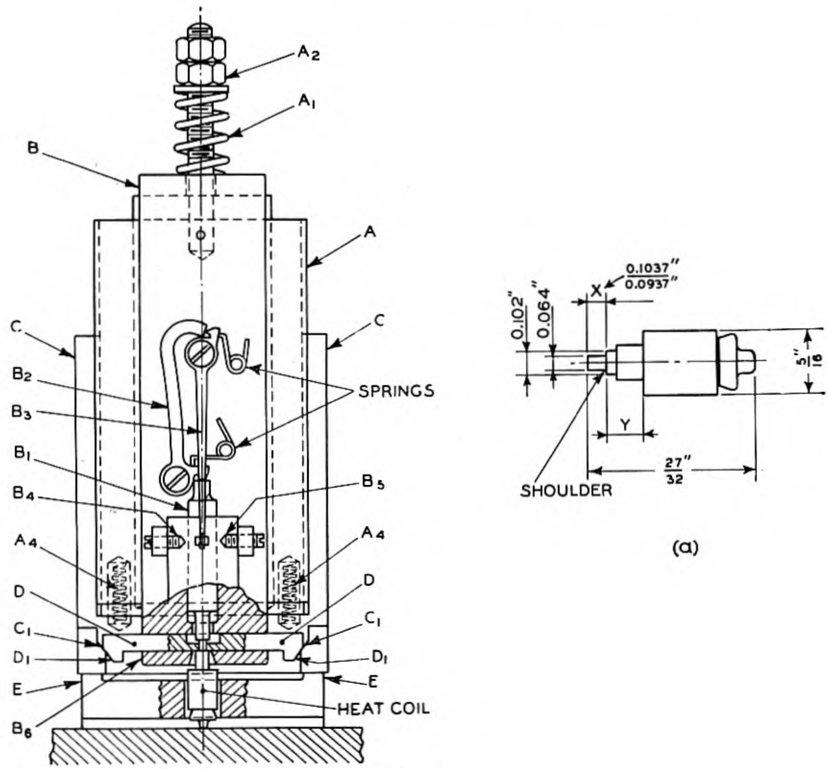

Fig. 4

soldered with low melting point solder a projecting pin. An excessive current through the coil melts the solder, allowing a contact spring to press the pin into the sleeve, which movement of the contact spring opens the circuit.

The machine gages the length of the pin X, the external length of sleeve Y, tests the strength of the soldered joint and measures the electrical resistance of the coil for high and low limits.

Referring to Fig. 3, there is an intermittently rotating disc D fitted with twelve chucks for holding the parts to be tested and a vertically reciprocating turret head *T* which carries the gages and contact fixtures for making the tests.

The hopper *H*, chain elevator *E* and feed tube *F* are shown in more detail in Fig. 5. The coils are carried up from the hopper by the

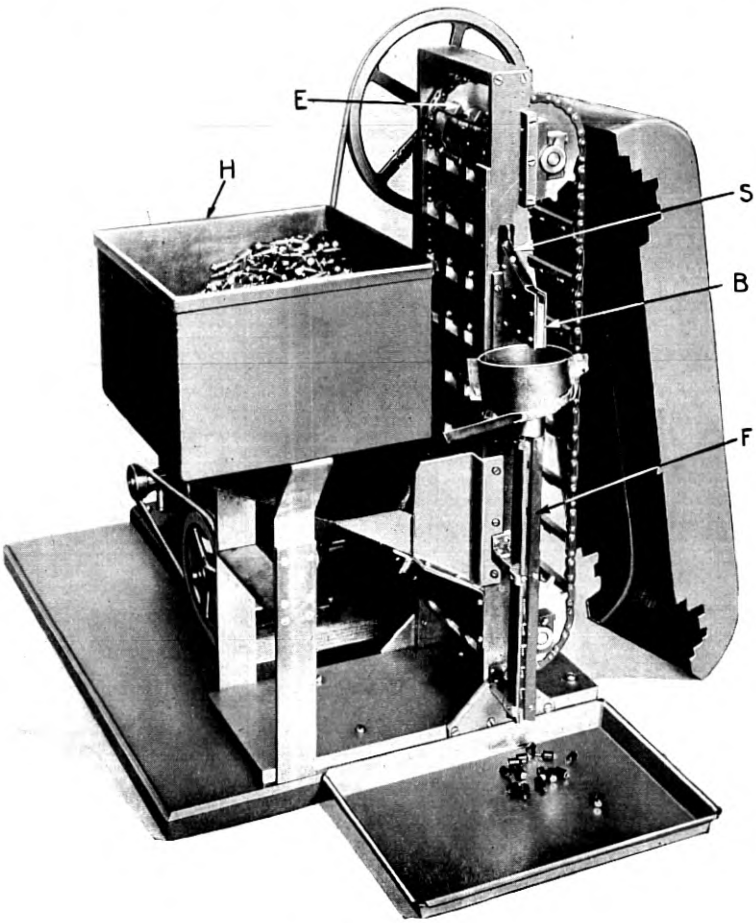

Fig. 5

elevator, two on each cross bar, and drop one after the other into the sloping chute *S*. Since the parts must be right end up for testing, the turning device *B* (shown in detail in Fig. 6) is placed at the end of the chute to turn over those pieces that are not already right end up. From the turning device the parts drop into the vertical feed tube *F*.

The chucks on the intermittently rotating disc take them one at a time from the feed tube and carry them under the gaging heads.

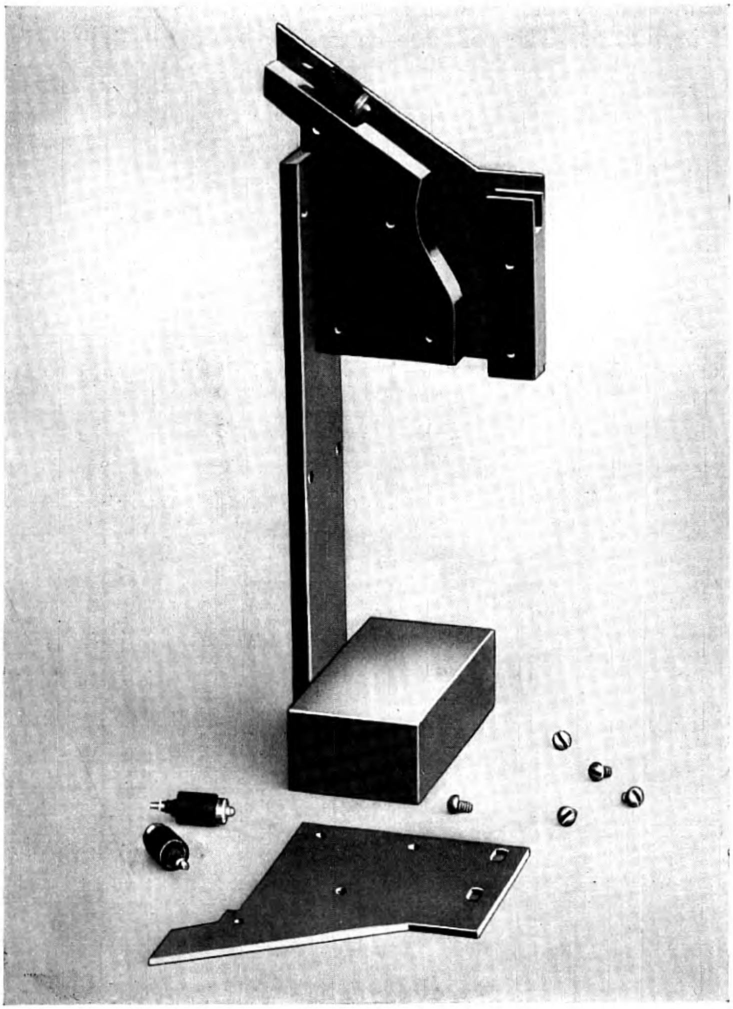

Fig. 6

In the first position the chuck picks a coil from the feed tube. In positions 2, 4, 6, 8 and 10 the coil is tested respectively for right end up, length of pin X, low limit of resistance, high limit of resistance and length of sleeve Y. At positions 3, 5, 7, 9 and 11 are located electromagnets, each controlled by the testing device in the position preceding it. These are for the purpose of ejecting defective parts so that if a part fails to meet the test at any position an electrical contact is closed which through the electro-magnets sets a trip at the next succeeding position of the chuck, and when the defective part reaches this position it is ejected from the chuck and through one of the tubes G falls into the proper compartment of the container I. At the 12th position the good parts are released from the chuck and fall into the pan K.

A multiple lever gage for this class of work is shown in Fig. 4, which shows the gage for length of pin X and also tests the strength of a

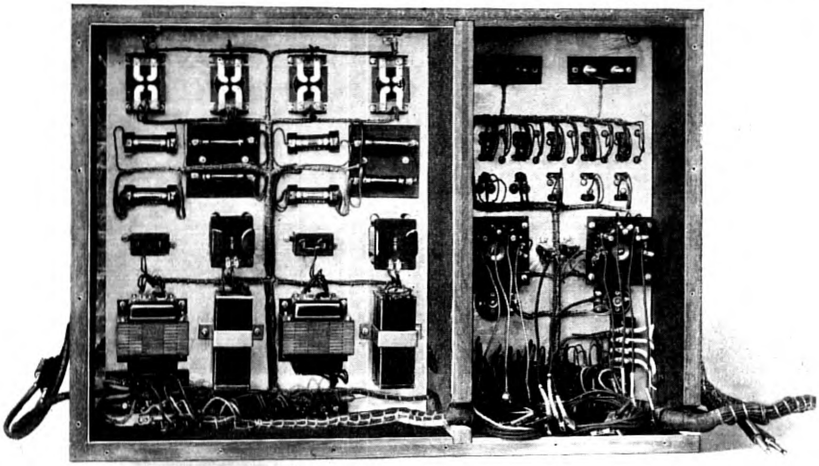

Fig. 7

soldered joint between pin X and sleeve Y. A is the body carrying two sliding members B and C. B carries the gaging mechanism and electrical contacts. B_6 is a preliminary centering guide for the heat coil. Centering slides D, D carried in B are normally held open by springs (not shown). A is normally lifted in C by springs A_4. Slide B is normally held down on A by means of spring A_1.

As the entire gage (A, B, C, D) descends, the heat coil is centered approximately by B_6. Then slide C is restrained by an anvil E, and as A continues downward the slides D, D are closed by the beveled surfaces C_1D_1, thereby centering coil and providing the gaging surface for the shoulder formed by the sleeve. As A continues downward, the gaging surfaces on D, D engage the shoulder and the pressure for operating the gage mechanism is transmitted from A to B through spring A_1, which limits the testing pressure applied to the soldered joint.

If the length of pin is within limits, the electrical circuit remains open and the coil passes on to the next test. If it is too short, the circuit is closed through lever B_3 coming in contact with B_4, and if too long the contact is made with B_5.

If the soldered joint fails, the effect is the same as a short pin. In either case the tripping magnet operates and at the next position of

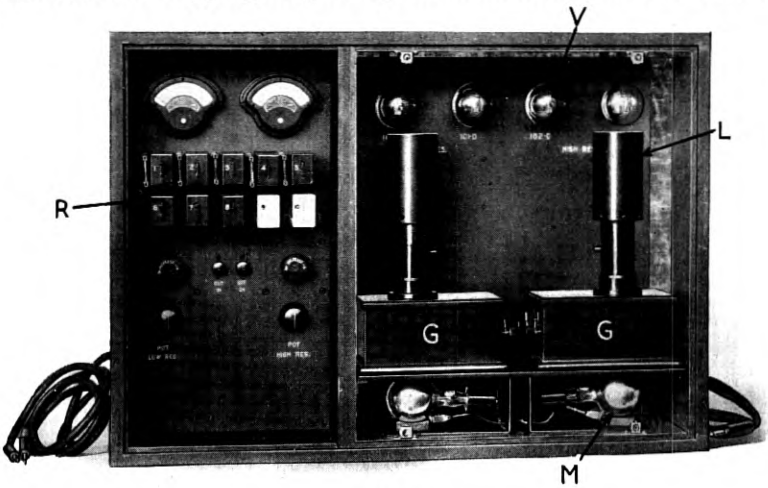

Fig. 8

the disc the defective part is released and drops into the proper container.

For testing the electrical resistance, contact is made with the coil terminals by the gaging machine and the resistance measurement proper is made by means of the apparatus shown in Figs. 7 and 8, which is essentially two Wheatstone resistance bridges, one for checking the resistance of the coil against a low limit and the other against a high limit. The galvanometers G, for indicating the balance of the bridges, each have a small rectangular, delicately pivoted coil which rotates between the pole piece of a strong magnet. The end of a long pointer attached to the coil is broadened out and contains a narrow slot which, in connection with a fixed slot, forms a shutter that passes or intercepts (depending on the position of the coil) a beam of light from a small lamp in the hood L passing to the photo-electric cell M.

The photo-electric cell is connected in the circuit of a vacuum tube amplifier, the tubes of which are shown at V. The position of the small shutter on the galvanometer needle is determined by the relative value of the resistance of the coil under test to that of standards contained in the bridge. If this resistance is too high in one case or too low in the other, the shutter is closed, preventing the light beam from reaching the photo-electric cell. This in turn through the action

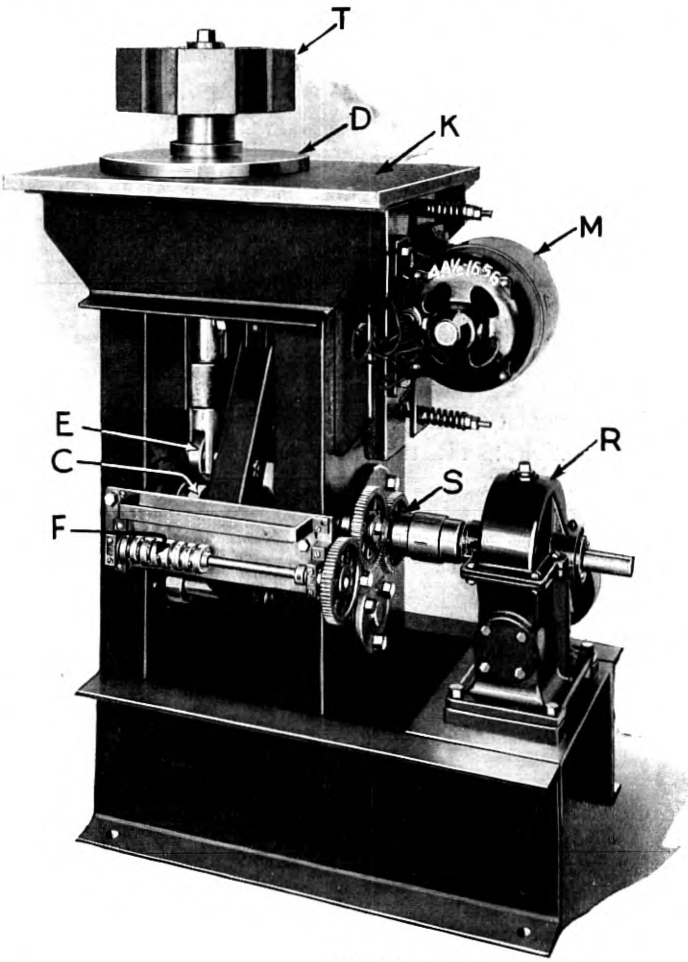

Fig. 9

of the vacuum tube amplifier and relays *R* actuates the trip on the machine which discharges the coil at the proper point.

While designing this machine much attention was given to producing a type that could be adapted readily to the testing of other parts requiring several operations.

Fig. 9 illustrates the fundamental parts of the machine. It is individually driven by the motor M belted to the reducing gear R which is attached to the main shaft S. The turntable D is given an intermittent motion by a Geneva gear and the turret head T has a vertical reciprocating motion from the cam C on the main shaft through the roller E. The cams F are used for operating a series of electrical contacts which work in synchronism with the other parts of the machine controlling the sequence of testing operations and the disposition of parts.

The frame is built of welded structural steel. The turntable may be fitted with a variety of chucks or holding fixtures and the turret with various forms of gages or testing apparatus. The cams and gear ratios may be changed to accommodate a wide range of testing requirements. Space is provided at K for a hopper or other feeding device. This type of machine is suitable for multiple tests on parts requiring special holding fixtures.

Multi-Unit Testing Machine—Semi-Automatic Feed

An entirely different type of gaging and testing machine is shown in Figs. 10 and 11, which, as illustrated, is equipped for testing porcelain

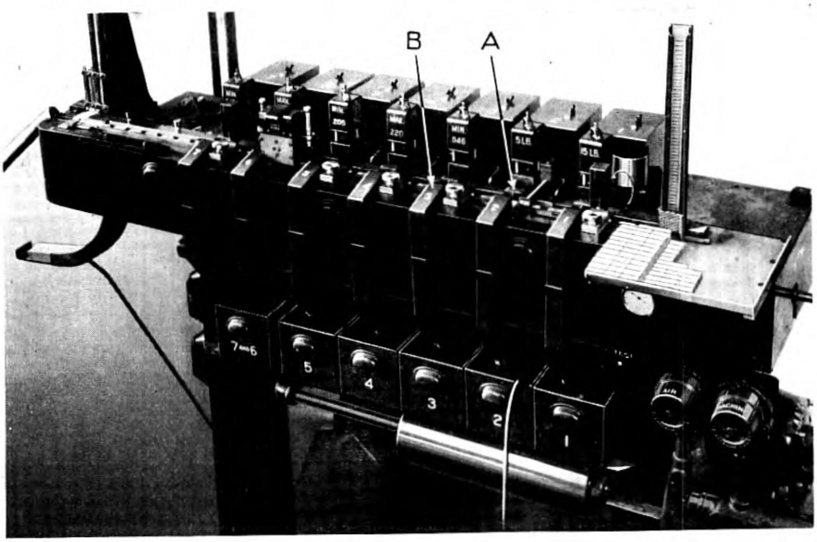

Fig. 10

protector blocks used for protecting telephone apparatus against high voltage electrical currents or static discharges.

These are porcelain blocks $1\frac{1}{4}$ in. x $\frac{3}{8}$ in. x $\frac{9}{32}$ in., having a recessed surface into the center of which is inserted a small carbon block having a face 0.0370 in. x 0.110 in., cemented in with a low melting point glass. The face of the carbon block is underflush with the rim of the porcelain block, Fig. 12 (*a*).

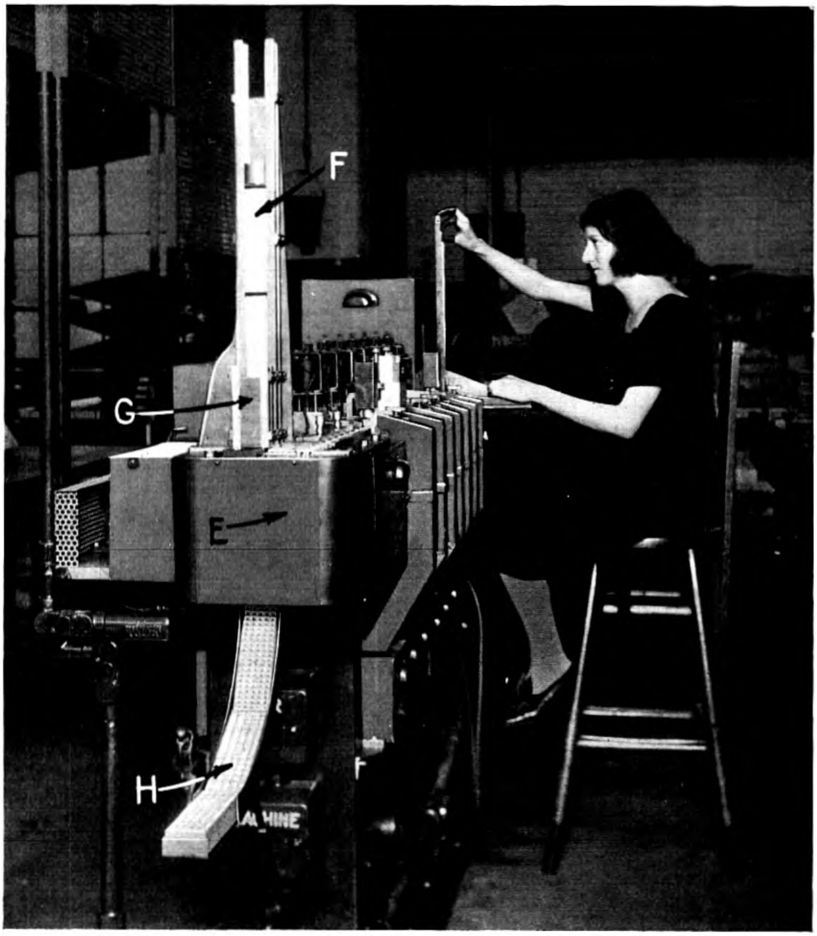

Fig. 11

As shown in Fig. 11, the blocks are being stacked by hand in the top of a vertical chute from which they are automatically fed to the machine at the bottom, but the feeding arrangement shown separately in Fig. 13 is now being added. This consists of a rotating disc having two V-shaped grooves in the surface and above it a stationary plate having a spiral slot. The operator places the blocks rear side up (by sense of touch) against the front side of the central opening of the stationary plate, three blocks being shown in this position at *A*. The disc carries them into the spiral slot at *B* and while they are passing the front opening in the top plate at *C* they are given a visual inspection.

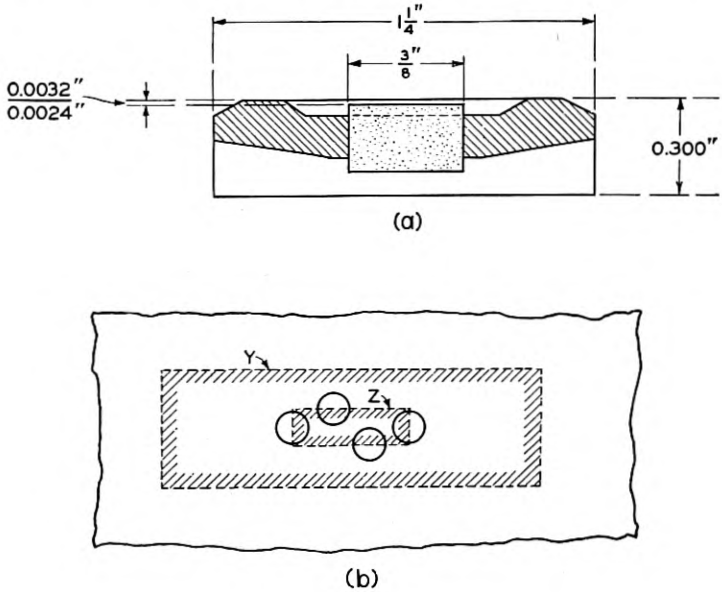

Fig. 12

During the second round they are turned over by the action of the two V grooves in the rotating disc and the radial motion given them by the spiral slot, and the front face is turned up for visual inspection during the second passage across the front opening at *D*. Any with visible defects are picked off by hand, while the others passing on through the last turn of the spiral at *E* are fed into the machine (Fig. 10) for the following gaging operations:

1. 15 lb. weight test for defective cementing.
2. 5 lb. weight test to detect misplaced inserts.
3. Gage height of back face of insert—minimum 0.046 in.
4. Gage thickness of block—maximum 0.220 in.
5. Gage thickness of block—minimum 0.205 in.
6. Gage the underflush dimension of insert which must be maximum 0.0032 in., minimum 0.0024 in., for at least half the area of the face of the carbon insert.
7. Gage the underflush dimension for minimum 0.0024 in. over entire face of insert.

The gage for operation No. 6 has four $\frac{1}{8}$ in. plungers (*P*, Fig. 15) arranged to make contact with the insert as shown at (*b*), Fig. 12, and the electrical contacts of the gage controlled by the plungers are connected to a bank of relays so that if any three or all four of the gage points are within the limits the block is passed, but if two or more are outside the limits the block is rejected. This gage, shown in Figs. 14, 15, 16, and 17, is without pivots, the moving parts being controlled

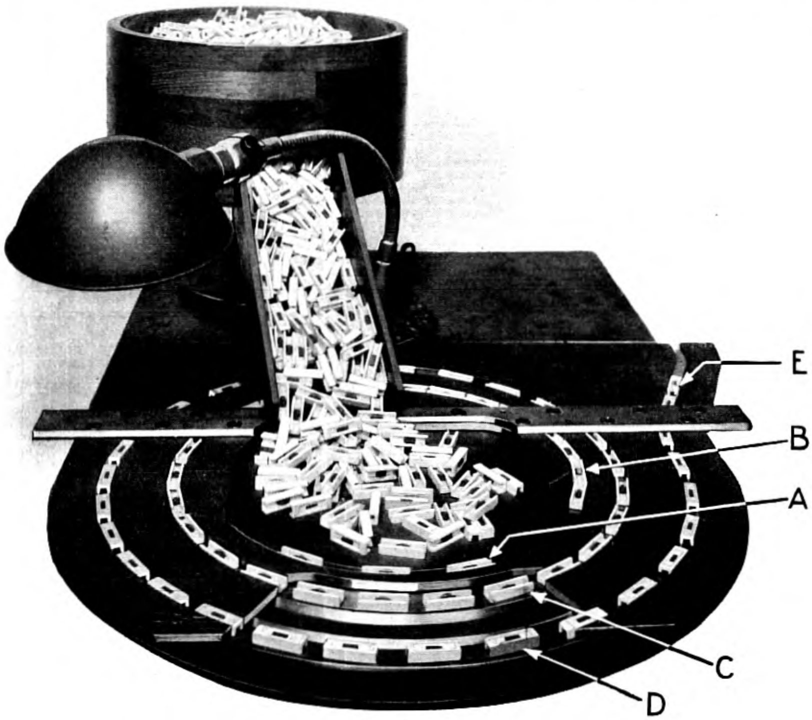

Fig. 13

by thin steel reeds as shown in Fig. 14. Fig. 15 shows a partial, and Fig. 16 a complete, assembly, while Fig. 17 shows the equalizing levers for centering the block in the gage. Using master steel gage blocks, the contact points are adjusted by the screws *A*, Fig. 16, to accept blocks if the underflush dimension is minimum 0.0024 in. and maximum 0.0032 in., and reject blocks when the dimension is 0.0023 in. or less or 0.0033 in. or more.

The single plunger gages used for operations 2, 3, 4, 5 and 7 are also of the reed type and are similar to that shown in Fig. 18. These have a sliding electrical contact instead of a point contact, the pointer *A* sliding on the surface of the insulated block *B* and making contact with the flush metal insert *C*.

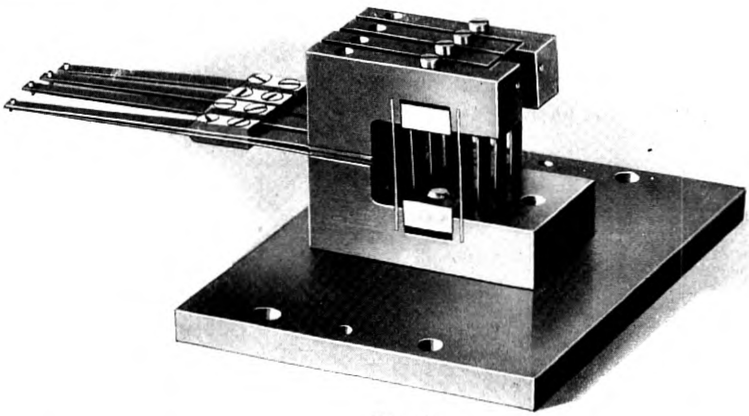

Fig. 14

Considerable experience has been gained in the design and use of the reed type gages and they are proving very satisfactory for a wide variety of uses. They are relatively inexpensive to build, require but

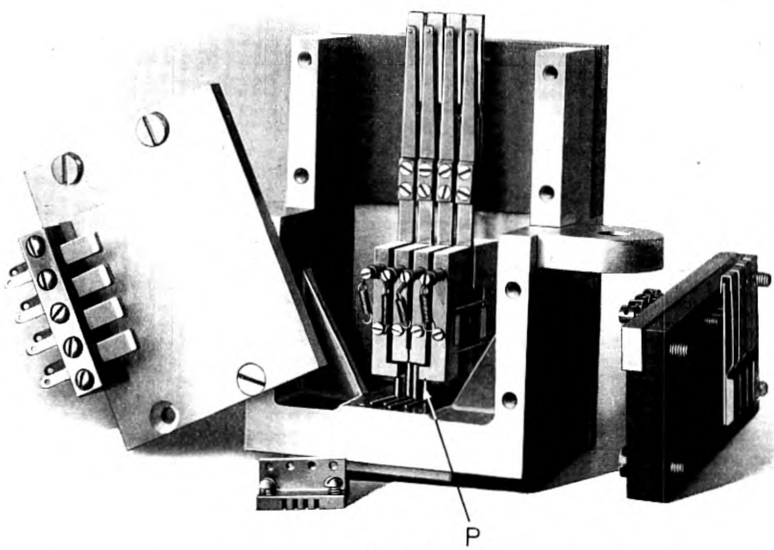

Fig. 15

little maintenance, the pressure on the gaging point may be kept low if desired and they are reliable in action.

Following each gaging operation the blocks pass between air blast tips *A*, Fig. 10, and the square tubes *B* marked 1 to 7 leading to receptacles bearing the same numbers. The electrical contact in each gage, which is closed by a defective block, sets a trip connected to the adjacent air tube so that, as the block passes, the air cock is opened for an instant and the defective block is blown out of the test line into the tube, through which it falls into the proper container. A small air compressor is installed as part of the machine.

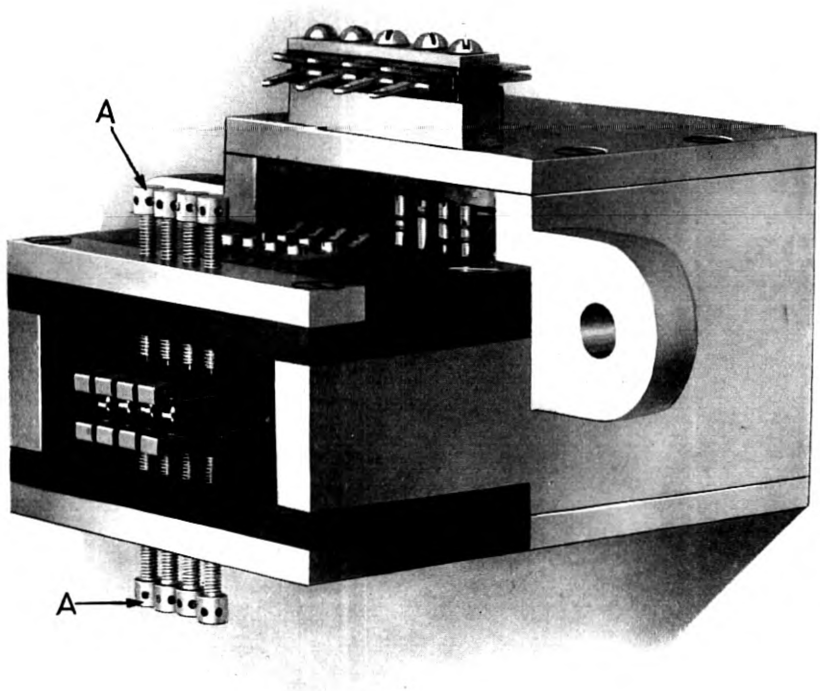

Fig. 16

The O.K. blocks pass along to the automatic packing attachment *E* (shown in more detail in Fig. 11), which places 100 of them in a box in layers of five each with cardboard separators between layers. The empty boxes are shown in the magazine *F*, the separators at *G*, and the filled boxes emerging at *H*.

The feed table shown in Fig. 13 will be placed at the same end of the machine as the packing attachment and the blocks will be carried from the feed table to the far end of the machine by a conveyor. By this arrangement one operator can feed the machine, make the visual inspection and remove the finished packages. This machine will effect a saving of $8,000 per year over the cost of hand gaging methods

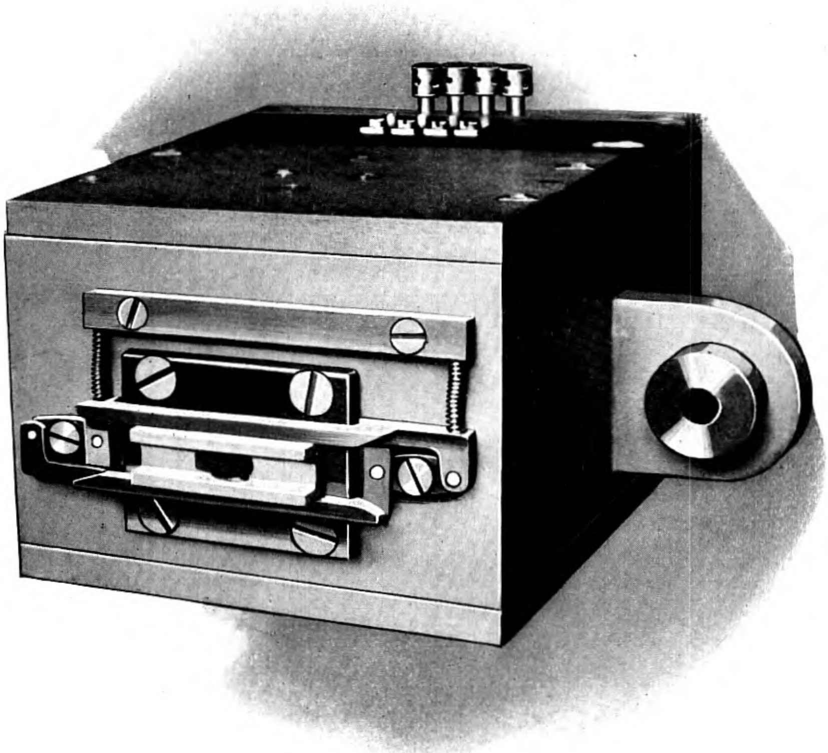

Fig. 17

on an output of 4,500,000 blocks. A similar machine is being built for another size of blocks.

Each gage with its associated equipment is an independent unit as shown in Fig. 19. The gages are located at *G*, either above or below the working surface. Relays and other electrical apparatus at *E*. The solenoid for opening the air cock is shown at *S* and the air blast tip at *T*. The connection for the electrical supply for each unit is made with the cord and plug *P*. The cams *C* and contact springs *D* operate in synchronism with the other parts of the machine and control details of the gaging operation and the air blast.

The main shaft of the machine can be seen at *A*, which drives the disc *B* through worm gears. When the units are attached the pin *K* engages a slot on a disc attached to the rear of the unit shaft *I*. Attached to the front end of the same shaft are two eccentrics *H* (shown in Fig. 20) which operate the feeding device located just back of the blocks shown in the illustration, Figs. 10 and 11.

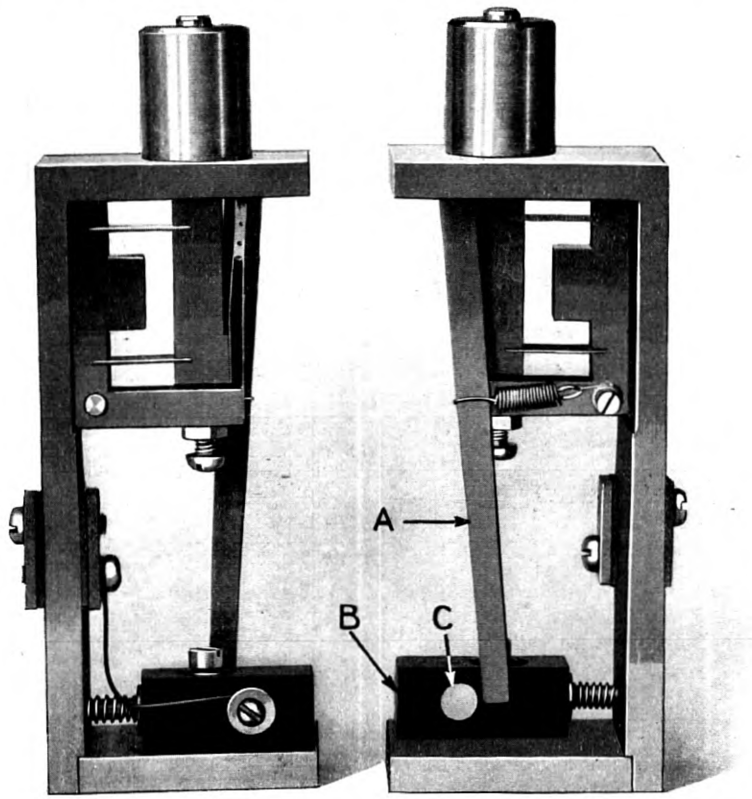

Fig. 18

The feeding mechanism (Fig. 20) is of the finger bar type consisting of two reciprocating bars *A* and *B*, both having the same travel.

Fig. 20 indicates the relative position of the driving parts. Finger bar *A* is operated by a bell crank gear segment and eccentric properly timed to step the protector blocks to the gaging position immediately prior to the gaging operation.

Feed fingers *C* are so located in this finger bar that the protector blocks are centrally located under the gage when the finger bar comes to rest at its forward position.

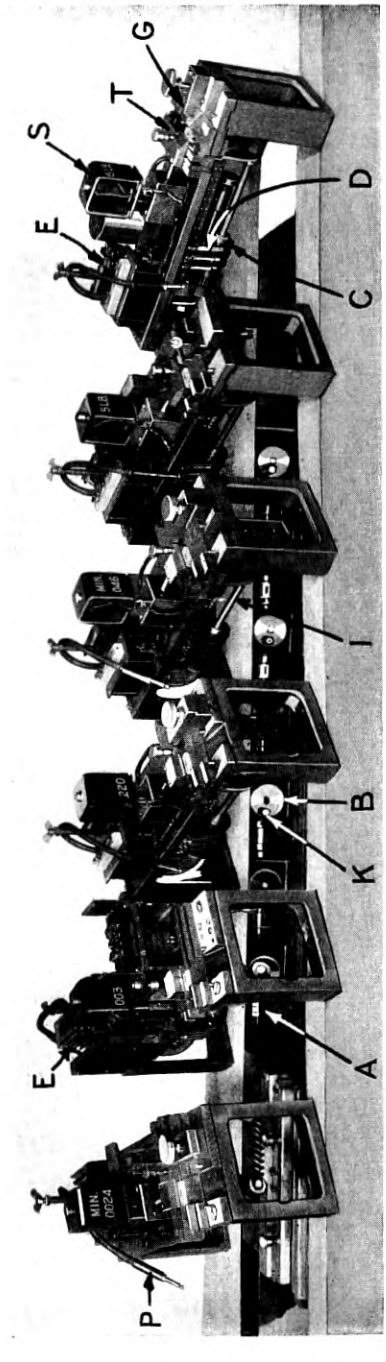

Fig. 19

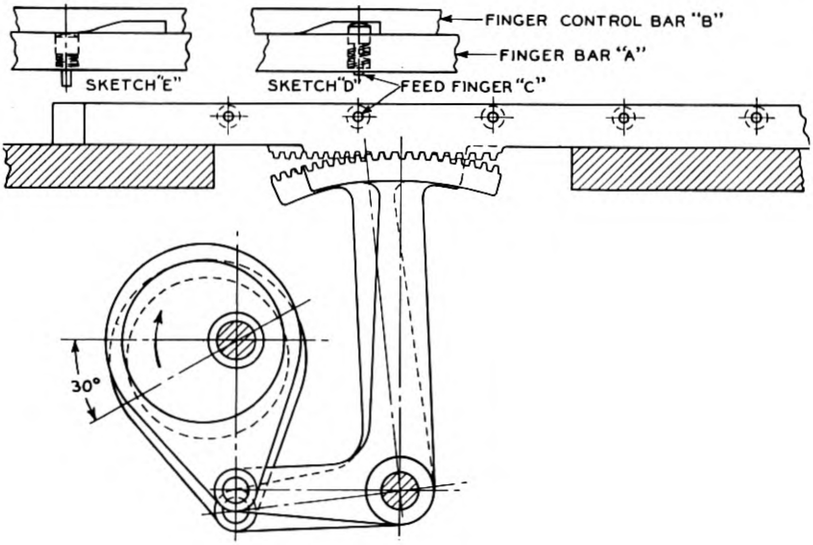

Fig. 20

The finger control bar *B* is likewise operated by a bell crank gear segment and eccentric but is set with a thirty degrees lag. The effect of this lag is illustrated in sketches *D* and *E* which show the manner in which the feed fingers *C* are withdrawn on the return stroke of the finger bar.

Economic Considerations

A comparison of hand versus machine gaging is given in Table I. In this particular case the quality of the inspection work was bettered approximately 100 per cent, while the cost was reduced 60 per cent. While this showing is rather better than the average, the tendency in most instances is in the same direction.

Like the turret machine previously described, this one was designed with the idea of making it readily adaptable for similar work on other parts. The number of units and thereby the number of operations on a machine may be varied greatly. A unit may be quickly removed for adjustment or repair and easily replaced. If the conditions warranted, spare units could be provided and an adjusted unit put in the place of a defective one in fifteen or twenty minutes.

The frame is built of welded structural steel. The machine is a complete unit with individual motor drive requiring only the attachment of the electric power supply.

The unit system for the equipment provides a wide latitude in the choice of gaging and testing fixtures to be used and the details of operating them.

TABLE I

COMPARISON OF THE ECONOMIC FACTORS ON TESTING OF PROTECTOR BLOCKS BY MACHINE METHOD AND BY MANUAL METHOD

	Machine Method	Hand Method	Remarks
Capacity	8,000,000 per year, 1 machine at 3,600 per hour	8,000,000 per year, 9 hand gages required	
Cost of Equipment	1 machine at $10,000	9 gages at $150—$1,350	Machine method requires $8,650 additional first cost, meaning an annual yearly charge, at 8%, of $670.
Cost of Labor	$1\frac{1}{2}$ operators at $1,920 = $2,880 per year (including loading)	6 operators at $1,920 = $11,520 per year (including loading)	Machine method gives a saving of $8,640 per year in labor costs.
Cost of Power	$40 per year	0	Machine method costs $40 a year additional for power.
Floor Space	Machine requires 35 square feet	6 operators require 90 square feet	55 square feet saved by machine.
Maintenance	Cost of maintenance at $130 per million blocks = $1,040 per year	Cost of maintenance of 9 gages per year = $1,560	$520 saved by machine method, nearly 30%.
Accuracy — Repeating results on parts that vary from tolerance limit by .0001 in.	95% (See note below)	45% Low degree of accuracy due to (*a*) plurality of gages, (*b*) plurality of operators	Degree of accuracy doubled.

Note: This means that a master gage or parts that are .0001 in. outside the tolerance limit will be rejected, in the first case, an average of 95 times in 100 trials, and in the second case 45 times. Parts that are .0001 in. within the tolerance limits will be passed as good in the same ratios. The disposition of parts that vary more than .0001 in. either way from the tolerance limits would follow the normal probability law. The figures given do not give any indication of the very small percentage of defective parts that would be passed as good or good parts classed as defectives, as these would depend upon the relative number of defectives and the distribution of their variations from the tolerance limit, as well as the precision of the methods given above.

The development of machine gaging has been greatly aided by the development of accessory parts, such as reliable indicating gages, chromium-plated parts, sensitive but sturdy relays, vacuum tube amplifiers and photo-electric cells.

Sampling methods or percentage inspection are applicable to parts that are made under conditions that may be considered approximately uniform or, as a statistician would say, under "a constant system of causes." Piece parts made in the punch press and screw machine are good examples of this. Many other classes of operations, particularly those in which some part is manual, produce parts which are not so uniform. As the variability increases, or as the requirements for precision become more exacting, the possibilities of sampling inspection become less attractive.

In many cases the conditions and requirements are such that only detail inspection or gaging is satisfactory.

In some instances automatic machine gaging of the entire output will cost less than a sampling system in which there must be included with the direct cost of inspection the cost of some additional supervision and control.

The possibilities so far as designs are concerned seem almost unlimited, so that the question of when to apply such methods becomes purely an economical one in which the number of parts to be handled, the difficulty, unpleasantness or tirescmeness of the operation, the precision required, and the cost of suitable labor become the controlling factors.

Aside from the question of cost, it is often a matter of great satisfaction to place an objectionable hand operation on the machine and release the labor for more pleasant and useful work.

Contemporary Advances in Physics, XVI

The Classical Theory of Light, Second Part[1]

By KARL K. DARROW

MEASUREMENT of wave-lengths is the subject which we shall now consider. So entitled, the topic seems unpromising, as some dry exercise in mensuration; but in truth it is distinguished for beauty and variety, and implicated with the whole of modern physics. This is not measurement of the lengths of palpable objects, as pieces of lumber or cloth, which are laid alongside of a yardstick or clamped in the jaws of a gauge. In optics, the methods of measuring wave-lengths are the methods of proving that waves exist, therefore of testing the undulatory theory of light. One could not reasonably ask for evidence of light-waves more convincing than the concord of the values obtained for the wave-length say of sodium yellow light, by all the diverse instruments which act by causing interference or diffraction: Newton's tapering film of air between a lens and a plate, Fraunhofer's grid of iron wires, the tilted mirrors of Fresnel, Michelson's echelon, and all the many gratings and interferometers continually in use in laboratories and classrooms. Wave-lengths of X-rays are computed from the diffraction-patterns imposed on X-ray beams by intercepting crystals, and these patterns were the evidence which showed some fifteen years ago that the rays are of the nature of undulations, though it could not disprove that in some paradoxical way they are also of the nature of corpuscles. From similar diffraction-patterns imposed by crystals on electron-streams it follows that these also are partly of the nature of waves, and again the patterns have supplied the values of the wave-lengths.

Moreover, evidence for waves and values of their lengths are only part of what a grating can supply. Once we are sure that we know the wave-length of a certain kind of light, we can send it against a grating and study the diffraction-pattern with the opposite intent: analyzing not the light but the grating, and deducing the widths and the spacings of the slits, if it is an alternation of slits and stops—the spacing and the shaping of its grooves, if it is an engraving on metal or glass—the arrangement of the atoms and of the electricity within the atoms, if it is a crystal. Therefore the methods for measuring wave-lengths of X-rays are also those for exploring the structures of solids and of the atoms of which these are composed. Remember

[1] Continued from the April, 1928, issue.

also that the instruments efficient in this field are the most delicate and accurate which have ever been made for any purpose; that they may be used to measure ordinary lengths and other physical quantities with an almost unbelievable precision; that the theory of relativity sprang from an experiment performed with one, and the only known way of measuring the diameter of a star involves the use of another. Surely, if this topic is not interesting, nothing in physics is interesting.

Methods of measuring wave-length are sometimes divided into those which operate by interference and those which utilize diffraction. Though to a thorough insight the distinction is only trivial, at the outset it is convenient. In an extreme example of what is specially called "diffraction," a single train of plane parallel waves is sifted through a sieve in the form of a grid or a sequence of slits; and each element of wave-front which passes through a slit evolves and spreads thenceforward according to the law of wave-propagation. Eventually—as a rule, in the focal plane of the lens beyond the grating—a region is reached where the light from the several slits intermingles; and here occur the variations in amplitude which disclose the waves and the wave-lengths. For, as I have said earlier, the eye perceives only the amplitude of the light-waves, and not their phase; therefore, in a plane-parallel beam where the phase is perpetually changing but the amplitude is everywhere the same, the eye receives a uniform impression, with nothing wavelike in it; and to make such a beam reveal that it is undulatory, we must cause the amplitude to vary from point to point. This is what we accomplish by breaking the beam into fragments, or lacerating it with obstacles, preferably with an obstacle having a periodic structure of its own, which is a grating. But it may also be accomplished by causing two plane-parallel beams to intersect one another under proper conditions. The region where they overlap is then a region of varying amplitude—indeed, the variations are as great as one can imagine; for if the beams are equally intense, there is a succession of parallel planes of no vibration and darkness, which separate spaces where there is vibration and light. The widths of these spaces, the *fringes*, may be computed from the wave-length, and reversely the wave-lengths from the widths, very simply and without any knowledge of the law of wave-propagation beyond the familiar expression for plane waves. Therefore this method of measuring wave-lengths by causing *interference* of two parallel beams is much the easiest to grasp; but it does not differ in principle from the method involving a grating, for that acts by interference between the beams from the various slits.

The Diffraction Grating

The ideal diffraction grating of theory is a sequence of equally-wide perfectly vacant slits, separated from one another by strips absolutely opaque and equal in width to one another though not necessarily to the slits. Actual gratings seldom resemble this picture, though Fraunhofer's first—the most important instrument, I suppose, in the story of spectroscopy—was an approximation to it which he made by winding a wire around and around a pair of screws held parallel and wide apart, soldering it in place and cutting away the alternate strands. So were some of his others, composed of gold-leaf mounted on glass and scratched along parallel lines with a diamond. So-called *reflection gratings* would also conform with the picture, if they consisted of bands of perfectly smooth reflecting metal separated by absolutely non-reflecting bands; for then the result would be the same as if the light came through the reflecting strips from a virtual image of the source located behind. Practical reflection gratings are not usually very like this conception, for the entire surface of the metal block is ploughed up into roughly-shaped furrows. In fact one could scarcely define the word "grating" less generally than as a periodically-repeated obstruction, or better yet a periodically-repeated device for perturbing the free onward flow of a beam of light.

Nevertheless the theory of the ideal grating contains most of what is required for the theory of the practical appliance. The reason is, that the action of the grating upon the light can be separated into two factors, each of which produces its own separate effect, each of which may be studied apart from the other. Commonly there is a set of maxima of brilliance in the diffraction-pattern; otherwise expressed, there are certain directions in which the intensity of the diffracted light is exceptionally great. From the locations of these maxima, the wave-length of the light is calculated. Now these locations are determined by the spacing of the units—be they slits and bars, furrows and ridges on a reflecting surface, planes of atoms in a crystal, or what not—whereof the exact repetition in sequence constitutes the grating. Thus if we know that a certain grating is ruled with 1000 "lines" to the inch, we can compute the wave-length of sodium light from the positions of the maxima in its diffraction-pattern, without knowing or caring whether the rulings are slits, grooves, triangular indentations, wavy ripples, or rough-bottomed troughs. If we know that in a crystal a certain grouping of atoms repeats itself one million times in a centimetre, we can calculate the wave-length of an X-ray beam or an electron-beam from the locations of its diffraction-maxima, without knowing anything about the arrangement of atoms in the group.

The contour of the rulings of a grating does, on the other hand, affect the relative intensities of the various diffraction-maxima and the details of the distribution of intensity throughout the diffraction-pattern. If they are grooves or troughs, their profiles in cross-section have influence upon the pattern; if they are slits with bars between, the ratio of width of slit to width of bar must be taken into account. In crystals the arrangement of the atoms in the groups controls the intensity-ratios among the diffraction-maxima and conversely is deduced from observations made on these. Even the distribution of electricity in the separate atoms of a crystal may be read from the details of the diffraction. These effects however can intrude upon the measurement of wave-lengths only in the cases—comparatively rare—in which some of the diffraction-maxima are actually blotted out, so that the uninformed observer may misinterpret those remaining. Except for cases such as these, one may derive the formula for computing the wave-length by assuming any convenient form of grating; and therefore we may think about a grid of slits and bars.

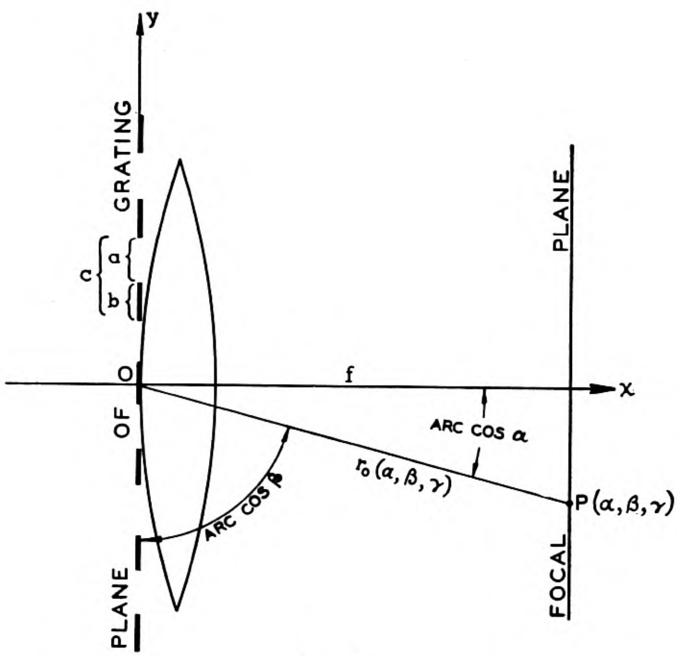

Fig. 1.

Consider then an alternation of slits of width a and bars of width b, occupying the plane $x = 0$. A beam of plane-waves, monochromatic and of the wave-length λ, travelling along the x-direction in the positive sense, shall fall normally upon it from behind. It is our object to determine the amplitude of the waves in the region in front of the grating. Suppose for instance that we select a plane parallel to the grating and at the distance x in front of it, and derive the formula for the amplitude at any and every point (x, y, z) of this plane. This formula is the description of the theoretical diffraction-pattern in the plane in question; and the actual pattern may be observed by setting up a screen or a photographic plate in the corresponding place.

We went through this process for a single aperture in the first part of this article; and there we found that the pattern is simpler (or, at least, more calculable) the farther the plane of observation is removed from the plane of the slit, being simplest when the two are infinitely far apart. To realize this case in practice we have only to set a lens immediately before the grating. Then, the diffraction-pattern appropriate to the plane at infinity—the so-called "Fraunhofer diffraction-pattern"—is transposed into the focal plane of the lens, where we must place the photographic plate in order to record it. Naturally it is reduced in scale and augmented in intensity when it is thus transposed, and for this as well as other reasons we had better express it, not in terms of the coordinates (x, y, z) of the points in the focal plane, but in terms of the direction-cosines ($\alpha = x/r$, $\beta = y/r$, $\gamma = z/r$) of the lines drawn to these points from the origin of coordinates.[1] Our formulæ for the diffraction-pattern in the infinitely-distant plane are in fact naturally expressed in terms of α, β and γ; and the lens may be regarded as an agency whereby that value of amplitude, which otherwise would have existed infinitely far away upon the line with direction-cosines (α, β, γ), is amplified by a constant factor and shifted inward along this line to the point where it intersects the focal plane.

We wish, then, to determine the vibration produced by a regular sequence of slits, all over the plane which is either infinitely distant or else the focal plane of the lens, according as the lens is absent or present.

Now we already have a formula for the vibration produced in that plane by any slit individually. It is the formula (93) of the first part of this article; to wit:

$$\begin{aligned} s &= \text{const. } (1 + \alpha)[C \sin (nt - mr_0) - S \cos (nt - mr_0)] \\ &= \text{const. } (1 + \alpha)\sqrt{C^2 + S^2} \sin (nt - mr_0 - \epsilon). \end{aligned} \tag{1}$$

[1] The origin should coincide both with the centre of the lens and with some point in the plane of the diffracting apertures. This is impracticable; but the error apparently does not make any trouble in practice.

Here the symbol s stands for the amplitude of the vibrating entity—whatever that may be—at the various points ("field-points") where the plane in question is intersected by the lines drawn from the origin with direction-cosines (α, β, γ); the symbols n and m for 2π times the frequency and 2π over the wave-length of the vibration, respectively; and the symbols C, S, r_0 and ϵ for various functions of (α, β, γ). In particular, C and S denote certain integrals extended over the slit, so that they involve the breadth of the slit as well as the variables (α, β, γ); this last is true of ϵ also; but r_0 denotes the distance from the origin of coordinates to the field-point, and thus involves the variables but not the breadth of the slit. As for the nature of the vibrating entity which is designated by s, I am keeping it intentionally vague. Suffice it to say that s is something of which the phase cannot be detected in any known way, but the amplitude controls the intensity of the light; the observed intensity being, according to the classical theory of light, proportional to the square of the amplitude.[2]

If therefore we were studying the diffraction-pattern of a single slit, we should be concerned only with the factor $(1 + \alpha)\sqrt{C^2 + S^2}$ in the expression for s. It would be short work to develop the expressions for C and S for a single rectangular aperture, finite or infinite in length; and having developed them, we should have solved the problem of the single slit; but in respect of our present purpose, it would be a detour. Remarkable as it may seem, the pattern of the single slit is only of secondary importance in determining that of a regular sequence of slits. When we undertake to sum up expressions such as (1) in order to compute the diffraction-pattern of such a sequence, we find the emphasis violently shifted. A new set of diffraction-maxima appear, and their positions are determined by the variation of the phase $(nt - mr_0 - \epsilon)$ from one slit to the next—in more general language, the variation which the phase undergoes in passing over one complete *period* of the grating-structure. Meanwhile the influence of the coefficients C and S, and that of the breadth of the slit which they involve, recede into the background. Not the features of the individual slit, but the interval at which one follows another, is now the dominant factor. This is the situation foreshadowed in the introductory pages.

To bring this out, let us orient the z-axis in the plane of the grating so that it runs parallel to the slits, which are of width a and are separated by bars of width b so that the period c of the grating is equal to $(a + b)$. The x-axis is to run, as heretofore, perpendicular to the surface of the grating and through the centre of the lens, so that it

[2] Or rather to the sum of the squares of the amplitudes of several quantities, any one of which separately satisfies the same equations as s.

intersects the focal plane (or the plane at infinity) at the point which is the centre of the diffraction-pattern. The y-axis is to lie in the plane of the grating perpendicular to the slits. The light comes up to the grating normally from behind, and therefore follows the x-direction. For reasons which will presently appear, it will suffice to calculate the diffraction-pattern over not the entire focal plane, but only the line where this is intersected by the xy-plane. For any field-point on this line $\gamma = 0$ and $\alpha = \sqrt{1 - \beta^2}$.

To the total vibration at any field-point, each slit now makes a contribution given by the expression (1). Numbering them in order, we may write for the contribution of the kth slit:

$$s_k = \text{const. } (1 + \alpha) A_k \sin \varphi_k, \tag{2}$$

in which A_k stands for the value of $\sqrt{C^2 + S^2}$, and φ_k for the value of $(nt - mr_0 - \epsilon)$, appropriate to the kth slit.

Now A_k has the same value for all the slits. This may be proved directly from the formulæ [3] for C and S, or indirectly by the following chain of reasoning. The function $\sqrt{C^2 + S^2}$ describes the diffraction-pattern formed by the single slit on an infinitely-distant screen, when there is no lens. Two similar slits a finite distance apart would produce two such patterns, one displaced by the same finite amount relatively to the other. But on the infinitely-distant screen the fringes and other details of the patterns are themselves infinitely broad, so that a finite displacement of one with respect to the other leaves them still practically—and, in the limit, exactly—in coincidence. This remains true when the patterns are transposed to the focal plane of the lens; those produced by a slit in one place coincide exactly with those which would be produced by an exactly similar slit lying anywhere else.[4] Therefore $\sqrt{C^2 + S^2}$ must be the same function of (α, β, γ) for every slit.

At first glance this argument seems to prove that the diffraction-pattern for the grating is merely that of the individual slit, multiplied manyfold; but that conclusion would in general be false, for we have not to add amplitudes but to compound vibrations with due regard to their relative phases. The phase φ_k which figures in equation (2) differs from slit to slit; and if these follow one another at equal intervals, φ_k changes from one to the next in equal steps.

[3] By operating on the expressions (presently to be derived) for C and S in the case of a rectangular aperture, one may show that, while each separately varies when the position of the rectangle with reference to the origin is changed, the sum of their squares remains the same. As any finite aperture may be regarded as a collection of finite or infinitesimal rectangles, the theorem is general. I am indebted to Mr. L. A. MacColl for working this out.

[4] The practical limitation to this statement would be set by the impossibility of making an ideally perfect lens of indefinitely great size.

To prove this, and to find the magnitude of these equal steps, one may proceed as follows. Omitting the lens again, consider in the grating any two consecutive slits k and $(k + 1)$, and on the very

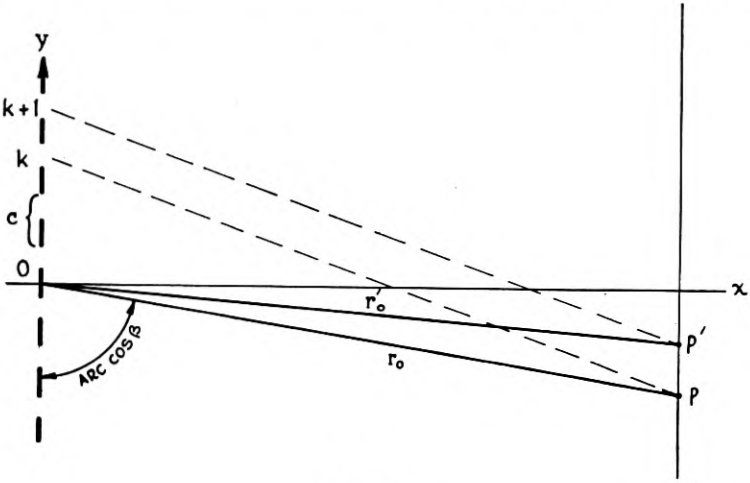

Fig. 2.

distant screen two field-points P and P' separated by the same distance c as separates corresponding points of the two slits—that is, the period of the grating. Write down successively the formulæ for the vibrations produced by k at P and by $(k + 1)$ at P'. They are, respectively:

$$A \sin(nt - mr_0 - \epsilon_k); \qquad A \sin(nt - mr_0' - \epsilon_{k+1}).$$

Since P lies in the same direction from k as P' from $(k + 1)$, these two are equal; hence:

$$\epsilon_{i+1} - \epsilon_i = m(r_0' - r_0). \tag{3}$$

Here the factor $(r_0' - r_0)$ on the right is the difference between the distances from the origin to P' and to P. In the limit when these distances become infinitely great, all the lines from the origin and the slits to P and P' become parallel and inclined to the plane of the grating by the angle of which the cosine is β; and the difference between the paths to P' and to P from the origin attains the limiting value $c\beta$. Hence in the limit:

$$\epsilon_{k+1} - \epsilon_k = mc\beta. \tag{4}$$

This is the "step" or difference in phase between the contributions of successive slits to the vibration at the field-point. The expression looks more familiar if we put θ for the angle between the normal to the grating and the direction from the grating to the field-point, so that $\beta = \sin\theta$; then:

$$\epsilon_{k+1} - \epsilon_k = mc \sin\theta = \frac{2\pi}{\lambda} c \sin\theta. \tag{5}$$

Thus we have arrived at the conclusion—indeed almost self-evident—that the consecutive slits of the grating supply to the total vibration at the field-point contributions which are exactly equal in magnitude and follow one another at equal intervals of phase.

Our problem therefore is to sum up the series of these contributions. The process is an easy one; but we shall be able to foresee the major feature of a diffraction-spectrum without even writing down the summation. For it is evident that there must be maxima of vibration-amplitude, maxima of diffracted intensity, at the field-points or in the directions where the contributions of all of the slits agree in phase—that is to say, differ in phase by integer multiples of 2π. Counting outward from the centre of the diffraction-pattern, or normal to the grating, the first of these maxima must lie in the direction for which the "step" in phase of equation (5) is equal to 2π; hence for this "first-order maximum":

$$\sin\theta = \lambda/c. \tag{6}$$

The second lies in the direction for which the step in phase is equal to twice 2π; the third in the direction for which the step is thrice 2π; and in general there is a sequence of maxima, the general formula for the nth of which is the celebrated "plane-grating formula":

$$\sin\theta_n = n\lambda/c, \qquad n = 1, 2, 3, 4 \cdots. \tag{7}$$

The symbol n stands customarily, and in this article henceforth shall stand, for the *order* of the maximum; from now on I will write $2\pi\nu$ for the quantity which before was denoted by n.

These are the great principal maxima of the spectrum cast by a diffraction-grating. There are others between, but in practice they are inconspicuous or invisible. Those of which I have just derived the locating formula are the maxima from which wave-lengths are computed. Let it be emphasized again that the formula was derived without taking into account the ratio of slit-width to bar-width, and that it does not involve the width of the individual opening, but only the spacing between corresponding points of consecutive slits. Indeed, if one examines the deduction, it will be seen that really nothing peculiar to a slit enters into it at all. All that is preassumed is that the grating sends to the field-point a series of component vibrations, equal in amplitude and stepped off equally in phase. Such is indeed the case when the grating is a series of windows letting light through towards the camera, separated by walls which intercept the light. Such is also the case when the grating is a series of mirrors reflecting light towards the camera, separated by windows which let it escape or by absorbing surfaces which swallow it up. Such is the case when the instrument is a surface of metal ploughed into furrows, so that the reflecting-power towards any assigned direction varies from point to point across the furrow, and varies periodically as one moves across the system of furrows. Such is the case if the waves traverse or are reflected from all points of the grating equally, but with phase-retardations which vary periodically across the grating-surface. Such is the case when the instrument is a surface containing oscillators able to vibrate in unison with the incident waves and able to radiate new waves because of their vibration, these oscillators being evenly spaced or else clustered in identical groups which themselves are evenly spaced. Such in fact is in general the case whenever the "grating" is any object with a periodic structure, the details of which are able in any manner known or unknown to perturb the passage of the waves; for anything which confuses or impedes the even onward progress of a train of waves, whether it be a vibrator which they set into oscillation or merely an inert impenetrable obstacle in their way, becomes thereby the source of a new system of undulations.

Wave-lengths of light therefore are determined by setting up in the path of the light-stream something which has a periodic structure, of which the period is known; locating the diffraction-maxima, if such there be; and using the formula (7), provided that the object is plane (another, which we shall eventually derive, is used if the object is three-dimensional and the waves travel across its structure). Location of one maximum would not as a rule suffice, for without further knowledge its order could not be identified. One must measure sufficiently many maxima to infer from the ratios of their values of $\sin \theta$ what their orders are. On the other hand, understanding of the precise mode and mechanism of the action of the elements of the grating upon the light is not required, desirable as it may be. Perhaps we do not properly understand how the atoms of a crystal scatter even X-rays; and certainly the founders of the wave-mechanics did not foresee that crystals scatter electron-waves. Yet Davisson and Germer determined the wave-lengths of these latter in 1927 with the same equation wherewith Fraunhofer in 1821 had ascertained the wave-lengths of the lines of the solar spectrum—the very equation,

$$\lambda = \frac{c}{n} \sin \theta_n,$$

taking for c the spacing between consecutive lines of atoms in the surface-layer of the nickel crystal which diffracted the electrons, where Fraunhofer had taken the spacing between the wires of the grid which was his primitive grating.

Next it is important to discover how distinct these maxima are; whether, when the amplitude of the vibration in the focal plane is plotted against θ, the peaks are broad and flattish or narrow and sharp. This too can be foretold without the labour of a complete solution of the problem. Taking any of the principal maxima—say, that of nth order, which is located at the angle $\theta_n = \text{arc sin } (n\lambda/c)$—let us inquire how near to it the amplitude will sink to zero.

Now the nth of the principal maxima is located by the condition that the phase of the contribution of every slit is $2n\pi$ in arrear of that of the slit preceding. If there are $2M$ slits altogether (it is convenient to suppose the total number to be even, though whether it is even or odd makes no appreciable difference), then at θ_n the contribution of the last slit is $(2M - 1)n \cdot 2\pi$ behind that of the first. Estimate now the vibration at the point in the focal plane —call its direction-angle $(\theta_n + \Delta)$—where the contribution of the last slit is $(2M - 1)(n + 1/2M)2\pi$ behind that of the first. It is readily shown that here the component vibration due to the last or $2M$th slit is exactly equal in magnitude and opposite in phase to that which is produced by the Mth of the slits; and in the same way every ruling of one-half of the grating may be paired off with the corresponding ruling of the other half, their effects destroying each other pair by pair. At the angle $(\theta_n + \Delta)$, therefore, there is darkness; and likewise at the angle $(\theta_n - \Delta')$, where in the focal plane or the infinitely distant plane the contributions from the first and the last slit arrive with a phase-difference of $(2M - 1)\left(n - \frac{1}{2M}\right)2\pi$.

The entire peak culminating in the nth principal maximum is consequently bounded by the directions $(\theta_n - \Delta')$ and $(\theta_n + \Delta)$; and it is easy to see that the greater the number of rulings (the spacing being supposed to remain the same) the narrower and sharper is the peak, and the more accurately can the location of its summit and therefore the wave-length be determined. Its breadth, in fact, varies inversely as the number of rulings or "lines." This is shown by writing down the formulæ for the angles corresponding to the minima which bound it. We have:

$$\sin(\theta_n + \Delta) = \left(n + \frac{1}{2M}\right)\frac{\lambda}{c}; \quad \sin(\theta_n - \Delta') = \left(n - \frac{1}{2M}\right)\frac{\lambda}{c},$$

whence, approximately,

$$\Delta = \frac{\lambda}{2Mc}\frac{1}{\cos\theta_n} = \frac{1}{2Mn}\tan\theta_n, \tag{8}$$

so that the narrowness of the peak, as one might say, is proportional to the order thereof as well as to the total number of lines in the grating.[5] If the grating were infinitely wide, an unlimited sequence of perfectly-evenly-spaced identical units, the peaks would be infinitely narrow; light of a definite wave-length would be diffracted only in certain perfectly definite discrete directions.

Alikeness, closeness, and multitude of rulings are therefore the desiderata of a grating; alikeness, because without it the first condition for the formation of sharp diffraction maxima would be lacking—closeness, so that the maxima of lower orders, the only ones sufficiently intense to be perceived, may be spread out widely enough for convenience of observation—multitude, so that the diffracted beams shall be narrow and sharp, easy to set upon and easy to discriminate from one another. The degree of closeness which is required depends upon the spectral range which is to be explored. Ordinary "optical" gratings ruled with a diamond on metal or on glass are acceptable throughout the visible spectrum and the range to which the title "ultra-violet" is commonly restricted, extending from the visible down to wave-lengths of the order of one hundred Angstroms. They are however too fine for the remoter infra-red, for the study of which coarse lattices of wire have been used; *a fortiori* they are much too fine for Hertzian or radio waves, for which it is no exaggeration to say that a colonnade might operate as a grating; and they are commonly considered much too coarse for X-rays, though during the last two or three years several men of science have achieved the great technical feat of forcing optical gratings to measure wave-lengths which formerly were thought accessible to crystals only. Crystals are too fine for the visible spectrum, and too coarse for certain of the gamma-rays which proceed from the collapsing nuclei of atoms undergoing transmutation. Crystals with spacings of unusual width from atom-plane to atom-plane are

[5] These facts are usually expressed as statements about the "resolving power" of a grating; for if the incident light contains two not very different wave-lengths, they will form two peaks of each order not very far apart, and the possibility of distinguishing these two—of "resolving" them, to use the technical term—will depend upon the narrowness of each. If arbitrarily one says that two such peaks are just distinguishable when the summit of one falls upon the minimum adjacent to the other—in which circumstance the difference $\delta\lambda$ between their wave-lengths may readily be proved equal to the quotient of the mean of their wave-lengths, λ, by $2Mn$—then by this criterion a grating is able in its nth order to discriminate two adjacent lines of the spectrum, if their wave-lengths differ by more than that amount; and by definition the resolving power of the grating in its nth order is $\lambda/\delta\lambda = 2Mn$, the product of the number of rulings by the order.

chosen for work upon the longer X-rays, as optical gratings are ruled with lines unusually far apart for work in the near infra-red.

The ruling of good gratings is an art; and those who have practiced it with conspicuous success are fewer far than those who have attained pre-eminence in music or in painting. Amateurs, mechanics, and professors figure upon the list, the first of all being Fraunhofer, who from a glazier's apprentice evolved into the founder of spectroscopy. After his gratings of wires and of scratches in a foil of gold-leaf, he invented the method of engraving with a diamond-point upon a surface of metal or of glass (he used the latter) which is followed to this day. He met and grappled with all the difficulties which were later to beset his followers, and described them in language which now sounds strangely modern. Then, as now, it was possible to rule tens of thousands of rough grooves roughly to the inch; the trouble lay, as still it lies, in making them identical and spacing them equally.

Equality of spacing depends upon a screw, which is turned through a prearranged angle and is expected to advance through a definite distance carrying the future grating with it, whenever the diamond has completed one ruling and is waiting to begin the next. Screws as manufactured are not good enough; and anyone who aspires to be a maker of gratings must first of all procure the best available, and then devote a long and tedious time—literally years—to making it still better. Primacy in the art passed to America in the eighties of the last century, because Rowland of Johns Hopkins developed with much labour a process for removing, or at least for mitigating, the imperfections of a screw. The greater the number of rulings to be laid down side by side, the longer the portion of the screw which must be made, as nearly as humanly possible, perfect; and Michelson has testified, from unrivalled experience of many years, that the time required for the process varies as the cube of the length of the screw and width of the planned-for grating. Increase of resolving-power thus is bought at an enormous price in patience and in perseverance. A research institute is as proud of a notable grating by Rowland or Michelson or Wood, as a picture gallery of an authentic Titian or Velasquez; and the promise of a new talent is not more joyfully received, than a rumour that someone is working to perfect a yet longer screw to make a yet wider grating.

Alikeness of successive rulings depends on the endurance of the diamond. The ruling-engine is sequestered in a well-insulated room, and after the temperature has settled down to constancy is set in motion by some device worked from outside, and left to do its task in solitude. If the diamond breaks, or suffers any great change in shape during the operation, the grating is good for nothing. This cannot be foreseen, it is not even known when it happens; to stop the process to see how things are going would be like digging up a seed to see how it is sprouting. The chance of such an accident is naturally greater, the more numerous the lines—another obstacle to the successful ruling of many-lined gratings of high resolving power.

A grating having been completed, it is removed from the engine and examined, to learn not merely whether it has been impaired by deformations of the diamond, but how—assuming it to have escaped that peril—the intensities of the various diffraction-maxima of different orders compare with one another. This is something which, as I have intimated, is controlled by the shape of the groove; this is the feature in which the individual units of the periodic structure manifest their quality. One shaping might obliterate all diffraction-maxima of even order; another might make the maxima on one side of the normal to the grating-surface stand out much more prominently than their companions on the other; still another could concentrate most of the diffracted light into one single beam. The maker of the grating cannot foresee, or can at best foresee only in part, what distribution of intensities he is going to get; for he cannot control the shape of the diamond-point, nor find it out by examination.[6] Having observed the distribution of intensities, however, he can deduce from it some facts about the shape of the grooves. This I suppose would be classified in most cases as useless knowledge; but the problem happens to be very nearly the same as that of determining the finer details of the arrangement of atoms in a crystal from the relative intensities of the various diffraction-beams which it produces when acting on an X-ray beam; and so I will devote a few paragraphs to it.

We return, then, to the grating of alternate slits and bars, to determine the influence of the ratio of slit-width to bar-width on the diffraction-pattern. Before making any calculations whatever, one striking prediction can be made directly. I have said that diffraction-maxima occur in every direction θ_n for which

$$\sin \theta_n = n\lambda/c, \qquad n = 0, 1, 2, 3, 4 \cdots,$$

because in every such direction the component vibrations arrive at the focal plane from the various slits with identical phase. But if for any of these directions the component vibrations are themselves

[6] He can control the result to a slight extent by varying the pressure with which the diamond bears upon the plate, ruling "with a light touch" or reversely; if he guesses the force just right, he may approach the condition of grooves separated by unbitten bands of smooth metal as wide as they, which resembles the theoretical case of an alternation of slits and bars of equal width.

non-existent, evidently the maxima in question are blotted out. This will happen, for example, to every maximum of even order, if bars and slits are equally wide. For, taking the direction $\theta_2(\sin\theta_2 = 2\lambda/c)$ as an instance: the contribution made to the total vibration by the upper half of each slit will be equal in magnitude and opposite in phase to that made by the lower half, and the total contribution of the slit will be zero. If in the spectrum produced by a grating the even orders are missing, or if—to say what would actually be noticed—the values of $\sin\theta$ for the present maxima stand in the ratios $1:3:5:7\cdots$ instead of $1:2:3:4\cdots$, the inference is that the grating has been so ruled that over half of every period the phase of the emerging (transmitted or reflected) light is constant, and over the other half no light comes forth at all; as for instance would be the case if half of every period were the unmarred surface of the metal, and the diamond had made the other half perfectly black. The reader may work out for himself what it must mean if every third, or every fourth, or every nth of the maxima is absent.

We return now to the expression (equation 2) for the contribution of a single slit or period of the grating and rewrite it, taking due account of our subsequently-gained knowledge that A_k is constant and φ_k increases by equal steps $mc\sin\theta = mc\beta$ from slit to slit:

$$s_k = \text{const.}\,(1+\alpha)\,A\sin(nt - mr_0 - \epsilon_0 - kmc\beta). \tag{9}$$

For convenience number the slits from 0 to $N-1$, representing by N their total number (formerly called $2M$, but now there is no reason for supposing it even), and locate the origin so that $mr_0 = \epsilon_0$. Gathering all the factors of the sine-function under a single symbol B, and writing out the expression for the summation of s_k from $k=0$ to $k=(N-1)$, we find for the resultant vibration in the direction θ:

$$\begin{aligned} s &= B\sum_{k=0}^{N-1}\sin(nt - kmc\beta) \\ &= B\sin nt(1 + \cos a + \cos 2a + \cdots \cos(N-1)a) \\ &\qquad\qquad - B\cos nt(\sin a + \sin 2a + \cdots \sin(N-1)a) \\ &= B{\textstyle\sum_c}\sin nt - B{\textstyle\sum_s}\cos nt, \end{aligned} \tag{10}$$

in which a stands for $mc\beta$ and $\sum_c$ and $\sum_s$ for the finite series of cosines and sines which are indicated.

For the amplitude of the vibration—the only thing which matters—we then have

$$D = B\sqrt{\textstyle\sum_c^2 + \sum_s^2}. \tag{11}$$

Now, as may easily be proved[7]:

$$\sqrt{\Sigma_c^2 + \Sigma_s^2} = \sin\left(\tfrac{1}{2}Na\right) : \sin\left(\tfrac{1}{2}a\right) \tag{12}$$

so for the amplitude of the vibration in the direction θ we have:

$$D = B\,\frac{\sin\left(\frac{1}{2}Nmc\sin\theta\right)}{\sin\left(\frac{1}{2}mc\sin\theta\right)}\,. \tag{13}$$

Here we have that product of two factors which was foreshadowed in the early pages of this article—one factor (the second) depending on the periodicity of the grating, and controlling the location of the diffraction-maxima; the other depending on the structure of the individual slit or groove or atom-row, and controlling their intensity.

The second factor displays the qualities which have already been deduced by simpler means, and others. It vanishes whenever $\frac{1}{2}Na$ is an integer multiple of π, except when simultaneously $\frac{1}{2}a$ is an integer multiple of π, in which exceptional cases the great principal maxima occur. These are not the only maxima, for between any two of them there are $(N - 1)$ equally-spaced minima (directions where $\frac{1}{2}Na$ is an integer multiple of π but $\frac{1}{2}a$ is not) and between these in turn there are $(N - 2)$ maxima of which the locations may be found by the usual method. These so-called "secondary" maxima are however faint and inconspicuous, having, according to Wood, but 1/23 the intensity of the principal peaks, unless the grating is composed of only half-a-dozen lines or fewer.

The first factor consists essentially of that function

$$(1 + \alpha)\sqrt{C^2 + S^2}$$

mentioned in equation (1) and earlier, which describes the diffraction-pattern of the single slit (or groove, or atom-row). Wherever that diffraction-pattern has a zero of intensity—the "centre of a black fringe," to use the common language—the intensity in the pattern of the grating is likewise forced to vanish. When the slit occupies half

[7] One method is based on the fact that Σ_c and $i\Sigma_s$ are respectively the real and imaginary parts of Σe^{ika}, so that

$$\Sigma_c^2 + \Sigma_s^2 = \left(\sum_{k=0}^{n-1} e^{ika}\right)\left(\sum_{0}^{n-1} e^{-ika}\right).$$

Further, by a well-known formula

$$\begin{aligned}\sum_{0}^{N-1} e^{ika} &= 1 + e^{ika} + (e^{ika})^2 + \cdots (e^{ika})^{N-1}\\ &= (1 - e^{iNa})/(1 - e^{ia})\end{aligned}$$

and there is a corresponding expression for e^{-ika}, multiplying the two of which together and taking the square root one arrives directly at the stated result.

the width of the period (slit plus bar) of the grating, the first of its black fringes falls square upon the second-order principal maximum of the grating spectrum, which is obliterated. This is a new way of expressing the fact already mentioned, that when the slits are as wide as the bars the diffraction-maxima of even order are absent.

More generally, the intensity at any of the principal maxima is proportional to the value of $(C^2 + S^2)$ appropriate to that direction—proportional to the intensity, in that direction, of the diffraction-pattern of the single slit; and from this we can understand how, from the relative intensities of the maxima of various orders, it is possible to deduce the breadth of the slit or something about the shape of the groove. If we had only a single slit, and could send through it light of known wave-length sufficiently intense to form a measurable diffraction-pattern, we could trace the curve representing observed relation between diffracted intensity and angle θ, and compare it with the predicted curves for various values of slit-breadth; the actual width of the slit would be the value for which the agreement was perfect. If instead we had a multitude of such slits equally spaced, the observed intensities of the diffraction-maxima would supply us, not indeed with the entire continuous curve of intensity-versus-angle for the single slit, but with as many points upon that curve as there were principal maxima within our range of observation; and these—if we had two or more—would be sufficient for the comparison with the theoretical curve for the single slit, out of which the width would be deduced. From this aspect, the function of the grating is to *enhance* the intensity, at certain discrete points, of the diffraction-pattern for the single slit. Of course, when we are interested in the breadth of the single slit or the shape of the single groove, we should prefer to observe the entire continuous diffraction-pattern produced by one alone. But it may be impossible to separate one from the rest; or if we could isolate one, it might be too small to transmit or scatter any perceptible amount of light. Such is the case with atoms.

The natural gratings which atoms form in crystals are three-dimensional, and to them the reasonings which are valid for plane gratings cannot be applied without some change; but the resemblance is very close. A beam of X-rays or electron-waves falling upon a crystal is spread out into a diffraction-pattern with strong maxima, of which the relative intensities depend upon the qualities of the individual diffracting units, the atoms or the groups of atoms which are repeated over and over again to form the crystal; while their directions depend upon the spacings between these identical groups, the periodicity of the crystal. From the directions of the principal diffraction-beams one may determine the spacings within the crystal if one knows the wave-length of the waves, or the wave-length if one knows the spacings. From the relative intensities of the beams one may deduce the distribution of the atoms within the groups, or rather the distribution of that which scatters the waves—commonly supposed to be mobile negative electricity, when the scattered waves are light; I do not know whether anyone has yet conjectured what it is that scatters the electron-waves.

The diffraction-beams proceeding from a crystal large enough to be manageable are very sharp, for the rows of atoms are far more numerous than the lines of the largest optical grating which can be made or hoped for. However, there is a limitation on their sharpness set by something to which an artificial grating is quite indifferent—the thermal agitation of the atoms, which has the same effect as though the widths of successive periods were variable and fluctuating. This effect is naturally more pronounced, the higher the temperature of the crystal; but the measurements show—for from the breadth of the diffraction-maxima it is possible to determine the mean amplitude of the temperature-agitation, another service of the crystal grating—that even at absolute zero it would not disappear, the atoms retaining a certain minimum amount of energy of vibration which apparently can never be taken from them, so long as they remain bound together in a crystal.

A few words, before leaving the subject of gratings, about the diffraction-pattern of a multitude of gratings oriented at random.

On an earlier page I said that, in computing the diffraction-pattern of a sequence of slits, we need determine it not for the entire focal plane, but only for a single line thereof—the line for which $\gamma = 0$, which is the line of intersection of the focal plane with the plane running normal to the slits and containing the infinitely-distant point-source of the parallel waves of light. The reason can now be stated. If we work out the expression $C^2 + S^2$ for a single long and narrow rectangular slit with its long sides are parallel to the z-axis, we find that the brighter parts of the diffraction-pattern form a long narrow band (criss-crossed with dark lines) with its length parallel to the y-axis and its breadth parallel to the z-axis. If the length of the rectangle grows infinitely long, the breadth of this band shrinks to zero; we have a single line of varying brightness parallel with the y-axis, which is the diffraction-pattern of the infinite slit. If instead of a single slit we have a regular sequence, their diffraction-pattern is still concentrated upon this line; it is the pattern which has just been computed, a function of the single variable β, or y, or θ; away from the line, the intensity is everywhere zero. Spectroscopists broaden this linear pattern in practice by using as source of light not a point, but a luminous line—an incandescent filament, for instance, or a slit backed by a flame—made parallel to the slits or rulings of the grating. Then the diffraction-pattern is spread into a band. If the light is monochromatic, one sees in the focal plane, at the positions of the principal maxima, not a sequence of brilliant points as the foregoing theory implies, but a sequence of brilliant lines—the lines of the spectrum.

Instead of these lines one will obtain circles, if one uses a point-source of light and a mosaic of gratings all lying side by side in a single plane and oriented every way. Each piece of the mosaic forms its own linear diffraction-pattern, perpendicular to the direction of its own rulings; and if the pieces are numerous enough, all of these are fused into a single circular pattern, each of the principal maxima standing forth as a brilliant ring. I am not sure whether this has been done with plane optical gratings; but the analogous method with X-rays and crystals is the familiar procedure known by the names of Debye and Scherrer and Hull, or as the "powder method." Being a case of diffraction in three dimensions, it is not entirely like my imaginary case of a mosaic of plane gratings. The resemblance however extends so far, that from the broadness of the rings one may infer the size of the tiny crystals which make up the three-dimensional mosaic, the "powder"; for the smaller these are, the fewer rows of atoms each contains, and the wider their diffraction-maxima must be. But it requires very fine grinding indeed, or the dispersion of the crystals as a colloid in solution, to make them so small that the broadening of the rings is noticeable.

What would be observed, if individual slits or apertures or atoms were dispersed completely at random over the plane or throughout space? If there were many apertures all alike and all similarly oriented, but with no regularity whatever in arrangement, the diffraction-pattern would be the same as that of any singly, though more intense. The water-droplets in misty air act thus in forming haloes. If atoms were truly spherical and could be crowded together into a dense mass without any regularity, the diffraction-pattern of the mass would be that of the individual atom, and would disclose the radial distribution of its scattering-power—whether that be negative electricity, or something else. Even if atoms are not spherical, one might expect to learn in this way the average distribution of scattering-substance over all the orientations. Experiments have been conducted for this purpose; but it is difficult to find a piece of matter in which the arrangement of the atoms is entirely irregular, that is, a perfectly "amorphous" substance; perhaps not even liquids satisfy this requirement.

Interference

When a pair of beams of light are projected together upon a screen, it is usually observed that the illumination resulting from them jointly is the simple sum of the illuminations which each produces by itself when the other is shut off. One may easily go through life without ever once finding this rule in default. Yet by intelligent design it is possible to contrive conditions in which the rule does not prevail; and actually two rays of light directed upon the same point may counteract one another and cause total darkness, and two perfectly uniform wide beams falling together upon a surface of frosted glass may decorate it with a pattern of dark fringes separated by light, dark circles alternated with bright, black networks upon a background of color—arabesques of shadow and light, more delicately shaded than anything achievable in pigment or stained glass. The brilliant and versatile Thomas Young, he who was the first to read the Egyptian hieroglyphics upon the Rosetta stone, was also the first to discover some of these lovely phenomena; a pair of exploits, which for eminence and diversity will probably never be surpassed. It happened that the first disclosure of the phenomena which demand the wave-theory of light coincided as accurately with the advent of the nineteenth century as the first realization of the necessity of quanta came at the dawn of the twentieth; for Young discovered the interference of light in 1800.

"Interference" is a name which Young selected; he said that in the conditions of his experiments beams of light interfere with one another. For the observer this was not, on the whole, an ill-chosen word, since the visible effect of the two lights conjointly is not the mere sum of the visible effects of each separately. True, it implies that the lights destroy or diminish one another, whereas in fact they are as likely to cooperate as to conflict, two equal beams combining into one of intensity as much as fourfold that of either. This is not serious; we are all accustomed to using the word *addition* to cover subtraction; and here the analogy is very close. The so-called "interference" is simply the necessary result of adding two vibrations with due regard to their direction and their phase. This is the method which was used to calculate diffraction-patterns; and in fact a diffraction-pattern is nothing but a special case of interference-pattern—not usually a simple one, for the vibrations which must be summed are very numerous, demanding integrations and long summations. The simplest interference-pattern occurs when two plane-parallel beams of light of equal amplitude intersect one another; and this we will now consider.

Designate by 2θ the angle at which the two beams are inclined to one another, and draw the x-axis to bisect it; then the two wave-functions are

$$s' = A \sin (nt - mx \cos \theta - my \sin \theta),$$
$$s'' = A \sin (nt - mx \cos \theta + my \sin \theta)$$

and their sum [8] is

$$s' + s'' = s = 2A \cos (my \sin \theta) \sin (nt - mx \cos \theta). \qquad (14)$$

We see immediately that this is a situation in which the wave-theory of light predicts a peculiar and characteristic variation of amplitude from point to point in space, which can be tested in detail, and of which a favorably-resulting test has evidential value; whereas in either beam separately the amplitude is constant, and nothing is observable which demonstrates that there are waves. Here, in the region where the beams overlap, the amplitude varies sinusoidally between zero and the maximum value $2A$; the distance between two consecutive loci of zero amplitude, which are planes perpendicular to the axis of y, being

$$d = \pi/m \sin \theta = \tfrac{1}{2}\lambda/\sin \theta. \qquad (15)$$

The presence of a series of equally-spaced planes of darkness, their separation varying inversely as the sine of the angle between the beams, is then to be taken as evidence that light is undulatory; and from their separation and the angle between the beams one may compute the wave-length of the light. A more thorough test, made by measuring the distribution of light-intensity between two such planes, would lead (anyway it ought to lead) to the conclusion already known, no doubt, to all the readers of this paper: that the intensity of the light varies as the square of the amplitude of the waves.

To produce this effect of interference, the two intersecting beams must have started from the same source of light, and at very nearly the same instant—that is to say, the optical paths from the source along the two beams to the region of overlapping must be the same within a few millions of wave-lengths, or a few hundreds of centimetres. By the wave-theory, this is easily understood. We must think that

[8] To add them thus implies that the quantity denoted by s is either a scalar, or a vector perpendicular to the xy-plane. Since light is not adequately described by either assumption, we must anticipate defects in the theory, more prominent the larger the angle θ. In practice θ is evidently always so small that there is no trouble from this source.

a beam of light from a flame or an arc consists of myriads of feeble beams each proceeding from a single atom. Each is divided—the methods of division are the methods of producing interference-fringes—and the separate parts are then caused to overlap. Each pair which came originally from a single atom produces a set of interference-fringes, and the fringes for all these pairs coincide in space. Each fraction of a divided beam may also interfere with a fraction of another, proceeding from another atom; but owing to the uncontrolled and uncontrollable phase-differences between the beams of a pair so formed, the fringes for these pairs do not coincide, and on the whole they efface one another. By the quantum-theory the explanation—not indeed of the fact that interference occurs only under these special conditions, but of the fact that it ever occurs at all—is not so easy. Indeed the fact commonly expressed by saying that light from a source is "coherent" with itself, has been regarded as the most difficult of all for the quantum-theory to explain.

To produce interference, then, we must divide a beam of light and cause its parts to cross each other's paths. The simplest of the devices which effect this were invented by Fresnel; a pair of prisms which turn two portions of the beam towards one another, and a pair of mirrors which reflect two portions across each other's routes. A single mirror indeed suffices; standing acoustic waves are produced thus, in Kundt's tube and otherwise, with values of the angle 2θ sometimes as great as 180°; but light-waves are so short that with so great an angle the distance between dark fringes would be too small to measure, if not indeed to perceive; and we must use the facility for expanding them which the factor $\sin\theta$ in equation (15) offers us.

The Interferometer

In the devices which I have thus far mentioned, the interference of overlapping wave-trains oblique to one another causes the formation of alternate zones of darkness and light in space; and the visible fringes are the cross-sections of these zones upon a screen set up to intersect them. There are however other instruments in which the overlapping beams are parallel to one another, the region which they occupy is not traversed by bands of light and shade, and a screen thrust across it shows uniform illumination; and yet when the eye or the camera is located in that region, fringes are produced upon the retina or on the plate by the action of the lens of either. These are not so easily understood as the earlier devices, and yet it is important to comprehend them, for the striking applications of interference have been made by means of such as these. Among them are the interferometer of Fabry and Pérot, and that of Michelson.

Imagine, at the outset, a pair of perfectly plane and parallel mirrors, onto which wave-trains of extended plane wave-fronts are falling from every direction. The mirrors must of course be semi-transparent, so that part of the light which falls first upon one—say, the upper—is reflected from it at once, and part goes on to meet and be reflected by the lower. Thus (as Fig. 3 shows more clearly than words) the

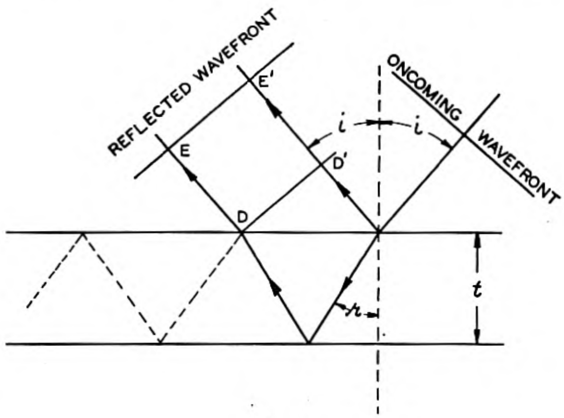

Fig. 3.

mirrors form out of each incident wave-train a first and a second reflected beam, which travel back through the space above the mirrors in the same direction, making according to the law of reflection the same angle i with the normal as the incident wave-train did. In truth there are not merely two reflected beams derived from each incident one, but an infinity thereof, owing to the multiple reflections which are indicated in the sketch. We need not however (as I shall presently show) take account of more than two; by combining the second reflected beam with the first we can predict the most important features of the interference.

It is necessary to be somewhat more precise about the nature of the mirrors. As good an example as any to begin with is that of the "thin plate"—a slab of some transparent substance, glass for instance, embedded in a transparent medium which I will take to be empty space. The mirrors, then, are the upper and lower sides of the plate. Denote by μ the ratio of the speeds of light in the environing medium and in the substance of the plate, by i the angle of incidence of any wave-train and by r the angle of refraction of its transmitted part; then as heretofore we have

$$\sin i = \mu \sin r. \tag{16}$$

The ratio A_2/A_1 of the amplitudes of the first and second reflected beams, and in general the ratios A_n/A_1 of the amplitudes of any of the reflected beams and the first, are determined altogether by μ and i. An important consequence of this will presently appear. One can however alter these ratios, e.g., by half-silvering the sides of the plate; and the formulæ which I am about to quote may be applied to the case of two half-silvered mirrors facing each other in air, by setting $\mu = 1$.

Isolate then in mind a single incident wave-train. Denote by i its angle of incidence upon the upper surface of the glass; by t the thickness of the plate. A wave-front of the oncoming wave-train is divided into two. During the time while the part which entered the glass is advancing to the lower side, being reflected, returning to and re-emerging from the upper side, the part which was first reflected goes on to the level EE' of Fig. 3. The emerging wave coincides with the first-reflected part of a new wave-front which was following along after the old one at the interval $E'D'$. In general, there is a phase-difference φ between these two. The condition for interference is ideally satisfied; two wave-fronts coincide. The amplitude A of the resultant light travelling away from the mirrors is obtained by the prior formula from the amplitudes A_1 and A_2 of the first and second reflected beams and their phase-difference φ:

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos \varphi. \tag{17}$$

It is now our affair to deduce the value of this difference in phase. This is a simple task, for the angle φ, expressed in radians or in degrees, is 2π or 360 times the number of wave-lengths intervening between the wave-front DD' and the wave-front EE' which, so to speak, has gone on ahead leaving part of itself behind to combine with DD'. We have therefore only to multiply $2\pi/\lambda$ or $360/\lambda$ into the distance $D'E'$; which distance is found [9] by very easy manipulations of trigonometry, aided by the equation (16), to be $2\mu t \cos r$; so that for the phase-difference in radians we have

$$\varphi = \frac{2\pi}{\lambda} 2\mu t \cos r + \varphi_0. \tag{18}$$

The constant φ_0 is inserted to leave room for the possibility that abrupt changes of phase may occur at the instant of reflection, unequal for the two reflecting surfaces. Experience shows that in this case of

[9] For the distance $D'E'$ is the difference between BD' and BE'. The latter is evidently $BD \sin i$, which is $2t \tan r \sin i$, which is $2\mu t \tan r \sin r$. The former is the distance cT traversed by light in vacuo during the time T while the beam which entered the glass is traversing its zigzag path BCD; this time is $(\mu/c)BCD$, which is $2(\mu/c)t \sec r$.

the plate the value π must be assigned to φ_0; as though the phase were unaltered at the first reflection, but reversed at the second. This is however a point of minor importance.

The equation (18) is one of the most important in optics; we shall encounter it repeatedly, even so far along as in the X-ray range.

By comparing (17) and (18) one sees that, if wave-trains of equal intensity fall upon a glass plate from all directions, those which depart in the various directions are not equally intense; their brightnesses depend upon their angle of reflection which is their angle of incidence, i. However they are not separated in space, and hence there are no bands of light and darkness. But if a lens be placed in the region above the plate, it will direct each of the beams to a separate point in its focal plane; and since the illumination at the point where a wave-train converges is proportional to the intensity of that wave-train, there will be fringes in that focal plane. If the lens is set with its axis normal to the planes of the mirrors, as when one looks straight at them with the eye, then the points where the wave-trains reflected at any angle i converge lie all upon a circle, its radius depending on i. The fringes are therefore circular; looking vertically down upon a thin plate, or photographing it with a camera pointed directly towards it, one sees or registers a system of concentric rings, their centre wherever the perpendicular dropped from the lens reaches the plate. These are said to be localized at infinity; but the term is not a good one, for the fringes are not at infinity; they are on the retina or on the camera film, formed by the lens in the focal plane thereof.

The values of i for which the amplitudes of the reflected beams are least or greatest may be determined by differentiating (17). If we may neglect the variation of A_2/A_1 with i (as usually but not always we may), the result is the expected one: least intensity and blackest point of a fringe corresponds to a phase-difference of π or an odd-integer-multiple thereof, greatest intensity and brightest point of a fringe occurs with a phase-difference of zero or any even-integer multiple of π. Thus one may compute the actual size of the circular rings to be produced when a given lens sorts out the rays reflected by a given plate, and test the predictions by experiment; or rather, to say what is really done, one may determine the wave-length of the light by measuring the diameters of the rings and comparing them with the formulæ. But this is not a customary way of determining wave-lengths.

At this point it is expedient to remark that the size of the fringes is not affected by the presence of those third, fourth, fifth and indefinitely many reflected beams which I excluded from the computation. For the phase-difference between each of these and the one reflected once-less-often is given by the first term on the right in equation (18), and hence is greater by 180° than that between the second and the first; and when i is so chosen that the second and the first are in opposite phase, then all the beams of higher order are in the same phase as the second and reinforce it, the reinforcement being just so great—in the case of the transparent plate—that the resultant of all these beams together is of the same amplitude as the first reflected beam. Therefore the minima, the centres of the dark fringes, are not shifted by the extra beams, but are rendered absolutely black.

The width of the fringes is greater, the thinner the plate—other things, naturally, being left unchanged. This is the reason why one needs quite a thin lamina of glass to see them well, and cannot get them at all with a windowpane. If the thickness of the plate is changing continually, one sees them narrowing or widening; one sees also a phenomenon much more striking, the generation of rings one after the other out of the centre of the fringe-system—if the plate is growing thicker; the reverse, if it is shrinking. Glass plates capable of shrinking or thickening at will are not as yet available; but at times the former case is realized by a soap-bubble on the verge of dissolution. Where the soap-film is about to give way, the fringes rapidly dwindle and are swallowed up into the central spot of the interference-system, which alternately turns dark and light, and finally goes black just at the instant before the bubble bursts. From this final blackness we infer that the value π must be assigned to the constant φ_0 of equation (3). Reflection from water to air and reflection from air to water are accompanied by phase-changes differing from each other by π, since the beams of light formed by two such reflections destroy each other when the reflecting surfaces are immeasurably close. But the bubble is too tender an instrument for practical use. The plate of variable thickness must be a slab of empty space between movable solid mirrors.*

The interferometer of Fabry and Perot is precisely such a thing: two half-silvered plates of glass facing each other across a narrow gap. The like-named instrument of Michelson is the same in principle; but by ingenious use of a third reflector, the two essential mirrors are transposed far apart—a most valuable feature, as we shall see. The incident wave-trains (coming from below, in Fig. 4) are divided by a semi-transparent mirror C inclined at about 45° to their path, and the fractions are sent off at right angles to one another, along the two

* Or else Lummer's device of a pair of wedges so proportioned, that a face of one may slide along a face of the other, while the two sides not in contact remain parallel to one another.

"arms" of the machine, at the ends of which they meet fully-reflecting surfaces A and B set normal to their courses. Reflected straight back along the arms, they are once more divided by the semi-transparent

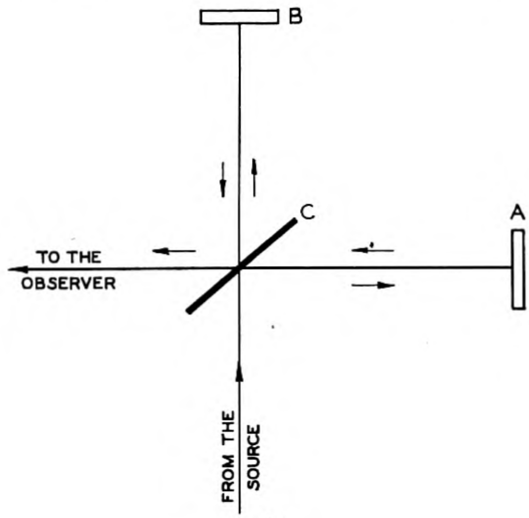

Fig. 4.

mirror, and the fractions which go towards the observer (left, in Fig. 4) are those which interfere. This seems complicated; but in essence it is not so. Everything happens as though the mirrors A and B were at the end of the same arm. Looking from the left through C, one perceives the mirror A and the virtual image in C of the mirror B; and this comes to the same as though C could be removed, and the horizontal arm of the machine swung into coincidence with the vertical arm in spite of the well-known prohibition against two objects occupying the same place at the same time. When the plane of A is accurately normal to the plane of B and the semi-transparent surface C is properly oblique, the virtual image of B is accurately parallel to A; and the observer sees circular fringes, which one by one emerge from a central spot or shrink down into it as either mirror is displaced along its arm.

This clever device of combining one real mirror with the virtual image of another, to form a thin plate of which one surface is impalpable, is of enormous value. One can make the virtual reflector go right through the real one, the fringes being swallowed up into the centre one after another, and then reborn in due order after the whole field of view goes black at the instant of coincidence. By moving one of the reflectors through a chosen distance and counting the fringes which are born or consumed during the motion, one may evaluate the distance in terms of the wave-length of the light, or the wave-length in terms of the chosen distance, according as the one or the other is independently known. For, returning to equation (18) and putting $\mu = 1$ (since the two surfaces of the "thin plate" are separated only by air) and $r = 0$ (since the light is normally incident), we see that φ is changed by 2π when the spacing between the reflectors is changed by $\frac{1}{2}\lambda$; but when φ is changed by 2π, a single ring is added to or subtracted from the system of annular fringes; hence the total number of rings created or destroyed during the motion of the mirror is equal to twice the number of wave-lengths comprised in the distance which it traverses. In this way Michelson counted, as the first step in his determination of the length of the standard metre, the waves of the red cadmium line covering the distance between the two ends of an "intermediate standard" or *étalon*, about half a millimetre long.

In practice the real and the virtual reflector are frequently not quite parallel with one another; and this is sometimes a convenience. If they intersect (another of the things which are not possible with a pair of real reflectors) and the lens of eye or camera is located vertically above the line of intersection, this line stands forth embodied as a fine straight black fringe—the *central fringe*—companioned on either side by a multitude of others.[10] If now either of the reflectors be set

[10] Imagine a pair of mirrors inclined to one another at a very small angle φ. Establish a coordinate-frame such that the z-axis is the line of intersection of the two mirrors, the x-axis lies in the plane of either. Locate the lens of the eye or the camera at any point, say P; drop the perpendicular from P to the zx-plane at P_0; let R stand for its length, and t_0 for the distance between the mirrors at its foot. Consider the pair of reflected wave-trains arriving at P from any direction, making an angle i with the aforesaid perpendicular. Denote by α and β the projections of i upon the xy-plane and the yz-plane respectively. Assume all these angles to be small. The pair of reflected wave-trains arise from the reflection, at first and second mirrors respectively, of a single primary wave-train which fell at the same angle i upon the mirrors at a point where the distance t between them is equal to $(t_0 + R\cdot\tan\alpha\cdot\tan\varphi)$; or, to first approximation, $t = t_0 + R\alpha\varphi$. The phase-difference between them, by equation (3), is equal to $(2\pi/\lambda)2t\cdot\cos i$. To first approximation we have

$$\cos i = 1 - \tfrac{1}{2}i^2 = 1 - \tfrac{1}{2}(\alpha^2 + \beta^2).$$

Hence to this approximation we have for the phase-difference:

$$\delta = \frac{4\pi}{\lambda}(t_0 + R\alpha\varphi)(1 - \tfrac{1}{2}\overline{\alpha^2 + \beta^2}).$$

Rearranging, and dropping the terms in $\alpha^2\beta$ and $\alpha\beta^2$, we get

$$\alpha^2 + \beta^2 - 2(R\varphi/t_0)\alpha - 2 + (\lambda\delta/4\pi)(2/t_0) = 0.$$

The loci of constant phase-difference, and hence the fringes, are circles centred upon the line inclined at angle $\alpha_0 = R\varphi/t_0$ to the perpendicular dropped from the lens to the mirrors. If this perpendicular meets the mirrors along their line of intersection, as assumed in the text, we have $t_0 = 0$; the centre of the circles is infinitely remote, the fringes are straight. If then either the lens or the line of intersection is shifted sidewise, the fringes march sidewise and acquire a curvature.—It should be realized that if the wave-fronts are wide and the mirrors not perfectly parallel, there are fluctuations of intensity along each wave-front; it may be necessary to narrow the aperture of the lens in order to avoid confusion due to these.

into motion, the fringes march off sidewise, growing more curved as they go; and the number passing any fixed marker set up in the field of vision is the double of the number of wave-lengths comprised in the distance through which the mirror moves.

Consider now for a moment what must be observed, if wave-trains of many wave-lengths fall upon the mirrors, instead of pure monochromatic light. Each kind of light produces its own pattern of fringes; but since the breadth of a fringe depends upon the wave-length, those of one kind cannot fall perfectly—light upon light and shade upon shade—upon those of another; and in most parts of the field of view, if not in all, the various patterns must blot one another out. Yet there is one exception; returning to equation (18) one sees that for any value of r the phase-difference φ must vary with λ, *unless* $t = 0$—in which exceptional case $\varphi = \varphi_0 = 180°$ whatever the wave-length.[11] When the real mirror and the virtual one coincide perfectly, the field of view is black, however many wave-lengths are contained in the incident light; and when the real mirror and the virtual one intersect, the line of intersection is marked with a black fringe. Moreover, in the neighborhood of the central fringe there is a brilliant display of colors. Words are too feeble to describe them, but there would be no great scientific advantage to be gained from a description; for the tint observed in any particular direction is not a pure prismatic hue, but results from mixture of the wave-lengths not completely extinguished by interference in that direction, and depends therefore upon the physiology of the eye. What interests us as physicists is the service of the central black fringe and the companioning glory of colors in marking the point and moment when the real and the virtual mirrors intersect or coincide. This service was essential in the measuring of the standard metre, a process which we will now examine.

The interferometer used in the process is sketched as seen from above in Fig. 5. The mirrors M and M′ are at the two extremities of the intermediate standard; they are made parallel with the greatest possible exactness, and the further is elevated above the level of the nearer. As for the other mirrors, D is the movable "reference" reflector; the purpose of N will appear directly; N′ may be ignored.

The instrument is now so adjusted that D is strictly parallel to the virtual image of N, and intersects that of M at a small angle. Red cadmium light being used, a part of the field of view is occupied by the circular fringes due to the cooperation of D and N and part by the

[11] Also φ is independent of λ if $\cos r = 0$, a case which may occur if we have a stratum of air between glass plates, and choose an angle of incidence near the angle of total reflection.

straight fringes due to that of D and M. Among these latter the central fringe marking the line of intersection of D with the virtual image of M could hardly be identified; but when the white light is turned on, it stands forth unique. It is made to coincide with some

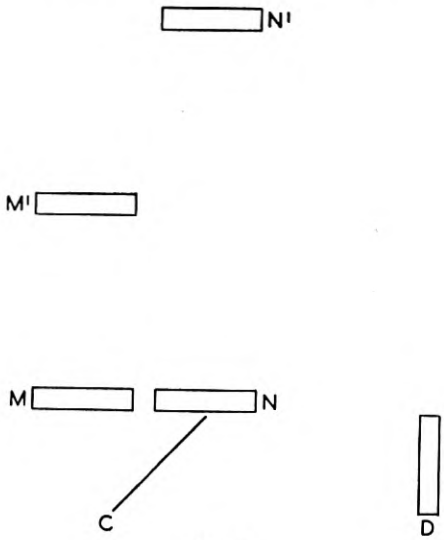

Fig. 5.

fiducial mark in the field of view, and the measurement commences. The red light being restored, the observer transfers his attention to the circular fringes formed by N and D, and counts them as they pass while he moves D along towards the virtual image of M′. Making occasional tests with the white light, he eventually notes the advent of the blaze of colours and the central black fringe which indicate the intersection of this image with D. When the black stripe coincides with the fiducial mark, the reflector D has moved through the length of the standard MM′, and this length has been measured by the count of the circular fringes which meanwhile were passing by in the other part of the field of view. The number of waves is, of course, not as a rule an integer; the fractional excess may be estimated quite accurately in various ways.

In the actual work, this shortest standard was about 0.4 mm. long, and comprised somewhat over 606 waves of red cadmium light. The counting of the 1212 fringes corresponding to its length was the only counting required; the rest of the process consisted of nine stages, the first of which was the type for all the others except the last. This second step was the comparison of the shortest standard with a second, made as accurately as possible to be twice as long; or rather, the shortest standard had been constructed with the deliberate aim of making it as nearly as possible just half so long as the second shortest. It was the office of the interferometer to determine how nearly this ideal had been attained; which was fulfilled by means of coincidences, detected as before through the coloured fringes.

Returning once more to Fig. 5, let M and M′ stand for the mirrors of the shortest standard, N and N′ for those of the second shortest. N and N′ are on a higher level than M and M′, and the field of view may therefore be divided into quadrants, in which appear the fringes—if and when there are any—formed by the collaboration of D with the virtual images of M and M′ and N and N′, respectively. The observer then goes through the following routine: (1) D is made to intersect the images of M and N; (2) D is drawn back till it intersects the image of M′; (3) the shortest standard carrying M and M′ is drawn back till the image of M again intersects D; (4) D is drawn back until it intersects the image of M′. The witness of intersection is always the central black fringe, appearing in the required quadrant or quadrants. Now if the distance NN′ is exactly twice the distance MM′, then when the observer completes the four stages of the routine D will intersect not only the image of M′ but that of N′; at the end of stage 4 the central fringe will appear in each of the upper quadrants, just above the point where at the beginning of stage 1 it appeared in each of the lower quadrants. If NN′ departs by a fraction of a wave from the doubled value of MM′, the central stripe in one of the upper quadrants will lag by a little behind that in the other. From the lag expressed in terms of the fringe-width, one may compute the difference between the length of the second standard and the doubled length of the first, and so obtain the number of waves comprised in the second. Now the second standard is made as nearly as possible of one-half the length of the third, the third of the fourth, and so on up to the ninth, which is made as nearly as possible one-tenth of the length of the standard metre. From each to the next the comparison is made in the same way. To show the remarkable reliability of Michelson's results I quote the three values which he obtained by three independent measurements of twice the number of waves of red cadmium light comprised in the length of the ninth standard:

310678.48, 310678.65, 310678.68.

After this point it remained to compare the ninth standard with a metre-rod, and the metre-rod with The Standard Metre. Returning for the last time to Fig. 5, we take M and M′ as the mirrors at the ends of the ninth standard. The steps are: (1) make M coincide with the end of the metre-rod, and D intersect the virtual image of M; (2) draw D back to intersect with M′; (3) draw MM′ back until M coincides with D; (4) draw D back to intersect with M′—and so forth ten times altogether, until for the last time D intersects M′, and M′ is very near the far end of the metre-rod. The discrepancy is again a fraction of a wave-length.

The result in which all this labour culminated was: **1,553,163.5** wave-lengths of red cadmium light are comprised in the length of the standard metre.

Such was the process of enumerating the millions of light-waves required to make up the length of that standard chosen for the measurements of common life, and so very ill-adapted to magnitudes of the scale of those in light—the distance between two scratches on the bar of platiniridium alloy, conserved in the vault at Breteuil with the care lavished upon a sacred relic. The achievement of Michelson was the bridging of a gap, or let me say a work of translation. Nearly all measurements of wave-lengths to this day are determinations of the ratio of one wave-length to another, as practically all measurements of objects an inch, a metre or a mile long are determinations of the ratios which they bear to metre-sticks. In dealing with tangible objects we use the language of the metre; in dealing with light-waves, we use effectively the language of another scale of measurement. Michelson was the first to make a supremely accurate translation from one to the other of these languages, so making it possible to express a measurement of either realm in the scale familiar for the other. Whether in addition he may be said to have replaced a standard essentially impermanent and transitory by one which in the nature of things is everywhere the same and forever immutable, is a question very likely never to be answered.

Harmonic Production in Ferromagnetic Materials at Low Frequencies and Low Flux Densities

By EUGENE PETERSON

SYNOPSIS: When a multi-channel communication circuit includes a non-linear element such as a ferromagnetic core coil, distortion of the wave form impressed upon the circuit is produced. In terms of the single frequency components, this distortion is manifested in the appearance of new components. This distortion may give rise to the reduction of quality in any channel, and it may also introduce crosstalk and interference, which consists of new frequencies not present in the impressed wave of any channel under consideration, produced by independent channels. In view of the recent increased use of multi-channel systems, it has become necessary to investigate the effects of this type of distortion, to determine the dependence of this distortion upon the properties of the magnetic materials constituting the cores of inductance coils and transformers, as well as upon the circuit impedances, and to determine those constants of core materials which are significant in the distorting process.

The behavior of magnetic materials to complex waves of magnetizing force is ordinarily a highly involved process, so that a direct correlation between distortion and some of the easily measured constants of materials is a matter of some difficulty. It has been established experimentally, as a confirmation of theoretical speculations, that the third harmonic e.m.f. generated by a sinusoidal wave of magnetizing force may serve as an index of the distortion with a complex wave of magnetizing foi ce. This relation is valid for low flux densities and for frequencies at which the screening effect of eddy currents is not important. The paper is therefore devoted to an investigation of the third harmonic production in its dependence upon the properties of hysteresis loops. These loop constants in turn are shown to be deducible from AC bridge measurements on a coil of known dimensions having a core of the magnetic material under investigation. The loop constants for a few materials are included in the text. An analogy exists between the treatments of hysteresis loop and of three-element vacuum tube characteristics which enables us to compare simplifying relations introduced by Rayleigh and by H. J. van der Bijl in the two cases.

The theoretical deductions are found to be in general agreement with experiment, and are applied to a number of cases of practical interest. These include the effects of air gaps and dilution, and the choice of core material in third harmonic production by inductance coils and transformers. Finally, the amount of third harmonic current flowing out of long lines is deduced with both lumped and continuous loading.

PART 1. HYSTERESIS LOOPS AND THEIR MATHEMATICAL REPRESENTATION

NEW and improved systems of multi-channel communication which have come into use during the past few years have imposed rigorous requirements on the circuit elements constituting the communicating link, and have made it necessary to investigate the degree of distortion which arises from the use of ferromagnetic apparatus. The distortion introduced may have two general effects: distortion of the signal in any one channel, which is usually the minor effect, and production of crosstalk and interference between the various channels to which the non-linear element is common. Such elements are, in general, ferromagnetic core coils or transformers. The distortion introduced by the non-linear relation between flux density and magnetizing force is therefore of fundamental importance in the design of iron core coils and transformers which carry simultaneously a number of communication channels.

The use of iron core coils and transformers in communication work is confined to comparatively low flux densities in contrast to the ordinary practice in power work, where operation usually occurs above the knee of the normal magnetization curve at a value of the order of a thousand times greater than that used in communication work. There are two main reasons for this restriction: losses are reduced, and the relation between flux density and magnetizing force approaches linearity so that distortion is minimized.

Under actual operating conditions in which the more important crosstalk effects arise, we have a complex wave of magnetizing force acting on the magnetic core. Non-linearity in the magnetic circuit gives rise to new frequencies which normally [1] are related to those impressed upon the circuit, being sums and differences of integral multiples of the originally impressed frequencies. These modulation products are all of odd order [2] when the core is unpolarized. On purely theoretical grounds we would expect to find relations between the amplitudes of the different frequencies resulting from any one order of modulation. To take the third order modulation products of two impressed frequencies (f_1, f_2) as an example, we would expect the amplitudes of the harmonics $3f_1$ and $3f_2$ to be related to the amplitudes of the other third order products: $2f_1 \pm f_2$, $2f_2 \pm f_1$. This has been confirmed by direct experimental test, so that we are enabled to use the generated third harmonic voltage as an index of the generated voltages corresponding to the other third order products. Accordingly, we shall deal in the following with the third harmonic produced by a sinusoidal wave of magnetizing force, and so avoid a more involved analysis.

The fundamental relation in the operation of ferromagnetic apparatus is of course the relation between the flux density B and the magnetizing force H. In contrast to the usual behavior of circuit elements, the relation between the two fundamental quantities—the independent and the dependent variables, H and B in this case—is a function not only of the value of the independent variable, but is also

[1] This is true in the low flux density region. At high densities and with highly reactive circuits as in magnetic modulators, other frequencies are sometimes found which correspond to natural oscillations of the coil and circuit.

[2] *Bell System Technical Journal*, Jan. 1928, pp. 110, 111.

a function of its previous history as manifested in the phenomenon of hysteresis. This complicates matters to an extent far greater than is the case with other circuit elements, and some analysis has been carried out in which the hysteresis loop has been replaced by the normal magnetization curve, or by some such single valued relation between the variables. For some purposes this convenient simplification—it cannot be called a close approximation for our present purposes—is satisfactory, while for others it does not begin to tell the story. Inasmuch as our aim here is to deal with the actual phenomena involved rather than to arrive at some arbitrary procedure for representing the facts, we shall in the following base considerations upon the hysteresis loop.

Fig. 1 will serve to illustrate the effect of previous history upon flux density. The main loop there, which extends between the two

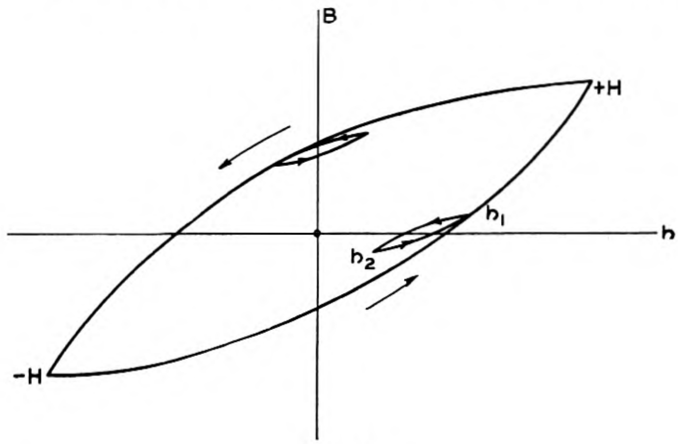

FIG. 1

limits of magnetizing force $-H$ and H, is obtained when the magnetizing force is varied cyclically between those two values in such a way that the magnetizing force has but one maximum and one minimum per cycle. In that case the B–H loop is traversed in the direction shown by the arrow. It is independent of frequency when the eddy-current losses in the iron are small, as we shall suppose them to be, and it is independent of the wave form of the magnetizing force so long as that wave form satisfies the condition we have laid down above. A sinusoidal magnetizing force, for example, satisfies that condition. When the magnetizing force contains components of such magnitude and phase that the wave form has multiple maxima or minima, the simple B–H loop no longer suffices to represent that relation, but auxiliary loops shown in the figure are involved.

Thus suppose a subsidiary maximum to exist at h_1; as the magnetizing force decreases, the flux density no longer follows the main loop but branches off on a subsidiary loop as indicated by the arrows. When the subsidiary minimum at h_2 is reached a new branch is started which completes the subsidiary loop, and which brings the magnetizing force back to the main loop at the point from which it originally diverged, and the main loop is thereafter followed until another maximum (or minimum as the case may be) of the magnetizing force wave is reached. For simplicity in the following we are going to deal solely with a sinusoidal wave of magnetizing force so that subsidiary loops are never called into play. With this understood, the relation between B and H is described by the simple loops of Fig. 2 over a

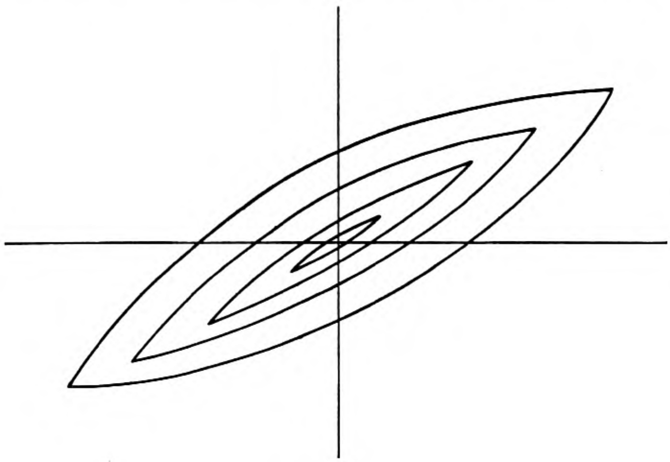

FIG. 2

certain range of magnetizing force, each loop being defined by a particular value of maximum magnetizing force. Each loop may be considered as constituted by two branches which join at the maximum field of the loop.

It is clear, therefore, that with a periodic magnetizing force having but one maximum and one minimum per cycle, the flux density depends upon three properties of the magnetizing force—the maximum value, the instantaneous value, and the sign of dh/dt, being located on the lower branch when dh/dt is positive, and on the upper when it is negative. Now it is of course evident that when a definite loop form is available, a numerical solution by graphical or step-by-step methods may be had. It is further evident that, in the case of a definite impressed magnetizing force, the B–H loop may be broken up into its harmonic components, as was done by S. P. Thompson.[3] Solutions in this form possess all the advantages and disadvantages of numerical ones in which any change of conditions leads to a new problem; the solution desired is a general one which will describe the phenomena in terms of coefficients characteristic of the magnetic material, and this type of solution is the subject of analysis in the following pages.

Perhaps the least difficult method of arriving at an analytical solution without making any assumptions as to the form of the loops is the following. A power series for each branch of any loop is formulated, and the two resultant equations are combined in a trigonometric series. In this way the solutions, each one valid over but half the cycle, are combined to represent the relation of B to H over the entire cycle.

The flux density on each branch of the loop may be defined in terms of two values of magnetizing force, one the maximum value of magnetizing force on the particular branch with which we happen to be concerned and the other the instantaneous value of the magnetizing force on that branch. The equation for either branch may then be expressed as a double power series in these two variables, the instantaneous magnetizing force (h) which will be expressed in gilberts/cm, and the maximum value of the magnetizing force (H), expressed in the same units;

$$B(h, H) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} h^m H^n, \tag{1}$$

where

$$a_{mn} = \frac{1}{m!\,n!} \frac{\partial B(h, H)}{\partial h^m \partial H^n} \bigg]_{0,\,0}. \tag{2}$$

The parallelism of this representation with that for the plate current-grid potential curves of a three electrode thermionic tube is evident,—in both cases double power series are involved.[4] The coefficients a_{mn} are derivatives which are evaluated at the point $h = 0$, $H = 0$, and it will be understood in (2) that these particular values are inserted after the derivatives have been taken, in quite the usual manner.

There is a further interesting parallel here to the vacuum tube case regarding simplification of the general relation. In the case of the vacuum tube a simplification of the double power series due to van der Bijl has been employed which represents the family of tube characteristics by an equation in a single variable. This simplification consists in assuming the amplification factor to be constant, and the important point for us here is that it is equivalent to the assumption

[3] "On Hysteresis Loops and Lissajous' Figures," *Phil. Mag.*, 1910.

[4] Peterson and Evans, "Modulation in Vacuum Tubes used as Amplifiers," *Bell System Technical Journal*, July, 1927.

that the different branches of a family have the same form, so that by suitable change of plate or of grid potential the plate current-grid voltage curves may be superposed. A relation of precisely the same form in the case of magnetic hysteresis loops was stated by Lord Rayleigh [5] upon examination of data obtained by a magnetometric study of the behavior of a single low permeability specimen. Examination of his data enabled Rayleigh to conclude that the branches corresponding to different loops of a family could be superposed when referred to a common loop tip, the branches all having the same parabolic form within the limits of accuracy of the measurements, over the range of magnetizing forces involved.

It is of course evident that even if this relation held at low fields in all materials it would break down at sufficiently high fields. Further, there is no *a priori* reason to expect this relation to hold for magnetic materials other than the one Lord Rayleigh investigated unless we restrict consideration to a very small range of magnetizing force. With these ideas in mind it seems the safer procedure to assume no such simple relation between the different loops of a family, however convenient it might be, and to treat the problem in more general terms; if any such simplifying relations exist they will be made apparent after application to definite materials. We may anticipate matters a bit to state at this point that certain materials seem to obey the relation while certain others seem to violate it within the range of forces involved in communication work, and further light is shed on the significant processes involved in harmonic production by treating the problem in this way.[6]

General Equations for Hysteresis Branches. The equations of both the upper branch family and the lower branch family have the form of equation (1)—we may designate the upper branches by B_1 and the lower branches by B_2—but the coefficients of the two series differ in general.

In order to put the equations in shape so that they may be of practical utility it is now necessary to determine the coefficients of the expressions for B_1 and B_2 so that they apply to a definite loop family, and this is accomplished by reference to some of the more general properties of the loops and of the normal magnetization curve. These properties are as follows:

[5] "Notes on Electricity and Magnetism, III," *Phil. Mag.*, 1887, V. 23.

[6] Unpublished work based on assumptions which include Rayleigh's relation was independently carried out by W. P. Mason of these Laboratories in 1922 and 1923. An account of some applications of Rayleigh's relation is to be found in an interesting paper by Jordan published in the *Elektrische Nachrichten Technik*, B. 1, H. 1, July 1924.

1. When both h and H are zero the flux density on either branch is zero.
2. The flux density on one branch with H and h given is equal and opposite in sign to the flux density on the other branch corresponding to the negative of h and to the same maximum force H.
3. The two branches corresponding to a definite H meet at the normal magnetization curve.

The application of these properties to the power series enables us to deduce relations between the coefficients as demonstrated in Appendix 1:

$$\begin{aligned} a_{01} &= a_{00} = 0, \\ a_{02} &= -a_{20}, \\ a_{03} &= -a_{21}. \end{aligned} \tag{7}$$

We shall find it sufficient to include the third degree terms for our work, so that we need not investigate relations between coefficients of higher degree. The loop equations are simplified by utilizing (7) and we can make a number of interesting deductions. Thus the equation for the normal *magnetization curve* is given by

$$B(H, H) = a_{10}H + a_{11}H^2 + (a_{12} + a_{30})H^3. \tag{6a}$$

In this equation a_{10} will be recognized as the initial permeability usually expressed as μ_0 since upon division by H we have for the *permeability*

$$\mu = a_{10} + a_{11}H + (a_{12} + a_{30})H^2 \cdots.$$

According to this equation the change of permeability with magnetizing force is linear at sufficiently small fields. The above equations are, in all rigor, infinite series, but for our purposes it will be found sufficient to consider only coefficients of the third and lower orders,—in some cases the second order will suffice.

An expression for the *remanence curve* of the loop family may be obtained by setting h equal to zero in the equation for the upper branch. In that case we find from (4a)

$$B(O, H) = a_{02}H^2 + a_{03}H^3 + \cdots, \tag{8}$$

hence for sufficiently small magnetizing forces the remanence increases as the square of the magnetizing force.

The *hysteresis loss per cycle per unit volume* may also be obtained directly from the branch equations (4a) and (5a). The loss in ergs, w, is equal to the area of the hysteresis loop divided by 4π. If now we consider the loop area to be built up of strips of infinitesimal width based on the h-axis, the height of any one of the strips is given as

$$B = B_1(h, H) - B_2(h, H)$$

and we have

$$w = \frac{1}{4\pi}\int_{-H}^{H} Bdh \tag{9}$$

so that, by Appendix 1, (9) may now be expressed as

$$w = \frac{2}{3\pi}(a_{02}H^3 + a_{03}H^4 + \cdots). \tag{10}$$

It is clear, therefore, that at sufficiently low fields the hysteresis loss varies as the cube of the magnetizing force, and diverges when the field is made sufficiently large—it seems to be in general agreement with experimental results on ballistic loops and, as will be pointed out later, is verified by impedance change data under alternating excitation. In view of the remanence curve equation, (10) may be rewritten as

$$w = \frac{2}{3\pi}HB(O, H). \tag{11}$$

Various approximations have been made in the past to hysteresis loop forms in order to obtain convenient expressions for the hysteresis loss. Thus if we consider the loop as an ellipse the loss becomes

$$\frac{1}{4}HB(O, H),$$

while if we consider the loop a parallelogram the loss is

$$\frac{1}{\pi}HB(O, H).$$

Both these expressions give too large a result since the coefficient $2/3\pi$ of the exact equation (11) is 0.212.

Branch Equations for Materials Obeying Rayleigh's Relation. Rayleigh's observation enables us to establish relations between the different coefficients involved in our development above when the loops of a family are similar in form. For the sake of completeness a derivation of these relations is given in Appendix 2; their validity may then be judged by test in specific cases. In the derivation we assume a power series expansion for one branch of the largest loop of a family referred to the tip, and assume that the smaller loops are of the same form. Then by referring the equations to the origin instead of to the loop tip, we arrive at the hysteresis branch equations.

By comparing the coefficients of this equation with those of the general equation (1), we can deduce relations between the coefficients. These are, from Equation 16 of Appendix 2,

$$\begin{aligned} &a_{01} = 0, \\ &a_{20} + a_{02} = 0, \qquad &a_{11} = 2a_{20}, \\ &a_{21} + a_{03} = 0, \qquad &a_{12} = a_{21} = 3a_{30}, \\ &a_{40} + a_{22} + a_{04} = 0, \qquad &a_{31} = a_{13} = 4a_{40} = 6a_{22}. \end{aligned} \tag{17}$$

The left column is in agreement with Equation (7), while the right column furnishes new relations between the coefficients. Thus the coefficients based on loop similarity are not inconsistent with those deduced under no assumptions, but the former involve additional relationships between coefficients which need not be satisfied in the general case.

Families of hysteresis loops obtained by the usual ballistic method have been examined for these relationships in the case of several materials with the following results for coefficients up to and including the second degree:

Hysteresis Loop Coefficients

	Silicon Steel	'B' Dust [7]	'C' Dust [7]
a_{10}	270	35.3	26.2
a_{11}	2600	1.53	0.59
$a_{11}/2$	1300	0.76	0.29_5
a_{02}	967	0.65	0.29
a_{20}	− 951	− 0.71	-0.28_5

The values tabulated were obtained by 'smoothing' data obtained from ballistic loops, which leaves a great deal to be desired from the standpoint of precision. The difference between a_{02} and $-a_{20}$ is an index of this lack of precision. The average deviation of $a_{11}/2$ from the mean of a_{02} and a_{20} (a relation deduced on the basis of similarity) is small for C dust and large for silicon steel while the third order coefficients show much poorer results. The lack of precision in the original data however, is such as to leave unsettled the question of loop similarity for the materials tested at the fields to which the coefficients apply; the experimental error is too great.

The use of ballistic methods for determining coefficients becomes even less satisfactory for materials with low hysteresis losses and cannot be used for analyses which have any aspirations to precision. It will be shown in the next section, however, that the significant coefficients enter into the impedance of a coil which employs the material under examination as a core, so that the desired coefficients

[7] Speed and Elmen, "Magnetic Properties of Compressed Powdered Iron," *A. I. E. E.*, V. 40.

may be obtained with the aid of AC bridge measurements under appropriate conditions for which eddy currents and winding capacities are unimportant. These measurements are of a higher order of precision than those obtained by the ballistic method, and will be treated in detail in the next section.

Part 2. Alternating Magnetization

Sinusoidal Magnetizing Force. The same magnetic characteristics which were the subject of the investigation of the last section are involved in alternating magnetization when the applied field has but one maximum and but one minimum per cycle, and when the eddy losses are not great enough to introduce screening effects. As one of the results of that investigation, we have arrived at equations for both the upper and the lower branch families, so that to obtain an expression for the flux density valid over the entire cycle, it is now necessary to combine the two branches by a Fourier's series in the usual manner.

Before doing this, however, it is necessary to express the branch equations for the flux density in terms of time, rather than as series of powers of the independent variable, the magnetizing force. To do this we substitute the equation for the magnetizing force [8]

$$h = H \cos pt \tag{18}$$

in the branch equations (4a) and (5a) of Part 1. The result of the substitution is to give the upper and lower branch flux equations in terms of powers of cos pt, which may be expressed in terms of multiple angles. These two equations are then combined in a Fourier series which is valid over the entire cycle:

$$B = \frac{b_0}{2} + \sum_{k=1}^{k=\infty} (b_k \cos kpt + a_k \sin kpt),$$

in which the coefficients are given by the usual expressions, equation 22a of Appendix 3. The coefficients $b_0, b_2, \cdots b_{2k}, a_0, a_2, \cdots a_{2k}$ are found to be identically zero on account of the symmetry of the loop family about the origin, while the fundamental and third harmonic coefficients are found to be as follows:

$$\begin{aligned} a_1 &= \frac{8}{3\pi}(a_{02}H^2 + a_{03}H^3), \\ b_1 &= a_{10}H + a_{11}H^2 + (a_{12} + \tfrac{3}{4}a_{30})H^3, \\ a_3 &= -\frac{8}{15\pi}(a_{02}H^2 + a_{03}H^3), \\ b_3 &= a_{30}H^3/4. \end{aligned} \tag{25}$$

[8] The purely sinusoidal magnetizing force may be obtained, despite the varying reaction of the iron core coil, by connecting a generator through a low pass or band pass filter having a high impedance outside the pass band, or by connecting a pure sine wave generator to the coil through a high impedance.

Details of the derivation are given in Appendix 3. Inasmuch as we assume the applied field to vary as cos pt, the b's are in phase with the applied field and the a's are in quadrature. The two a's, it is observed, are connected by a constant of proportionality ($a_1 = -5a_3$) and depend upon the remanence; or upon what is the same thing, the hysteresis loss divided by the magnetizing force.

If we expanded the loop equations to higher powers we should find that a_1 and a_3 cease to be linearly proportional. The coefficient b_1 is observed to have its first three terms identical with those of the normal magnetization curve, but the fourth term differs by precisely the amount b_3.

The voltage existing across a coil enclosing a core characterized by the above coefficients is

$$E = nA10^{-8}\frac{dB}{dt}, \tag{26}$$

where n is the number of turns enclosing the core, A is the core area in cm², B is the total flux density and E is the total generated potential. Carrying out this operation we have

$$E = nA10^{-8}(pa_1 \cos pt + 3pa_3 \cos 3pt \\ - pb_1 \sin pt - 3pb_3 \sin 3pt) \tag{26a}$$

in which the voltage components in phase with the current depend on the hysteresis coefficients, and the quadrature components depend only on the coefficients of the normal magnetization curve. The dependence of each of these components upon the applied field is clear from Equation (25). To take the two third harmonic components it will be observed that a_3 starts to vary with the square, while b_3 starts to vary with the cube, of the applied field. It follows therefore that at sufficiently small amplitudes the third harmonic is produced by the a_3 term and not by the b_3 term, which means that *under the conditions noted, harmonic production is due to hysteresis and not primarily to permeability change.*

From the two fundamental components of voltage across the coil an expression for the inductance and resistance offered to the flow of alternating current may be deduced. The inductance, it is easy to see, is obtained directly from the magnetization curve,—the d.c. permeability of the normal magnetization curve therefore coincides with the a.c. permeability at small fields. The resistance may be obtained from the expression derived above for hysteresis loss per cycle per cc. If we multiply that value by the volume of iron in the coil, by the frequency, and by 10^{-7} to convert ergs to watts we have the loss per second, and this may be equated to the square of the effective fundamental current multiplied by the hysteresis resistance, or

$$wfA\pi d10^{-7} = I^2R_h/2.$$

Here d is the diameter in cm, I is the peak value of the fundamental current, f is the frequency, and R_h is the hysteresis resistance. This may be solved for the hysteresis resistance in terms of the r.m.s. value of fundamental current $\bar{I}$, which is at low fields,

$$R_h = 1.2 \cdot 10^{-8} n^3 Af\bar{I}a_{02}/d^2. \tag{27}$$

The hysteresis resistance at small fields thus varies linearly with the applied current.

It is easy to derive the relation between the hysteresis resistance and the generated third harmonic voltage at low fields since they involve the same constants a_{02} and a_{03}; from (26a) and (25) we have for the r.m.s. third harmonic voltage

$$\bar{E}_3 = 3pnA10^{-8}a_3/\sqrt{2} = 0.72 \cdot 10^{-8} fn^3 A\bar{I}^2 a_{02}/d^2.$$

Comparison with (27) shows that

$$\bar{E}_3 = 0.6R_h\bar{I}. \tag{28}$$

This simple relation is valuable in obtaining an idea of the harmonic production in a specific coil through resistance measurements, and more than that, it enables us to determine the coefficients significant in the distorting process for the material under test, so that the harmonic production in a coil of any dimensions enclosing the same core material may be calculated. The degree of precision ordinarily attainable is brought out for two materials in Fig. 3, and the agreement is observed to be within the experimental error in those two representative cases.

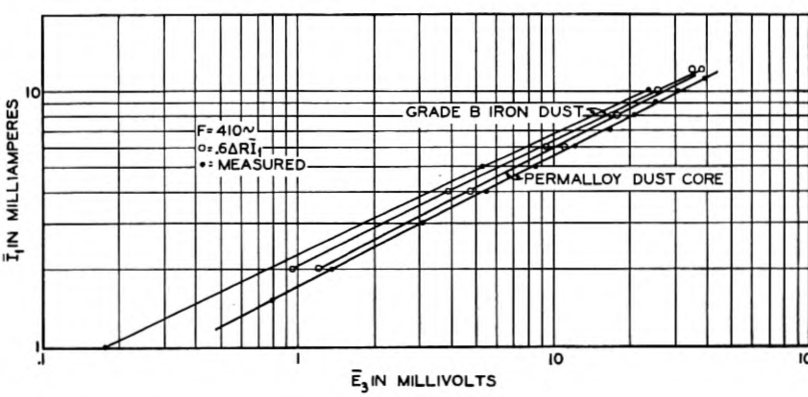

FIG. 3

These relations provide us with an expression for the ratio a_{11}/a_{02} which may be obtained from impedance measurements. It will be recalled that Rayleigh's relation would give this ratio the value two, and a.c. bridge measurements may be used to check this relation. The change of resistance with current is given by (27), and the change of reactance with current may be calculated from (25) and (26), since we have

$$\Delta X = a_{11}H^2pnA10^{-8}/\bar{I}\sqrt{2}$$
$$= 1.42\cdot 10^{-8}n^3Af\bar{I}a_{11}/d^2.$$

Hence if we denote R_h by ΔR, we may write

$$\frac{a_{11}}{a_{02}} = 0.85\frac{\Delta X}{\Delta R}.$$

The results obtained on some dust cores of different composition and two solid materials may be tabulated as follows:

Material	Number of Specimens	a_{11}/a_{02}
Grade 'B' Iron Dust	10	2.65
Grade 'C' Iron Dust	1	2.63
Permalloy Dust [9] 'A'	3	2.10
" " 'B'	3	1.81
Iron Dust No. 1	1	2.80
Iron Dust No. 2	2	2.55
Perminvar [10]	1	3.0
Alloy 'D'	2	1.84

The method of analysis and the results obtained above may be summarized as follows. Starting with the general development of the hysteresis branches in a double power series and making use of only the most fundamental experimental observations, equations have been derived for the normal magnetization curve, the remanence characteristic, and the hysteresis loss characteristic in terms of the original hysteresis branch coefficients by combining the equations for the two branches in a trigonometric series. The series for the normal magnetization curve is found to start with the first power of the magnetizing force, the remanence starts with the second power, and the hysteresis loss starts with the third power, the curves varying in accordance with their respective first terms when H is small. These results seem to be in general agreement with experience. For large values of H the shapes of the different curves depend of course upon the size and sign of the coefficients involved, about which nothing can be said until the coefficients have been evaluated from experimental data.

[9] Shackelton and Barber, "Compressed Powdered Permalloy," A. I. E. E. Convention, February 1928.

[10] Elmen, "Magnetic Properties of Perminvar," *J. F. I.*, September 1928.

In the case of a sinusoidal magnetizing force the fundamental and third harmonic flux densities were derived with the aid of the trigonometric series development. These were shown to depend in a simple way at low fields upon the normal magnetization and hysteresis loss characteristics. The in-phase fundamental voltage component depends directly upon the hysteresis loss per unit current, while the fundamental component in quadrature varies with the magnetization curve at low forces. The in-phase third harmonic varies with the in-phase fundamental at low forces, and the third harmonic in quadrature comes in only with larger forces in a manner which depends upon the magnetization characteristic. The range of forces over which our equations are valid depends simply upon the number of terms taken in the development of the branch equations, and is not necessarily restricted to small forces. The hysteresis loop coefficient enters into the impedance offered to the fundamental frequency by a coil enclosing the core material in question in such a way that the third harmonic produced may be deduced from the change of resistance with current. Further, the ratio of the change of reactance with current to the change of resistance with current may be used to provide a test of Rayleigh's relation, which is found to hold for some materials, while it is invalid for others. The precision obtainable in the evaluation of hysteresis loop coefficients is much greater by the a.c. bridge measurement under proper experimental conditions than by the analysis of ballistic loops. Incidentally attempts have been made to obtain B–H loops by AC methods with the aid of the Braun tube, for example,[11] but the precision attainable is not sufficiently high for our purpose.

Complex Magnetizing Force. Our preceding analysis has furnished us with the fundamental voltage drop across a coil enclosing the iron core under consideration, together with the third harmonic voltage generated in the coil winding due in general partly to the non-linear B–H relation and partly to the effect of hysteresis. The generated voltages corresponding to the fifth, seventh, and higher orders are also calculable by the same methods but will not be specifically considered since they are smaller than the third harmonic at low fields. This generated third harmonic voltage exists in its entirety across the coil winding only when the impedance of the external circuit is much higher than that of the coil at the harmonic frequency, a condition not usually satisfied in telephone circuits. A current of the third harmonic frequency then flows in the circuit, its amplitude and phase depending evidently upon the generated third harmonic voltage—that is, the coil structure and core material—as well as upon the total

[11] Peterson, *Phys. Rev.*, Vol. 27, No. 3, p. 320.

circuit impedance to the third harmonic which includes that of the coil. We have now to determine the coil impedance to a complex magnetizing force constituted by the fundamental and the third harmonic. It is evident at the outset that if we restrict consideration to a very small third harmonic, the impedance to the fundamental cannot be very materially altered.

In general the fundamental and third harmonic may exist in any phase but for our present purpose we shall take the magnetizing force to be

$$H = H_1 \cos pt + H_3 \cos 3pt, \tag{29}$$

in which the phase angle is assumed zero. This seems to be an arbitrary assumption, but it may be shown that the results are not materially different for other phase displacements, and there is some gain toward simplicity of treatment by taking the phase angle zero. The equations of Part 1 may then be carried over without change, and details of the analysis are given in Appendix 4.

The ratio of the resistances to the third harmonic and to the fundamental is 0.77, from the relations deduced in the appendix. Some experimental work has been done to test the validity of the expressions derived above for the reactance and resistance components of the third harmonic, the results of which are given in Fig. 4. It is seen that the check for the inductance to the third harmonic is very good but that the resistance component appears to be substantially that of the fundamental rather than 77 per cent of it, as the analysis indicates.

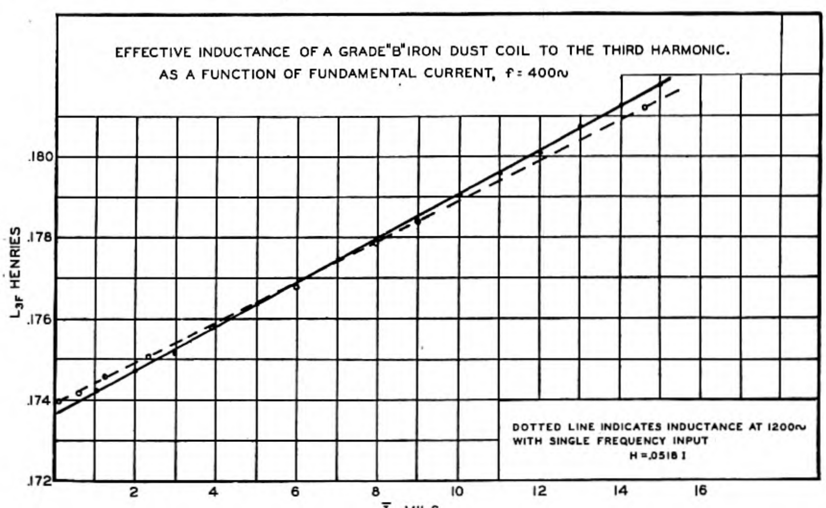

FIG. 4*A*

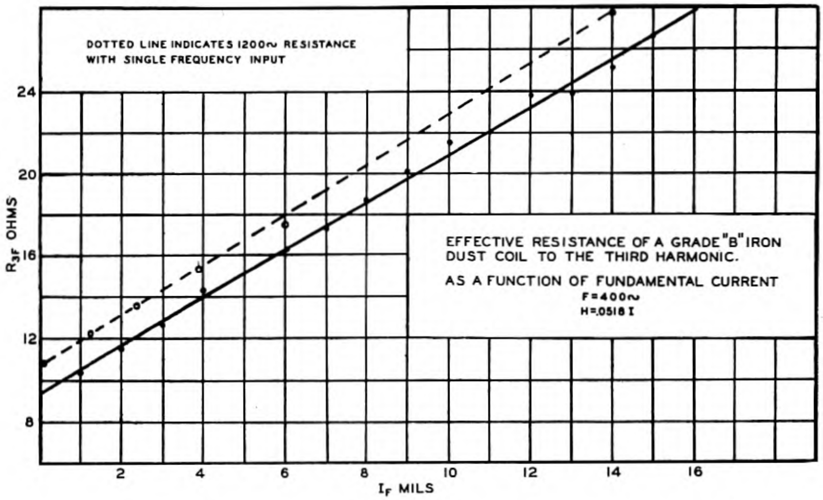

Fig. 4B

The experimental method for obtaining this result—the impedance to a small third harmonic in the presence of a relatively large fundamental—is illustrated in Fig. 5. The method consists in measuring the third harmonic current by means of a current analyzer [12] for a number of circuit conditions in which the fundamental current is

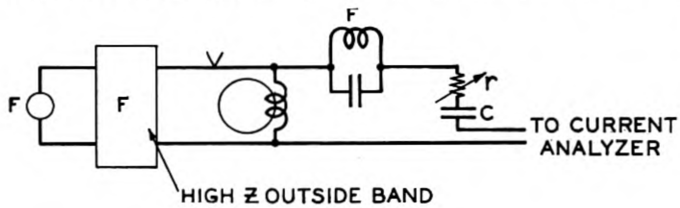

Fig. 5

maintained constant. The circuit is first tuned to the third harmonic by varying the capacity C in the third harmonic circuit, and the current is then measured for a series of values of the series resistance r. A shunt resonant circuit tuned to the fundamental is inserted in the third harmonic path so as to separate effectively the third harmonic circuit from that of the fundamental. With this precaution taken to avoid harmonic production in the analyzer and to maintain the fundamental current constant while r and C are varied, the inductance to the third harmonic is obtained from the resonating capacity, and the resistance is determined as that value of r for which the third harmonic current falls to half its maximum.

[12] For details of current analysis see the paper by A. G. Landeen, *B. S. T. J.*, April 1927.

Part 3. Application of the Analysis

Air-Gaps and Dilution. Under certain conditions improvement in the operation of iron core coils toward freedom from harmonic production may be attained by inserting air-gaps in the magnetic path. The expressions which we have derived up to this point are valid for a material having the constants assigned, and the question now arises as to the parameters which characterize the operation of the iron core including air-gaps, and their relation to the parameters for the original core without air-gaps. With these relations given, our previous work may be applied to cores with air-gaps.

In establishing the correspondence between the parameters for the two cases, it is instructive to use two methods—one a direct attack,[13] the other resting on an analogy with non-linear vacuum tube circuits.[14] We may determine the effects sought for by the direct method on consideration of a single branch of a hysteresis loop, which is expressed by equations (4*a*) or (5*a*) of Part 1,

$$B(h, H) = \sum\sum a_{rs} h^r H^s. \tag{36}$$

Now with an air-gap in the magnetic circuit, the magnetomotive force effective is that applied, reduced by the drop across the air-gap, or

$$\begin{aligned} m' &= m - \rho\varphi, \\ M' &= M - \rho\Phi, \end{aligned} \tag{37}$$

in which mM, $\varphi\Phi$ are instantaneous and maximum values, respectively, of the impressed m.m.f. and flux, and in which ρ represents the air-gap reluctance

$$\rho = \lambda/A, \tag{38}$$

λ being the length of air-gap and A the core cross-section.

In order to apply (37) we re-express (36) in terms of magnetomotive force and flux as follows:

$$\varphi(m, M) = \sum_r \sum_s \frac{A a_{rs}}{l^{r+s}} m'^r M'^s, \tag{39}$$

where l is the length of magnetic circuit in the iron. Equations (37) may now be substituted in (39) to yield

$$\varphi(m, M) = \sum_r \sum_s \frac{A a_{rs}}{l^{r+s}} (m - \rho\varphi)^r (M - \rho\Phi)^s. \tag{40}$$

[13] Due to Mr. H. P. Evans.

[14] See Appendix 5.

This equation may be solved for φ by identifying coefficients when we substitute the solution

$$\varphi(m, M) = \sum_r \sum_s \frac{A a_{rs}'}{l^{r+s}} m^r M^s,$$

which is equivalent to

$$B(h, H) = \sum_r \sum_s a_{rs}' h^r H^s. \tag{41}$$

Carrying out the operations, the primed coefficients are determined in terms of the original coefficients and the core structure as follows:

$$\begin{aligned} a_{10}' &= a_{10} \Big/ \left(1 + \frac{\lambda}{l} a_{10}\right), \\ a_{02}' &= -a_{20}' = a_{02} \Big/ \left(1 + \frac{\lambda}{l} a_{10}\right)^3, \\ a_{11}' &= a_{11} \Big/ \left(1 + \frac{\lambda}{l} a_{10}\right)^3. \end{aligned} \tag{42}$$

It is clear that the effect of an air-gap is to diminish the higher order coefficients to a greater extent than those of lower order; no new coefficients are introduced. If we refer to a constant flux density, then the impressed force is

$$H' = H\left(1 + \frac{\lambda}{l} a_{10}\right) \tag{43}$$

and the modulation voltage becomes proportional to

$$a_{02}' H'^2 = \frac{a_{02} H^2}{1 + \frac{\lambda}{l} a_{10}},$$

so that the harmonic e.m.f. has been reduced in the ratio of $1 + \frac{\lambda}{l} a_{10}$ to unity, which is precisely the ratio of initial permeabilities.

Dilution. If the magnetic material is diluted with a non-magnetic substance (insulation) so that the effective air-gap length is λ, the above equations for the flux density still hold. The effective cross-sectional area, however, is reduced to

$$A' = A\left(\frac{l - \lambda}{l}\right)^2, \tag{44}$$

which is ordinarily of small account compared to the other factors involved.

Choice of Core Material. The ideal characteristics for a core material in respect to its freedom from harmonic production may be determined by a consideration of the circuit problem. The third harmonic driving e.m.f. is obtained from the last section as

$$E_3 = pnAa_{02}H^2 10^{-8},$$
$$\doteq \frac{n^3 A}{d^2} a_{02} I_1^2, \tag{45}$$

in which n represents the number of turns enclosing the core of area A, d is the mean diameter of the toroidal core and a_{02} is the hysteresis coefficient.

Suppose that we have to design a coil of fixed inductance in which the harmonic is to be a minimum, so that

$$L = K \frac{n^2 \mu A}{d} = \text{const.} \tag{46}$$

We proceed to the consideration of a number of special cases subject to condition (46).

Case 1—L **Fixed,** n **Variable.** From (46)

$$n = \text{const.} \left(\frac{d}{\mu A} \right)^{1/2},$$

which may be substituted in (45) to give

$$E_3 \doteq \frac{1}{\sqrt{Ad}} \frac{a_{02}}{(\mu)^{3/2}}. \tag{47}$$

Hence in a coil of fixed inductance, fixed core area, and fixed core diameter, minimum harmonic voltage is produced with a material for which $a_{02}/\mu^{3/2}$ is minimum.

Case 2—L **Fixed,** A **Variable.** From (46)

$$A = \text{const.}\ d/n^2 \mu,$$

which, upon insertion in (45), yields

$$E_3 \doteq \frac{n}{d} \frac{a_{02}}{\mu}, \tag{48}$$

in which the modulated voltage varies linearly with a_{02}/μ.

Case 3—L **Fixed,** d **Variable.** By a similar procedure, we obtain an expression for the generated third harmonic voltage

$$E_3 \doteq \frac{1}{nA} \frac{a_{02}}{\mu^2}. \tag{49}$$

For further work we suppose the coil reactance considerably greater than the circuit resistance at the fundamental frequency, and consider the case of an input transformer in which the secondary is practically open-circuited.

Case 4—**Input Transformer.** In accordance with the above assumption the fundamental current through the coil is determined by the coil reactance x or

$$I_1 = \frac{E}{x} \doteq \frac{d}{n^2 \mu A}.$$

Putting this in (45), we find

$$E_3 \doteq \frac{1}{nA} \frac{a_{02}}{\mu^2}, \tag{50}$$

which is identical with Equation (49).

In the four cases treated above it has appeared that there are three quantities which characterize the modulating properties of a ferromagnetic coil under different conditions—a_{02}/μ, $a_{02}/\mu^{3/2}$, a_{02}/μ^2. The values of these ratios have been determined for a number of materials and are tabulated below.

	a_{02}/μ	$a_{02}/\mu^{3/2}$	a_{02}/μ^2
B dust	18×10^{-3}	3.1×10^{-3}	0.52×10^{-3}
C dust	11	2.2	0.42
Permalloy dust [15]	17	2.1	0.24
Perminvar [15]	2.1	0.1	0.0045
Silicon Steel	2500	210	16

These results apply only when eddy currents are negligible; as the frequency is raised eddy currents effectively screen the core and the harmonic flux is reduced to a greater extent than is the fundamental. This effect, as is well known, depends upon the specific resistivity of the core material and will not be evaluated here.

It has been pointed out that under different circuit conditions an index of the modulation is given by the ratios a_{02}/μ, $a_{02}/\mu^{3/2}$, a_{02}/μ^2, depending upon specific conditions. For diluted materials, these ratios referred to the undiluted material become

$$\frac{a_{02}}{a_{10}\left(1 + \frac{\lambda}{l} a_{10}\right)^2}, \quad \frac{a_{02}}{\left[a_{10}\left(1 + \frac{\lambda}{l} a_{10}\right)\right]^{3/2}}, \quad \frac{a_{02}}{a_{10}^2\left(1 + \frac{\lambda}{l} a_{10}\right)},$$

and the utility of air-gaps toward the reduction of harmonic is made evident quantitatively in every case.

In order to test these relations a comparison was made of the

[15] Laboratory specimens.

coefficients for specimens of grade "B" and grade "C" dust which represent different dilutions of electrolytic iron. The coefficients used were those tabulated in Part 1. In accordance with equations (42) above we may write

$$
\begin{aligned}
a_{10}\Big/\left(1+\frac{\lambda_1}{l}a_{10}\right) &= 26.2, & a_{10}\Big/\left(1+\frac{\lambda_2}{l}a_{10}\right) &= 35.3,\\
a_{11}\Big/\left(1+\frac{\lambda_1}{l}a_{10}\right)^3 &= 0.59, & a_{11}\Big/\left(1+\frac{\lambda_2}{l}a_{10}\right)^3 &= 1.53, \quad (51)\\
a_{02}\Big/\left(1+\frac{\lambda_1}{l}a_{10}\right)^3 &= 0.29, & a_{02}\Big/\left(1+\frac{\lambda_2}{l}a_{10}\right)^3 &= 0.65.
\end{aligned}
$$

From these, we should have equality of the ratios

$$0.59/1.53, \qquad 0.29/0.65, \qquad (26.2/35.3)^3,$$

or

$$0.37, \qquad 0.45, \qquad 0.41,$$

which are evidently in fair agreement. It is clear then that the properties of diluted materials may be calculated from the characteristics of the original material, at least when the process of dilution does not change the intrinsic properties of the magnetic material involved.

Applications to Transformers. The third harmonic flux component may be obtained when the fundamental magnetizing force is given, and the latter is easily obtained in toroidal core inductance coils from the relation

$$h = 0.4ni/d, \tag{52}$$

where both h and i refer to the fundamental frequency. In a transformer, however, the net magnetizing force is obtained as the sum of two components, one due to the primary and the other produced by the secondary, so that some further investigation is required before the net magnetizing force in the core may be calculated in terms of the transformer constants and the primary current.

Net Magnetizing Force. To obtain the net magnetizing force which determines uniquely the generated flux components we are required to obtain the primary and secondary currents $(i_1 i_2)$, to multiply each current by the number of times it encircles the core, to add the two products, and finally to multiply the sum by $0.4/d$:

$$H_{\text{net}} = 0.4(n_1 i_1 + n_2 i_2)/d. \tag{53}$$

To evaluate (53) then we shall solve for the primary and secondary fundamental currents in terms of the circuit constants and the applied potential.

The transformer circuit equations from which the currents may be evaluated are the familiar ones

$$\begin{aligned} E &= Z_1 i_1 + jpMi_2, \\ O &= Z_2 i_2 + jpMi_1, \end{aligned} \tag{54}$$

in which we assume the transformer to be an idealized structure—the capacity and leakage effects of actual transformers will be assumed combined with the connected impedances for simplicity—they will therefore not be dealt with explicitly. From the second of (54) is obtained the relation between the two currents

$$i_2 = -jpMi_1/Z_2 \tag{55}$$

which may be substituted in the first of (54) to give the usual expression for the primary current

$$i_1 = E \Big/ \left[R_1 + \frac{p^2M^2}{Z_2^2} R_2 + j\left(X_1 - \frac{p^2M^2}{Z_2^2} X_2 \right) \right]. \tag{56}$$

An expression for the net magnetizing force in terms of i_1 may be had putting (55) and (56) in (53)

$$H_{\text{net}} = \frac{0.4n_1 i_1}{d}\left(1 - \frac{jpMK}{Z_2}\right), \tag{57}$$

in which K represents the turns ratio:

$$K = \frac{n_2}{n_1} = \frac{\sqrt{X_2}}{\sqrt{X_1}}. \tag{58}$$

If we make the convenient assumption, which is closely approximated under working conditions, that the coupling is perfect (no leakage) we have

$$pM = \sqrt{X_1 X_2} \tag{59}$$

and (57) may be simplified to the form

$$H_{\text{net}} = \frac{0.4n_1 i_1}{d} \frac{R_2}{Z_2}. \tag{60}$$

This equation shows that the net magnetizing force may be obtained from that of the primary alone by applying the reduction factor R_2/Z_2. Now in well designed transformers the winding reactances are much greater than the resistances to which they are connected, and in this case

$$X^2_{1,\,2} \gg R^2_{1,\,2}. \tag{61}$$

Applying this further simplification to (60) we have finally for the net magnetizing force

$$H_{\text{net}} = \frac{0.4n_1i_1}{d}\frac{R_2}{X_2}, \tag{60a}$$

in which the reduction factor applied to the force due to the primary alone is seen to be the ratio of resistance to reactance in the secondary circuit.

This equation with the aid of (56) may now be put in terms of the primary voltage—a fixed quantity—rather than in terms of the primary current which is variable according to the circuit resistances used. Applying the assumptions (59) and (61), Equation (56) may be written

$$i_1 = E/(R_1 + R_2/K^2), \tag{56a}$$

which may be put in (60a) to express the net force explicitly in terms of the circuit constants:

$$H_{\text{net}} = \frac{0.4n_1E}{dX_1}\left(\frac{1}{1 + K^2R_1/R_2}\right) \tag{62}$$

In view of (58). When the transformer is terminated in its normal resistances the expression within parentheses reduces to the value one-half. It is of interest in this connection to compare the field here with that produced with the secondary open; in the latter case it is clearly

$$0.4n_1E/dX_1,$$

which is twice as great as the field in the properly terminated transformer. This result is otherwise evident, for in the properly terminated transformer but half the generator voltage is applied across the primary.

Third Harmonic. According to Equation (28) the amplitude of third harmonic voltage generated in a coil of n turns encircling a core subjected to a magnetizing force of H_{net} gilberts/cm is

$$E_3 = \frac{8}{5\pi}10^{-8}a_{02}pnAH^2_{\text{net}} \tag{63}$$

and the generated primary or secondary voltage is obtained by attaching appropriate subscripts to n. Because of the coupling, the two circuits react on one another at the third harmonic frequency as they did at the fundamental frequency, and to find the two third harmonic currents we must go through a procedure analogous to that followed for the fundamentals except that here we have an independent generator voltage effective in each of the two windings.

Indicating quantities referring to the third harmonic frequency by the subscripts p, s for primary and secondary circuits, respectively, we have the circuit equations for the third harmonic currents:

$$\begin{aligned} e_p &= Z_p i_p + j3pMi_s, \\ Ke_p &= Z_s i_s + j3pMi_p, \end{aligned} \tag{64}$$

and by equating the two expressions for e_p we obtain a relation between the two currents:

$$i_s = \frac{KZ_p - j3pM}{Z_s - j3pMK} i_p. \tag{65}$$

Putting i_p in terms of e_p, and using (59) and (61) we find

$$i_s = e_p K R_p / R_s X_p (1 + K^2 R_p / R_s). \tag{66}$$

Suppose now that we are dealing with pure resistance loads so that

$$R_1 = R_p, \qquad R_2 = R_s.$$

We may then substitute (63) for e_p in (66) to obtain the third harmonic current in the secondary.

From (63)

$$e_p = \frac{8}{5\pi} 10^{-8} a_{02} p n_1 A \frac{0.16 n_1^2 E^2}{d^2 X_1^2} \left(\frac{1}{1 + K^2 R_1 / R_2} \right)^2, \tag{67}$$

and by substitution in (66)

$$i_s = \frac{E^2 K R_1 \alpha}{3 X_1 R_2 \left(1 + K^2 \dfrac{R_1}{R_2} \right)^3}. \tag{68}$$

If we consider the factor containing the resistances,

$$F = \frac{R_1}{R_2} \frac{1}{(1 + K^2 R_1 / R_2)^3}, \tag{69}$$

it is zero for R_1/R_2 zero or infinite, and reaches a maximum under the condition

$$\frac{\partial F}{\partial (R_1/R_2)} = 0. \tag{70}$$

The third harmonic secondary current is maximum when

$$K^2 = \frac{R_2}{2R_1}. \tag{71}$$

Now K^2 is fixed by the turns ratio (58), so that we may say the secondary third harmonic current is maximum when the primary resistance is made half its nominal value, or when the secondary resistance is made twice its nominal value. This is well borne out by Fig. 6 due to A. G. Landeen taken under conditions to which the above discussion applies.

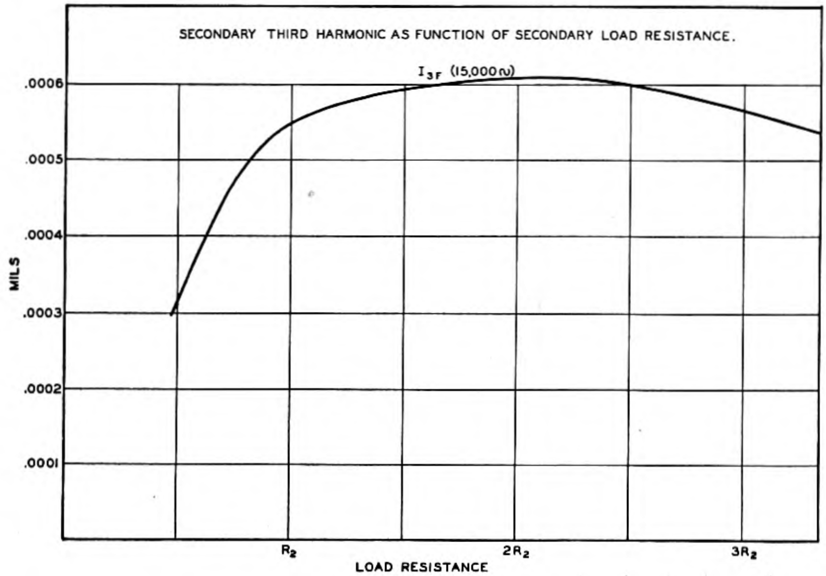

Fig. 6

At first sight the result obtained, in which the secondary harmonic current increases as the secondary resistance is increased for all values of secondary resistance below twice the nominal secondary resistance, seems a bit strange since the fundamental secondary current decreased under the same conditions. A little thought however shows that the increase of secondary resistance acts in two ways; first to reduce the harmonic current directly, and second to increase the net magnetizing force in the core which in turn increases the generated harmonic. Thus with zero secondary resistance the net magnetizing force is zero and no harmonic flux can be produced, while with large secondary loads the flux density in the core or the net magnetizing force reaches an asymptotic maximum so that the generated third harmonic increases slowly if at all while the reduction in secondary current by the resistance is continuously increasing. Under the assumptions noted Equation (68) is valid for any circuit condition and it will be observed that i_s decreases with the primary resistance below the optimum point. The value of α is explicitly

$$\alpha = 2.72 \times 10^{-10} \frac{a_{02}}{p^2 L_1^3} \left(\frac{n_1^3 A}{d^2} \right), \tag{72}$$

in which the bracketed expression coincides with the variable factor in equation (45). Hence when L_1 is constant the choice of core material follows the rules laid down for inductance coils. When the inductance is not constant, however, the factor becomes proportional to

$$\alpha = \text{const.} \frac{da_{02}}{n_1^3 \mu^3 A^2} \tag{73}$$

with the understanding that the turns ratio, resistances and frequency are fixed. Thus the secondary harmonic current with a given material is reduced by increasing the turns and core area and reducing the diameter. As far as core materials go, we have for the significant ratio a_{02}/μ^3 the values:

Material	a_{02}/μ^3
Silicon Steel	130×10^{-4}
Permalloy Dust [16]	2.4×10^{-4}
B Dust	5.3×10^{-4}
C Dust	4.3×10^{-4}
Perminvar [16]	0.05×10^{-4}

The great superiority of perminvar indicated above is restricted of course to fields of the order of less than 0.1 gilbert/cm. above which it tends to become smaller; at a field of 0.7 for example the above factor for perminvar would be multiplied by the factor three. Permalloy dust is observed to be approximately twice as good as the iron dust cores.

A very important question to answer with regard to transformer cores is this—what benefit can be gained as to harmonic production by inserting air-gaps, or by diluting the core material, leaving everything else unchanged. This is evidently to be answered by an investigation of the ratios a_{02}/μ^3 and a'_{02}/μ'^3, the primes referring as before to the diluted material. These relations are given by Equation (42) in which μ and a_{10} are interchangeable:

$$a_{10}' = a_{10} \Big/ \left(1 + \frac{\lambda}{l}\, a_{10}\right) = \mu',$$

$$a_{02}' = a_{02} \Big/ \left(1 + \frac{\lambda}{l}\, a_{10}\right)^3.$$

These permit us to evaluate a'_{02}/μ'^3:

$$\frac{a_{02}'}{\mu'^3} = \frac{a_{02}}{\left(1 + \frac{\lambda}{l}\, a_{10}\right)^3} \, \frac{\left(1 + \frac{\lambda}{l}\, a_{10}\right)^3}{a_{10}^3} = \frac{a_{02}}{\mu^3}, \tag{74}$$

[16] Laboratory specimens.

which clearly indicates that no change is produced in the secondary third harmonic current by introducing air-gaps or by diluting the core material, the change in core area being negligible. This relation has been verified experimentally.

The discussion above considers a constant potential applied to a transformer circuit and the relations derived are somewhat changed when we consider constant current or constant potential transformers. We seldom have to do with constant current transformers but constant potential transformers are met with frequently in vacuum tube circuits as input transformers or interstage transformers. The above discussion may be applied to this case by taking the primary generator resistance much lower than its nominal value—a condition, incidentally, which works toward the suppression of harmonic.

The conclusions are also somewhat altered when we have networks offering different impedances to the harmonic and the fundamental. Thus it is evident that the third harmonic current can be eliminated from both primary and secondary circuits by inserting a high series impedance to the harmonic frequency, or by encircling the core with a winding connected to a network which has a very low impedance to the third harmonic and high impedance to the fundamental. Further, in single frequency transmission the result may be equally well obtained by shunting a series tuned circuit around the primary or secondary winding.

Applications to Harmonic Production in Loaded Lines

If distortion takes place at any one point of a loaded line, the amount of distorted current received at the far end depends upon the amplitudes of the currents producing it, and upon the line attenuation from the point of origin to the far end. The phase similarly depends upon the phase of the fundamentals, and upon the phase shift of the line. When we have a number of distorting sources at different points along the line, the net distorted output is obtained by combining vectorially the currents due to the individual centers of distortion, since it may be assumed that no interaction exists. In a uniformly loaded line we may think of these sources of distortion as being uniformly distributed, but the amount of distortion introduced is, on the contrary, not distributed uniformly on account of the line attenuation which reduces the distortion generated at the far end of the line.

It is apparent, then, that if we are to calculate the net distortion introduced by the line, a complete specification of the phase shift and attenuation is required, together with a knowledge of the law of production of the distortion. This last also requires a specification of amplitude and phase angle in their relation to the fundamentals producing the distortion.

A situation somewhat analogous to the one under discussion which is less involved, however, is found in the diffraction of light, in which the illumination at a point proceeds from a number of coherent sources at various distances from the point. An important difference in the two cases is the attenuation of the transmitting medium which is ordinarily small in the optical case, so that the results of the optical investigations cannot be taken over directly.

In the following we propose to calculate the third harmonic output currents with the aid of our previously established relations. The law of harmonic production has been quite definitely established in the low frequency-low flux density range. The driving third harmonic voltage of frequency *3f* produced by a fundamental current of *rms* value $\bar{I}$ is given by (25) and (26*a*) as

$$E_3 = 0.72 \times 10^{-8} a_{02} \frac{n^3 A}{d^2} f \bar{I}^2 = M\bar{I}^2, \tag{75}$$

while the phase angle of the third harmonic generated potential is related to that of the fundamental current by the expression

$$\theta_3 = 3\theta_1. \tag{76}$$

The above data are sufficient for a solution of the problem of distortion in continuously loaded lines when the propagation constant is known as a function of frequency.

The fundamental current at any point distant x from the sending end of the continuously loaded, properly terminated line is

$$I = I_0 \epsilon^{-Px}, \tag{77}$$

where

$$P = \alpha + j\beta.$$

The third harmonic driving e.m.f. dE_3 generated in a length dx of the line may be written with the aid of (75), (76), and (77) as

$$dE_3 = MI_0^2 \epsilon^{-(2\alpha + j3\beta)x}. \tag{78}$$

Writing the line impedance as Z_0, the resulting current at x is

$$dI_3 = \frac{MI_0^2}{2Z_0} \epsilon^{-(2\alpha + j3\beta)x}. \tag{79}$$

With the line parameters a function of frequency we may write for the propagation of third harmonic current

$$i_3 = I_3 \epsilon^{-(\gamma + j\delta)x}. \tag{80}$$

The distortion at x produces a third harmonic current at the receiving end

$$di_3 = \frac{MI_0^2}{2Z_0}\epsilon^{-(2\alpha+j3\beta)x}\epsilon^{-(\gamma+j\delta)(l-x)}. \qquad (81)$$

The total third harmonic current at the receiving end may now be obtained by integrating (81) over the length of the line l, which gives us

$$i_3 = \frac{MI_0^2}{2Z_0}\frac{\epsilon^{-(\gamma+j\delta)l}}{[(2\alpha-\gamma)+j(3\beta-\delta)]}(1-\epsilon^{-[2\alpha-\gamma+j(3\beta-\delta)]l}). \qquad (82)$$

Inasmuch as we are concerned with the output amplitude, the above expression may be put in a somewhat more convenient form:

$$|i_3|^2 = \left(\frac{MI_0^2}{2Z_0}\right)^2 \frac{\epsilon^{-2\gamma l}}{(2\alpha-\gamma)^2+(3\beta-\delta)^2}$$
$$\times [1+\epsilon^{-2(2\alpha-\gamma)l} - 2\epsilon^{-(2\alpha-\gamma)l}\cos(3\beta-\delta)l]. \qquad (83)$$

Thus when l is zero the harmonic vanishes as it should, and as l increases the current passes through maxima and minima determined by the cosine term. If the attenuation is not very great, the maxima occur approximately at the line lengths

$$l = \frac{(2n-1)\pi}{3\beta-\delta},$$

where n is a positive integer. These distances correspond to odd half wave-lengths, as is true of the optical case. As l is increased the bracketed expression approaches unity and the current falls exponentially. Before this point is reached, however, the current increases in certain regions as the line length is increased. The fact that in an actual line the parameters vary with frequency means that these maxima and minima will vary in position according to the frequency, so that a maximum for one frequency may well coincide with a minimum for another frequency.

In the case of lumped loading, the integrations used for continuous loading are replaced by summations, as was done by Mason in the unpublished investigation previously cited.[4] If the spacing between coils is x_1, and if we have n coils for which $nx_1 = y$, the received third harmonic current at the end of a properly terminated line may be written

$$i_3 = \frac{MI_0^2}{2Z_0}\frac{\epsilon^{-(\gamma+j\delta)l}\{1-\epsilon^{-[2\alpha-\gamma+j(3\beta-\delta)]y}\}}{(\epsilon^{[2\alpha-\gamma+j(3\beta-\delta)]x_1}-1)}, \qquad (84)$$

which is to be compared with (82) for the continuously loaded line.

Appendix 1: Simplification of Loop Equations

From the first of the three properties mentioned we have $a_{00} = 0$, and from the second property

$$B_1(h, H) = - B_2(- h, H) \tag{3}$$

whence, if we write

$$\begin{aligned} B_1(h, H) = {} & a_{10}h + a_{01}H \\ & + a_{20}h^2 + a_{11}hH + a_{02}H^2 \\ & + a_{30}h^3 + a_{21}h^2H + a_{12}hH^2 + a_{03}H_3 \cdots, \end{aligned} \tag{4}$$

then

$$\begin{aligned} B_2(h, H) = {} & a_{10}h - a_{01}H \\ & - a_{20}h^2 + a_{11}hH - a_{02}H^2 \\ & + a_{30}h^3 - a_{21}h^2H + a_{12}hH^2 - a_{03}H^3 \cdots. \end{aligned} \tag{5}$$

From the third property, the two branches meet at the loop tip which lies on the magnetization curve for which $h = H$, or

$$B_1(H, H) = B_2(H, H). \tag{6}$$

From the two equations (4) and (5) we have by virtue of this relation

$$a_{01}H + (a_{20} + a_{02})H^2 + (a_{21} + a_{03})H^3 = 0$$

and since this relation holds for every value of H, the coefficients of each power of H must be zero so that

$$\begin{aligned} a_{01} &= 0 = a_{00}, \\ a_{02} &= - a_{20}, \\ a_{03} &= - a_{21}. \end{aligned} \tag{7}$$

The final expressions for the branches are then evidently obtained by putting (7) in (4) and (5) which become

$$\begin{aligned} B_1(h, H) = {} & a_{10}h \\ & - a_{02}h^2 + a_{11}hH + a_{02}H^2 \\ & + a_{30}h^3 - a_{03}h^2H + a_{12}hH^2 + a_{03}H^3, \end{aligned} \tag{4a}$$

$$\begin{aligned} B_2(h, H) = {} & a_{10}h \\ & + a_{02}h^2 + a_{11}hH - a_{02}H^2 \\ & + a_{30}h^3 + a_{03}h^2H + a_{12}hH^2 - a_{03}H^3. \end{aligned} \tag{5a}$$

Appendix 2: Rayleigh's Relation

If we suppose the lower branch of any loop, when referred to the tip of the largest loop considered, to be given by the equation

$$B_1 = \mu_0 h_1 + v h_1^2 + \lambda h_1^3 + \omega h_1^4 \tag{12}$$

we may refer the family of branches to the origin by the transformation

$$\begin{aligned} h &= h_1 - H, \\ B &= B_1 - B_m, \end{aligned} \tag{13}$$

if $B_m H$ refer the midpoint of the largest loop to the tip. Then

$$B_m = \mu_0 H + 2vH^2 + 4\lambda H^3 + 8\omega H^4. \tag{14}$$

Putting (13) in (12)

$$B + B_m = \mu_0(h + H) + v(h + H)^2 + \lambda(h + H)^3 + \omega(h + H)^4,$$

whence, subtracting (14),

$$\begin{aligned} B' = \mu_0 h + v(h^2 + 2hH - H^2) + \lambda(h^3 + 3h^2H + 3hH^2 - 3H^3) \\ + \omega(h^4 + 4h^3H + 6h^2H^2 + 4hH^3 - 7H^4), \end{aligned} \tag{15}$$

which represents the hysteresis branch equation referred to the origin, on the basis of loop similarity.

The coefficients obtained by the two methods may now be compared. Thus identifying coefficients of (15) with those of the general equation (1)

$$\begin{array}{llll} a_{10} = \mu_0, & a_{11} = 2v, & a_{02} = -v & a_{12} = 3\lambda, \\ a_{20} = v, & a_{21} = 3\lambda, & a_{03} = 3\lambda, & a_{13} = 4\omega, \\ a_{30} = \lambda, & a_{31} = 4\omega, & a_{04} = -7\omega, & a_{22} = 6\omega. \\ a_{40} = \omega, & & & \end{array} \tag{16}$$

Appendix 3: Alternating Magnetization, Sinusoidal Magnetizing Force

The resulting expression is simplified if we make the following substitutions

$$\begin{aligned} \alpha &= a_{02}H^2 + a_{03}H^3 = B(O, H), \\ \beta &= a_{10} + a_{11}H + a_{12}H^2, \\ \delta &= a_{30}H^3. \end{aligned} \tag{19}$$

It may be noted that α is the remanence and that β is an approximation to the permeability, in fact the permeability is given as the sum of β and δ. With (18) and (19) inserted in the branch equations, then, we have

$$\begin{aligned} B_1(H \cos pt, H) &= \alpha + \beta \cos pt - \alpha \cos^2 pt + \delta \cos^3 pt, \\ B_2(H \cos pt, H) &= -\alpha + \beta \cos pt + \alpha \cos^2 pt + \delta \cos^3 pt. \end{aligned}$$

For convenience we shall express these relations in terms of multiple angles, and we have for the equation of the upper loop family

$$B_1(H \cos pt, H) = -\alpha/2 + (\beta + 3\delta/4) \cos pt - \alpha/2 \cos 2pt + \delta/4 \cos 3pt.$$

If we write

$$\begin{aligned} A &= \alpha/2 = (a_{02}H^2 + a_{03}H^3)/2, \\ B &= \beta + 3\delta/4 = a_{10} + a_{11}H + a_{12}H^2 + 3a_{30}H^3/4, \\ C &= \ = -A, \\ D &= \delta/4 = a_{30}H^3/4, \end{aligned} \tag{20}$$

the final form for the loop equations is

$$\begin{aligned} B_1(H\cos pt, H) &= \quad A + \mathrm{B}^{17}\cos pt + C\cos 2pt + D\cos 3pt, \\ B_2(H\cos pt, H) &= -A + B\ \cos pt - C\cos 2pt + D\cos 3pt. \end{aligned} \tag{21}$$

We are now in position to combine the two equations of (21) in a Fourier series valid over the entire cycle as

$$B = \frac{b_0}{2} + \sum(b_k \cos kpt + a_k \sin kpt), \tag{22}$$

where

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_0^{2\pi} f(pt)\sin kpt\, d(pt), \\ b_k &= \frac{1}{\pi}\int_0^{2\pi} f(pt)\cos kpt\, d(pt). \end{aligned} \tag{22a}$$

For our particular case we have, since $B = B_1$ for the first half of the cycle, and $B = B_2$ for the second:

$$\begin{aligned} \pi a_k &= \int_{-\pi}^{0} B_2(h, H)\sin kpt\, d(pt) + \int_0^{\pi} B_1(h, H)\sin kpt\, d(pt), \\ \pi b_k &= \int_{-\pi}^{0} B_2(h, H)\cos kpt\, d(pt) + \int_0^{\pi} B_1(h, H)\cos kpt\, d(pt). \end{aligned}$$

These integrals may be simplified considerably when we take advantage of the fact that both B_1 and B_2 are even functions of the time as given by (21). Thus

$$\begin{aligned} \pi a_k &= \int_0^{\pi} [B_1(h, H) - B_2(h, H)]\sin kpt\, d(pt), \\ \pi b_k &= \int_0^{\pi} [B_1(h, H) + B_2(h, H)]\cos kpt\, d(pt). \end{aligned}$$

Referring to (21) we may then write

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^{\pi} (A + C\cos 2pt)\sin kpt\, d(pt), \\ b_k &= \frac{2}{\pi}\int_0^{\pi} (B\cos pt + D\cos 3pt)\cos kpt\, d(pt). \end{aligned} \tag{23}$$

[17] This coefficient is not to be confounded with the general expression for flux density.

Upon integration of (23) the coefficients for the fundamental and third harmonic flux components are found to be as follows:

$$
\begin{aligned}
a_1 &= \frac{4}{\pi}(A - C/3), \qquad & b_1 &= B, \\
a_3 &= \frac{4}{\pi}\left(\frac{A}{3} + \frac{3C}{5}\right), \qquad & b_3 &= D,
\end{aligned}
\tag{24}
$$

which, by reference to (20), may be put in terms of the branch coefficients.

Appendix 4—Impedance Reaction to a Small Third Harmonic in the Presence of a Large Fundamental

We have for the two hysteresis branch equations from Eqs. (4*a*), (5*a*)

$$
\begin{aligned}
B_1(h, H) &= \quad B(O, H) + \beta h + \gamma h^2 + a_{30}h^3 + \cdots, \\
B_2(h, H) &= - B(O, H) + \beta h - \gamma h^2 + a_{30}h^3 + \cdots,
\end{aligned}
\tag{30}
$$

in which

$$
\begin{aligned}
\beta &= a_{10} + a_{11}H + a_{12}H^2, \\
\gamma &= - (a_{02} + a_{03}H).
\end{aligned}
\tag{31}
$$

Putting (29) in (30) we get

$$
\begin{aligned}
B(h, H) = A &+ B \cos pt + C \cos 2pt + D \cos 3pt + F \cos npt \\
&+ G[\cos (n+1)pt + \cos (n-1)pt] \\
&\qquad + J[\cos (n+2)pt + \cos (n-2)pt]
\end{aligned}
\tag{32}
$$

in which the coefficients have the following significance

$$
\begin{aligned}
A &= B(O, H) + H_1^2\gamma/2, \qquad & F &= \beta H_3 + 3a_{30}H_1^2H_3/2, \\
B &= \beta H_1 + 3a_{30}H_1^3/4, \qquad & G &= \gamma H_1 H_3, \\
C &= \gamma H_1^2/2, \qquad & J &= 3a_{30}H_1^2H_3/4. \\
D &= a_{30}H_1^3/4,
\end{aligned}
\tag{33}
$$

The coefficients of the Fourier Series for the output wave may now be obtained as before by combining the two equations (30) since each one is operative during one-half the cycle. There results an expression similar to the one obtained in the single frequency case, and since we have

$$
I = b_0/2 + \Sigma a_k \sin kpt + \Sigma b_k \cos kpt
$$

the coefficients are evaluated from the expressions

$$
\begin{aligned}
a_k &= \frac{2}{\pi}\int_0^{2\pi} (A + C\cos 2pt + G(\cos 4pt + \cos 2pt)) \sin kpt\, d(pt), \\
b_k &= \frac{2}{\pi}\int_0^{2\pi} (B\cos pt + (D + F)\cos 3pt \\
&\qquad + J(\cos 5pt + \cos pt)) \cos kpt\, d(pt).
\end{aligned} \tag{34}
$$

Upon integration we find

$$b_1 = B, \qquad b_3 = F. \tag{35}$$

Comparing these two coefficients we see that at low amplitudes we may write

$$b_1 = \mu H_1, \qquad b_3 = \mu H_3,$$

in which the permeability is the same to the two components, and is determined by the fundamental amplitude. For the dissipative terms we find

$$
\begin{aligned}
a_1 &= \frac{4}{\pi}(A - C/3 - 2G/5), \\
a_3 &= \frac{4}{\pi}(A/3 - 3C/5 + 6G/35),
\end{aligned} \tag{35'}
$$

but some care is required in interpreting these expressions. Inasmuch as we are primarily interested here in determining the dissipative component to a third harmonic magnetizing force of amplitude H_3, we are required to select from a_3 only those terms containing H_3, which means the single term $24G/35\pi$. The other terms take care of the harmonic producing properties of the core and do not affect the impedance to the third harmonic. The impedance term for the third harmonic comes down to

$$\frac{24}{35\pi}(a_{02}H_1H_3 + a_{03}H_1^2H_3),$$

which may be written as

$$\frac{24}{35\pi}H_3\frac{B(O, H)}{H_1}.$$

This may be compared with the corresponding term for the fundamental given by (25).

Appendix 5. Effect of Air-Gap by Vacuum Tube Analogy

In the elementary treatment of non-linear two element vacuum tube circuits, approximate solutions are obtained in the form

$$J = J_1 + J_2 + \cdots J_n,$$

where J represents the variable part of the space current on the basis that J_n represents the nth approximation, but that the series converges rapidly so that we need consider only the first two terms to arrive at a substantially accurate result. The expressions derived for the first and second approximations are known to be

$$J_1 = a_1E/(1 + a_1R),$$
$$J_2 = a_2E^2/(1 + a_1R)^3,$$

where the 'a' coefficients describe the tube characteristic

$$J = a_1v + a_2v^2,$$

R being the external plate circuit resistance, v the tube potential, and E the circuit e.m.f.

Turning now to the equations for the hysteresis loop branches, we have from (4a)

$$B = a_{10}h - a_{02}h^2 + a_{11}hH + a_{02}H^2.$$

Hence by the analogy between flux and current, and between reluctance and resistance, the first order terms are reduced by the factor $1/(1 + a_1R)$ which corresponds to $1/(1 + \lambda a_{10}/A)$, and the second order terms are reduced by the cube of this factor, which yields the same results (42) as the laborious direct method.

Airways Communication Service [1]

By EDWARD B. CRAFT

THE present development of air transport is bringing out its need for adequate communication in much the same manner as the earlier development of railway operations disclosed for that industry the necessity of special communication services if speed and density of traffic were to be obtained with safety. The electric telegraph by a most fortunate coincidence was available just at the time the railways required it; and as the demand for speed became pressing the telephone was perfected. Today the railways of the country, in general, use the telegraph for administrative messages, where a written record is wanted, and use the telephone for despatching, where speed and accuracy are primary requirements.

By another fortunate coincidence, radio appears to be available just at the time it is needed for communication with aircraft in flight. Radio in the form of either telegraph or telephone has been highly developed for communication between points on the surface of the globe. For communication between aircraft and airports it is available in principle although not yet so well developed. During the war, both in this country and abroad, radio equipment of relatively crude design was installed in aircraft and proved of great utility. Since the war, radio telegraphy for aircraft has been further developed by the naval and military services, but radio telephony has received less attention, probably because of the inherent difficulties and lack of a pressing demand.

Following the remarkable success of the Air Mail and the passage of the Air Commerce Act of 1926, we are now fairly launched into an era of air transport of mails, express and passengers. National Airways, laid out and equipped by the Department of Commerce under authority of the Air Commerce Act, already compare in extent with the main trunk line mileage of the railways. Scheduled flying over these airways goes on by night as well as by day. A commercial degree of reliability and safety has been reached in so far as the airplane and its engine are concerned and, when surprises due to bad weather can be eliminated, the safety of air transport should compare favorably with that of other forms of transportation.

Although weather is beyond our control, meteorological science is able to forecast its major phenomena with a high degree of precision,

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provided data describing present and past weather conditions can be collected from a sufficient number of places. The progress of a weather disturbance can be tracked and the time of its arrival at a given point predicted. By means of a suitable communication system weather reports from observers located along and near an airway can be collected; and it should be possible, therefore, to reduce materially the weather hazard of air transport.

A full-scale meteorological experiment of this nature is now being conducted in California by the Weather Bureau with the cooperation of the Guggenheim Fund for the Promotion of Aeronautics and of the Pacific Telephone and Telegraph Company. Meteorologists at the Oakland and Los Angeles airports receive several times a day, by long distance telephone, weather data from observers at a large number of selected points in the state. After an exchange of these collected data, these meteorologists forecast flying-weather for aviators starting out over the airway between these airports. The experiment will be continued until the value of the special weather service can be estimated.

Since the communications problem of safe air transport presented features which in a number of respects were unique, it was referred by the Interdepartmental Committee on Aeronautical Meteorology to experts of the American Telephone and Telegraph Company and Bell Telephone Laboratories. What was desired was the collection of reports from a considerable number of widely distributed observers in a relatively short interval of time, say, from twenty observers in twenty minutes. Naturally, it is not commercially practicable to call the party desired, set up the connections, have him answer and give his data all in the space of one minute. However, an equivalent result has been obtained by evolving a special telephone procedure for the purpose. At the appointed time a team of long-distance telephone operators call up successively the listed observers. Each as he answers is asked to hold the line and wait his turn when the operator connects him to the airport meteorologist.

It has been found by trial that the weather data can be reported and recorded in thirty seconds. Consequently, the list of observers can be readily gone through if one minute each is allowed. To the Los Angeles and Oakland airports about forty observers are now reporting weather five times a day. These collected reports are exchanged between airports; and airplanes starting over the airway are provided with a forecast of the weather they may expect enroute and upon arrival.

On the basis of these forecasts, it is hoped that the pilots may be able to avoid bad weather by choosing an alternative route or by selecting the terminal field where weather conditions are more propitious. Both Los Angeles and the San Francisco Bay region have several airports and there are two routes between them, one up the valley via Bakersfield, and the other the more direct line to the west. The experiment will be carried on for a full year and so cover the complete cycle of the seasons. On the basis of the demonstrated value of this service to the users of the airway, the matter of its continuance or possible extension to other airways can then be decided by the Weather Bureau. Unfortunately, however, California weather is proverbially good, and the experiment will, therefore, be concerned mainly with local fog and visibility conditions. It is possible also that interests other than aeronautical may discover advantages in a short range forecast of local weather. If so, the value of the experiment will be correspondingly increased.

Weather data are also being collected in the east from observers in New Jersey and Pennsylvania by the meteorologist at Hadley Field who employs a somewhat similar method of sequence operation of the long-distance telephone lines.

In addition to the problem of collecting weather data, there is the closely related matter of distributing local weather reports and forecasts between airports. This is "point-to-point service." It may be accomplished by a special radio-telegraph network, by commercial telegraph or by long-distance telephone, and over private or leased wires either by telephone or by telegraph. Local conditions, volume of traffic and economic considerations, in general, determine which type of service should be provided.

Besides its use for weather messages, point-to-point communication between landing fields along an airway is desirable for following the progress of an airplane with its passengers and cargo. Such a despatching service is somewhat analogous to that of a railway and is a necessity if scheduled connections with trains and other aircraft are to be met. Also, there is the necessity of accountability for mails and express; for example, on departure the landing fields ahead must be informed not only of the fact of starting but of what mail is on board. Upon landing there must be a message announcing the event. In this way the progress of a plane can be followed by the terminal airports.

Although air transport of passengers has not yet reached a large volume in this country, European experience indicates that we soon will be concerned with communication problems having to do with passengers' convenience and comfort. Train and bus connections, hotel accommodations and meals, will have to be arranged for by the traffic department of an air transport company.

Point-to-point communication facilities are also required for the general administrative business of the airway and of the air transport companies.

Along our present airways at short intervals are intermediate landing fields upon which planes may land when forced down by weather or mechanical trouble. Such landings, however, are infrequent and will presumably become increasingly rare; but when a forced landing does take place instant communication with the nearest airport is urgent on account of passengers, mail, and the air transport company itself. Telephones are now provided at these intermediate fields by the Department of Commerce and kept available for such emergency use. The same telephones can be used, of course, for the routine collection of weather data by the airport meteorologist.

On some airways communication between terminal landing fields or airports is now handled by radio telegraph and on others by long-distance telephone. Neither system is ideal for the purpose. Radio telegraph is slow and is often unreliable in times of bad static when weather messages become urgent. It also utilizes radio ether channels which are needed for communication with planes. Moreover, a telegraph operator must be constantly listening throughout the twenty-four hours even though messages come infrequently. Commercial wire telephone service on the other hand although generally fast and reliable provides no written record of the messages, nor does it economically repeat messages at such other and distant airports as may be interested. Weather conditions at Cleveland, for example, are of interest both to New York and to Chicago airports. Likewise the time of departure of the New York air mail from Chicago is of interest to all landing fields enroute.

An ideal system which is instantaneous and reliable, repeats messages at all airports, is free from interference, takes up no radio channels, and furnishes a permanent record of all messages at all airports, is the telephone-typewriter service. Telephone-typewriter systems make possible the instantaneous transmission of communications between distant offices and provide simultaneously each office and any desired intermediate stations with typewritten copies. This service has been used for a good many years by the principal press associations and is now being extended rapidly to serve the needs of our larger business organizations.

To utilize the telephone-typewriter system along an airway requires only the installation of keyboard transmitting apparatus and tape printing apparatus at terminal fields and their interconnection by a private or leased wire circuit. Then anyone familiar with a typewriter may type a message which will appear on the tape fed automatically from the apparatus at every other connected point. The message is automatically and permanently recorded under the control of the sending station. Constant attendance or listening-in is, therefore, not required; and operators at the various receiving points are thus free to attend to telephone calls from intermediate fields, to operate radio beacons and lights, and to carry on whatever duties may be assigned to them.

Telephone-typewriter service has been initiated by the Department of Commerce at Hadley Field, at Cleveland, at Chicago and at San Francisco, where in each place the local radio stations, weather bureau offices and the airport offices are all interconnected. It is planned, at a later date, to equip experimentally some airway with complete telephone-typewriter service between airports.

When an aviator leaves an airport he should be given information of the weather along the route ahead of him and a forecast of the nature of probable changes during the time of his flight. If general weather conditions are settled, or if his flight is a short one, a forecast is entirely adequate. However, for long flights and at times of uncertain and threatening weather, it is important that the pilot be continuously advised by radio of the weather conditions he may encounter during his flight. In particular, reports of the visibility and landing conditions at the airport where he expects to land and storm warnings should be sent him. Weather and landing advice can be broadcast from each airport along the airway. Provision of radio transmitters at airports and receiving sets in the planes will make possible a simple one-way system of communication and will permit any number of planes in the air to be advised without confusion.

The Department of Commerce, in its program of Aids for Air Navigation plans to install radio-telephone transmitters at principal terminal fields to broadcast, to planes in flight, weather and landing information. In addition, there will be a radio-beacon service to assist pilots in finding the landing field.

European practice, however, has not developed a broadcasting service along this line but has evolved a two-way system in which the pilot of the airplane talks with the nearest airport. Such a system has obvious advantages where it is desired by an air transport company to instruct or control rather than merely inform its aviators. The obvious disadvantage lies in the fact that on a single radio channel the airport can converse with only a single airplane at a time. On the London-Paris airway, it is reported, the practice has recently been adopted of communicating on one channel by radio telegraph with the large planes which carry a radio operator and on another channel by radio telephone with the smaller planes.

Two-way communication has the great and obvious merit of permitting a pilot to discuss the weather outlook with an airport meteorologist, to consider alternative landing places in view of such factors as his remaining fuel supply or the direction of wind, and to decide if necessary on a change in landing place and to be assured of arrangements there for the care of his passengers and mail. It seems reasonable, therefore, to predict that operators of air transport fleets will require two-way communication with their planes in flight, although taxi services and private owners without ground organization along the airway may, in general, be content with a public one-way broadcasting service.

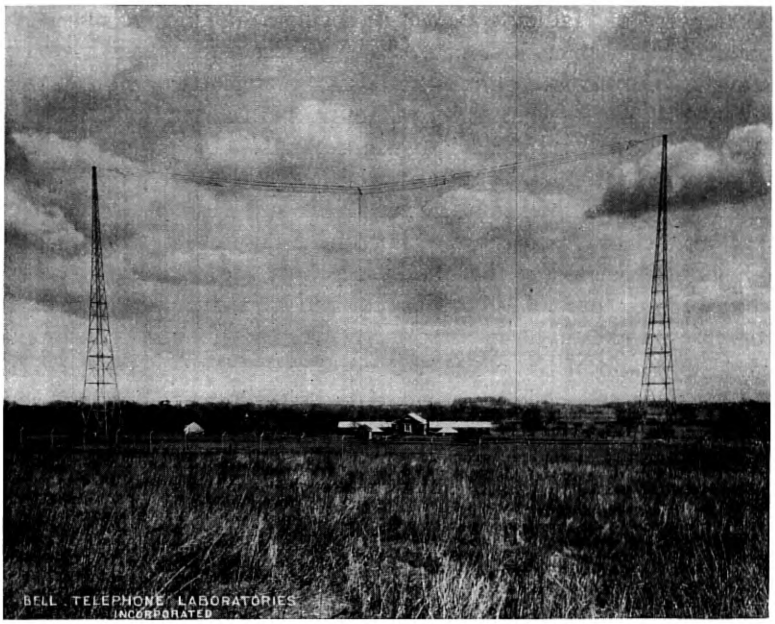

FIG. 1. THE WHIPPANY RADIO LABORATORY.

Whether one-way or two-way communication is desired for plane-to-ground use it appears that radio telephony as distinguished from telegraphy will be essential. Radio telegraphy requires on board the plane the individual attention of a special radio operator for sending and receiving. Although very large multi-engined passenger planes will certainly carry a relief pilot in flight, it is doubtful whether good commercial pilots can be made into good telegraphers and vice versa. For long distance over-sea flights and for expeditionary purposes the radio telegraph has, without doubt, preponderating advantages of longer range with the same transmitter power and of intelligibility through a higher level of interfering signals and acoustic noise on board, aside from its convenience in communication enroute with surface vessels. For regular service on established airways, however, the telephone is undoubtedly superior.

The perfection of facilities for communicating weather and landing information to planes in flight, which will enable them to operate with safety under relatively unfavorable meteorological conditions, will greatly stimulate the demand for improved aids to navigation. It seems to be established that flying under conditions of poor visibility, when landmarks are totally obscured and beacon lights are useless, requires some form of radio goniometry if the pilot is to find his way through.

A number of systems have been proposed for this purpose. The London-Paris Airway is equipped with radio direction-finding equipment on the ground by means of which the position of planes can be determined on request. The disadvantages of this arrangement lie mainly in its relative slowness and its lack of traffic capacity. The radio beacon of the type being developed by the Bureau of Standards, giving an equi-signal zone which can be observed by the plane, is free from these objections. It is, however, subject to the disadvantage that it indicates a straightline course which cannot always coincide with the airway and is of little value if detours are required to avoid storm centers and foggy areas.

Another system, a recent development of the British Royal Air Force, employs a rotating loop transmitter at the ground station and indicates the bearing of the plane with respect to the transmitter by means of a special stop watch. This system is relatively slow but permits the pilot to navigate as he would if one or more beacon lights were visible. All of these various methods of goniometry have special advantages and disadvantages, and occupy more or less of the valuable and restricted ether space. The evolution of the system which is most satisfactory will be a matter of time and will require close cooperation on the part of all factors in the industry.

Bell Telephone Laboratories, at its radio station at Whippany, New Jersey, has erected an experimental two-way radio-telephone system and radio beacon. In connection with this apparatus it utilizes a Fairchild Cabin Monoplane with Pratt and Whitney wasp engine. The plane has been carefully bonded and shielded and is equipped with radio field-measuring apparatus of the Laboratories' design. With this plane exact measurements can be made at various altitudes under different weather conditions of the efficiency of radio transmission from the Whippany transmitter. In addition the plane carries radio transmitting and receiving sets of experimental design.

FIG. 2. THE CABIN MONOPLANE FOR EXPERIMENT IN AIRWAYS COMMUNICATION.

It is, in fact, a flying radio laboratory in which the engineers may experiment under actual flying conditions.

Whether a radio beacon service and a radio telephone service at all the various airports over the country can be made practicable is largely a question of available ether channels. By international agreement, the frequency band 285–315 kcs. (1050–950 meters) is reserved for radio beacons, both marine and air service. For "air mobile service exclusively" there is reserved the band 315–350 kcs. (950–850 meters) in which the 900 meter wave (333 kcs.) is reserved as an air service calling wave and is not to be assigned.

Radio telephony requires a band of frequencies sufficiently wide to include the "side bands" of speech frequency. For distinct transmission of speech, neglecting certain requirements of musical quality, this might require a minimum of 6,000 cycles. In this band reserved for "air mobile service exclusively" there is room, therefore, for but three telephone channels above and three below the calling wave, or a total of six channels. Assuming that a beacon requires a channel width of but 300 cycles, there are altogether for marine and airport use one hundred beacon channels in the band 285–315 kcs.

FIG. 3. CABIN LABORATORIES OF THE MONOPLANE.

The band reserved for beacons is already partly occupied by marine beacons, and near the coast difficulty may arise in finding clear channels for airport beacons. Although it is probable that, by a proper geographical distribution of frequencies, there may be worked out without undue interference an adequate beacon service we can make no assumption that any extra space can be found in the beacon band for radio telephony.

A radio telephone system with a sufficiently powerful transmitter and sufficiently sensitive receiver to give reliable communication for 100 miles will give fair communication for perhaps 200 miles, and its carrier wave will interfere with reception for a much greater distance. To avoid interference due to the beating of carrier frequencies, airports within a few hundred miles of one another may be assigned to different frequency channels, but serious difficulty is at once apparent from a map of the National Airways. Within 800 miles of Chicago, for example, there are over fifty terminal fields or airports. It would seem obviously impractical to assign the available six telephone channels to cover the eastern and central United States without serious interference. By restricting power as much as possible and by other means yet to be devised, it may be found possible to assign the same wave-length to airports relatively nearer together. For the distribution of weather information only, however, the airways may well find insufficient the frequencies in the exclusive band, 315–350 kilocycles.

On certain main routes, air transport companies will eventually require two-way telephone despatching systems of their own to control plane movements. These systems will consist of radio stations situated at the various airports along the route and interconnected by suitable wire lines. The frequency channels required for such services cannot be found in the 315–350 kilocycles band which, as just indicated, is apparently inadequate for the public services of weather broadcasting from airports. Further channels in the shortwave region appear to be necessary.

In the short wave region Bell Telephone Laboratories have initiated an additional development project. In cooperation with the Boeing Air Transport Company, the Laboratories have undertaken to survey the Chicago-San Francisco Airway and to develop a system of two-way telephony between planes in flight and terminal landing fields on this route. The Boeing Company planes and landing fields will be equipped with experimental radio apparatus and a joint full-scale experiment will be conducted during the winter of 1928–29. From this work it is hoped to determine for an air transport company the requirements for a two-way radio telephone service. The investigation will furnish the basis for offering such facilities to other air transport operators.

This development of two-way radio-telephony on short waves is entirely distinct from the government's program of Aids to Air Navigation. That service contemplates one-way radio telephony and direction finding on long waves. The government service is to be available to all flyers who equip themselves to receive it. The two-way system is for private communication and despatching service of air transport companies which wish to control their planes in flight, and to remain in constant communication with their pilots and passengers.

Also, although not yet required, it can safely be predicted that at busy airports there will soon arise a need for radio means to control precedence in the take-off and landing of airplanes. This virtually amounts to traffic control and can be accomplished by low-power two-way radio telephone. Planes wishing to land may announce themselves and remain aloft until directed by the airport manager in the control tower to land at a designated part of the field.

In all these present and future problems, it is the policy of the American Telephone and Telegraph Company and the Bell System to assist by developing ways and means for making available to commercial aviation the best possible communication service.

Abstracts of Bell System Technical Papers Not Appearing in this Journal

Influence of Carbon and Silicon Variations in Grey Cast Iron.[1] D. G. ANDERSON and G. R. BESSMER. In this short article the author gives the results of a series of tests of grey cast irons with different carbon and silicon contents. Three series were run in each of which the silicon content was kept constant and the amount of carbon varied. The results indicated that with two percent silicon the carbon content may be reduced without materially increasing the amount of combined carbon. This results in some improvement in the physical properties of the iron.

Strength-Tests of Telephone Materials.[2] J. R. TOWNSEND. Static tests, such as the ordinary tension or torsion tests, have fallen somewhat into disrepute during the last ten years, the author claims, as the ultimate strengths obtained from them are not always indicative of the forces materials will withstand in actual service. Their place is being taken by repeated-stress tests in which the sample is subjected to conditions more nearly representing those met in ordinary service. In illustration the author mentions several tests of this class being applied in Bell Telephone Laboratories on cable sheath material and springs.

The Reduction of Atmospheric Disturbances.[3] JOHN R. CARSON. In the decade or so during which the problem of eliminating or at least reducing atmospheric disturbances has been given serious and systematic study we have learned, more or less definitely, what we can and cannot do in this direction. For example, we know that there are definite limits to what can be accomplished by frequency selection. We know that directional selectivity is of substantial value, particularly when the predominant interference comes from a direction other than that of the desired signal, and we can calculate pretty well the gain to be expected from a given design.

The object of this note is to analyze another arrangement which provides for high-frequency selection plus low-frequency balancing after detection. The broad idea of balancing out the interference is old, but no general analysis of the arrangement seems to have been made. Furthermoıe the principle of balance has recently acquired

[1] "Fuels and Furnaces," Vol. VI, No. 7, pp. 957 and 972, July, 1928.

[2] "Instruments," Vol. 1, No. 7, pp. 313–315, July, 1928.

[3] *Proceedings of the Institute of Radio Engineers*, July, 1928, Vol. 16, No. 7, pp. 966–975.

fresh interest due to the system disclosed by Armstrong [4] in which high-frequency selectivity and low-frequency balancing are essential features. Armstrong's scheme is treated in more detail in the latter part of this paper.

The conclusions of this study are entirely negative, that is, no appreciable gain is to be expected from balancing arrangements. This is quite in agreement with the conclusion drawn over ten years ago as a result of a rather extended experimental study made in the Bell System. In fact, as more and more schemes are analyzed and tested, and as the essential nature of the problem is more clearly perceived, we are unavoidably forced to the conclusion that static, like the poor, will always be with us.

Thermostat Design for Frequency Standards.[5] W. A. MARRISON. A means for maintaining constant temperature is described in which those temperature variations which are essential for operation of the controlling element are prevented from reaching the controlled chamber by a wall of material especially chosen for the purpose. Such a wall 1 cm. thick, consisting of alternate thin layers of felt and copper, will reduce temperature variations having a period of one minute or less by a factor of 10,000 to 1.

Technical Considerations Involved in the Allocation of Short Waves; Frequencies between 1.5 and 30 Megacycles.[6] LLOYD ESPENSCHIED. This short paper discusses the relation between frequency and distance of transmission for short waves in so far as it affects allocation. A table is given in which the entire short-wave field from 10 to 200 meters is divided into three major bands each containing numerous sub-bands. For each sub-band the number of channels theoretically possible is given and also the number of channels being used at the present time. Factors affecting the separation of channels are also listed.

Effect of Street Railway Mercury Arc Rectifiers on Communication Circuits.[7] CHARLES J. DALY. This paper describes the effects experienced on the telephone circuits from two mercury arc rectifier substations recently installed in Bridgeport, Conn., and shows in table form the relative magnitude of the interfering effects between rotating equipment and mercury arc rectifiers as a means of energizing the street railway system. The method and the type of apparatus used to reduce the effects experienced from the rectifiers are also described.

[4] *Proceedings of the Institute of Radio Engineers*, Jan., 1928, Vol. 16, No. 1, p. 15.
[5] *Proceedings of the I. R. E.*, Vol. 16, No. 7, pp. 976–980, July, 1928.
[6] *Proceedings of the I. R. E.*, Vol. 16, No. 6, pp. 773–777, June, 1928.
[7] *Journal of the A. I. E. E.*, Vol. XLVII, No. 7, pp. 503–506, July, 1928.

Compressed Powdered Permalloy—Manufacture and Magnetic Properties.[8] W. J. SHACKELTON and I. G. BARBER. The paper gives a brief description of the manufacture of magnetic cores of compressed permalloy powder followed by information covering their magnetic properties with particular reference to their use in loading coils. Production of the powder, and its insulation, pressing and annealing, are discussed. Under magnetic properties, permeability, core loss, and modulation are treated. Curves are given illustrating the characteristics of interest in connection with the design and application of loading coils; and comparisons to corresponding characteristics of compressed powdered iron are made throughout.

Thermal Agitation of Electric Charge in Conductors.[9] H. NYQUIST. The electromotive force due to thermal agitation in conductors is calculated by means of principles in thermodynamics and statistical mechanics. The results obtained agree with results obtained experimentally.

Time-Lag in Magnetization.[10] RICHARD M. BOZORTH. An investigation has been made of the time-lag in magnetization in a permalloy wire to determine whether lag can be satisfactorily accounted for as due to eddy-currents alone or whether permalloy shows a marked magnetic viscosity such as has been observed by Ewing in iron wires. Eddy-current lag has been calculated approximately in a manner which takes into account the changing slope of the magnetization curve. A comparison of the calculated and observed magnetization-*vs.*-time curves indicates that the effect is well accounted for as eddy-current lag alone. The eddy-current lag has also been calculated for an iron ring, for which the time-lag has been reported recently in a number of papers by Lapp. The time-lag which he observed is satisfactorily accounted for as eddy-current lag instead of as magnetic viscosity as he had supposed.

Thermal Agitation of Electricity in Conductors.[11] J. B. JOHNSON. Statistical fluctuation of electric charge exists in all conductors, producing random variation of potential between the ends of the conductor. The effect of these fluctuations has been measured by a vacuum tube amplifier and thermocouple, and can be expressed by the formula $\bar{I}^2 = (2kT/\pi)\int_0^\infty R(\omega)\,|\,Y(\omega)\,|^2 d\omega$. I is the observed current in the thermocouple, k is Boltzmann's gas constant, T is the absolute

[8] *Journal of the A. I. E. E.*, Vol. XLII, No. 6, pp. 437–440, June, 1928.

[9] *Physical Review*, Vol. 32, No. 1, pp. 110–113, July, 1928.

[10] *Physical Review*, Vol. 32, No. 1, pp. 124–132, July, 1928.

[11] *Physical Review*, Vol. 32, No. 1, pp. 97–109, July, 1928.

temperature of the conductor, $R(\omega)$ is the real component of impedance of the conductor, $Y(\omega)$ is the transfer impedance of the amplifier, and $\omega/2\pi = f$ represents frequency. The value of Boltzmann's constant obtained from the measurements lies near the accepted value of this constant. The technical aspects of the disturbance are discussed. In an amplifier having a range of 5,000 cycles and the input resistance R, the power equivalent of the effect is $\overline{V}^2/R = 0.8 \times 10^{-16}$ watt, with corresponding power for other ranges of frequency. The least contribution of tube noise is equivalent to that of a resistance $R_e = 1.5 \times 10^5 i_p/\mu$, where i_p is the space current in milliamperes and μ is the effective amplification of the tube.

The Voltage-Current Relation in Central Cathode Photoelectric Cells.[12] THORNTON C. FRY and HERBERT E. IVES. This paper presents a theoretical basis for the interpretation of the experimental results described in the paper which follows. It considers a source of photoelectrons located on the inner of two concentric spheres; derives the trajectory of an electron shot off at any angle with any speed; and then makes use of this information to compute the current which would be received by a small collector located anywhere on the outer sphere upon very general assumptions as to the directional distribution and velocity distribution of the photoelectrons. This theoretical study is followed by graphical presentation of results computed for several typical cases of special interest in connection with the experimental study.

The Distribution in Direction of Photoelectrons from Alkali Metal Surfaces.[13] HERVERT E. IVES, A. R. OLPIN and A. L. JOHNSRUD. An experimental study of the distribution in direction of photoelectrons emitted from alkali metal surfaces irradiated by light incident at various angles and polarized in different planes. The alkali metal surfaces used were of two sorts: (1) liquid alloys of sodium and potassium, (2) thin films of potassium or rubidium on polished platinum. In all cases the alkali metal surface was at the center of a large spherical enclosing anode, provided either with collecting tabs at various angular positions or with an exploring finger. It is found that the emission closely obeys Lambert's law, but that the ellipse by which the emission is represented, in polar coordinates, is more elongated normally to the surface for perpendicularly incident light than for obliquely, when the direction of the electric vector is in both cases parallel to the surface, and still more elongated for obliquely incident light with the

[12] *Physical Review*, Vol. 32, No. 1, pp. 44–56, July, 1928.
[13] *Physical Review*, Vol. 32, No. 1, pp. 57–80, July, 1928.

electric vector in the plane of incidence. The distribution curves are all perfectly symmetrical about the normal to the surface, showing no tendency to follow the direction of the electric vector.

Oscillographic Observations on the Direction of Propagation and Fading of Short Waves.[14] H. T. FRIIS. The short-wave transmission path is generally but not always located in the vertical plane through the transmission and receiving points.

Direction finding depends upon determining the direction of the wave at the receiving point; it does not give accurate results when the twilight zone is in the way of the wave path.

The angle between the earth and the direction of short-wave propagation varies continuously and the changes in this angle are much larger than the changes in angle of propagation in the horizontal plane.

The observations are consistent with the view that the fading is mainly caused by wave interference.

An Improved Permeameter for Testing Magnet Steel.[15] B. J. BABBITT. The increasing use of cobalt steel in the manufacture of permanent magnets has created a need for a permeameter that is capable of determining accurately the magnetic properties of such steel in bar form. The common commercial permeameters are not capable of producing the high magnetizing forces required for this purpose. Commercial permeameters are chiefly of two types, the yoke type and the Burrows type. The latter is difficult to operate and requires an experienced operator for a reasonable output; it cannot be adapted to the testing of cobalt steel unless it is practically rebuilt throughout. The yoke type of permeameter may be adapted to the testing of cobalt steel by the use of extensions to the poles so that the distance between them is much less. In this way the greater part of the magnetomotive force is distributed over a short portion of the magnetic circuit and the magnetomotive force per centimeter is correspondingly greater. The permeameter that is described below has been developed by the Magnetic Materials Division at the Hawthorne Works of the Western Electric Company to overcome the chief objections common to present commercial permeameters.

Corrosion of Cable Sheath in Creosoted Wood Conduit.[16] R. M. BURNS and B. A. FREED. This paper deals with the identification of a corrosion of lead cable placed in creosoted wood conduit, and with the

[14] *Proceedings of the Institute of Radio Engineers*, May, 1928, Vol. 16, No. 5, pp, 658–665.

[15] *Journal of the Optical Society of America and Review of Scientific Instruments* Vol. 17, No. 1, pp. 47–58, July, 1928.

[16] *Journal of the A. I. E. E.*, Vol. XLVII, No. 8, pp. 576–579, August, 1928.

determination and application of methods of allaying it. The trouble was experienced mainly on the Pacific Coast where, although Douglas fir conduit was introduced about 1911, the first case of corrosion which could definitely be ascribed to the creosoted conduit did not occur till 1921.

A search for the cause of the trouble led to making systematic analyses of the air present in the conduit and these analyses revealed the presence of acetic acid in sufficient amount to account for the corrosion in the presence of carbon dioxide which was also shown to be present.

After much experimenting a method was developed to stop the corrosion by pumping ammonia into the ducts. Results have been very satisfactory and seem to indicate that a single treatment is sufficient.

Small Samples—New Experimental Results.[17] W. A. SHEWHART and F. W. WINTERS. This article reviews briefly the Theory of Errors of Averages, paying particular attention to some of the most recent work in connection with small samples. New empirical results are presented showing the advantage that arises from the use of the latest error theory and pointing out the effect of the limitations imposed upon it. The information contained in this paper indicates that further theoretical studies are necessary in order that the application of small sample theory may give more accurate solutions to the problems that arise in practice.

[17] *Journal of American Statistical Association*, New Series, No. 162 (Vol. XXIII), pp. 144–153, June, 1928.

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