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Theory and Design of Uniform and Composite Electric Wave-filters

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THE electric wave-filter, as regards its general transmission characteristics and its extremely important rôle in communication systems, is well known. Its physical theory was discussed in detail in the preceding number of this Journal by its inventor, G. A. Campbell. In the present paper it is proposed to present systematic general methods of wave-filter design, together with representative designs, which have been developed in connection with the practical utilization of this device in the plant of the Bell System.

First is considered a general theory of design combining physical and analytical considerations which gives explicitly the structure of a uniform type of wave-filter having any preassigned transmitting and attenuating bands as well as desirable impedance and quite arbitrary attenuation characteristics. Next, this theory is applied to the design of a low-and-band pass wave-filter from which are derived design formulæ for all the practical uniform wave-filter structures in present use, belonging to the classes low pass, high pass, low-and-high pass, and band pass. Then the subject of composite wave-filters is taken up, these being non-uniform wave-filter networks obtained by combining sections of wave-filters having equivalent characteristic impedances but different propagation constants. Among others, a superior advantage of composite over uniform wave-filters is shown to be their greater flexibility of design, as a result of which composite wave-filters are often the only means of meeting severe design requirements. Many of the methods here used are found to have further application in general recurrent network design.

The ideal toward which wave-filter design is usually directed is a finite network having any preassigned transmitting and attenuating bands, zero attenuation and a terminal characteristic impedance equal to any one preassigned constant resistance in all transmitting bands, and infinite attenuation throughout all attenuating bands. Due to such an

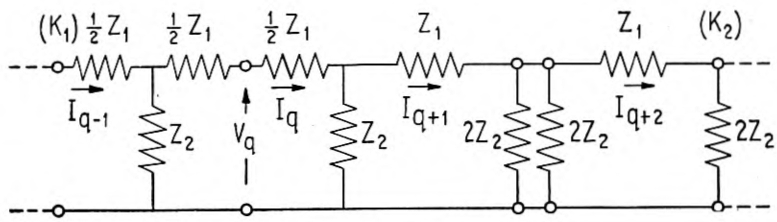


Fig. 1—Ladder Type Recurrent Network

impedance characteristic, at frequencies in the transmitting bands there would be no loss of transmitted energy if the network were inserted between two resistances, a generator and a receiver, each having a constant resistance of this same magnitude, approximately the case of two transmission lines. The infinite attenuation of the network to currents of all other frequencies would effectively prevent energy transmission through it.

Practically, such an ideal has not been attained but the methods developed here lead to designs which can approximate it rather closely. No attempt will be made to give the construction of wave-filter elements minimizing energy dissipation, as we shall be concerned mainly with a determination of the magnitude and locations of the elements in the network. It may be stated, however, that the less dissipative the elements the more nearly will the ideal of free transmitting bands be reached.

Part I. Theory of Design

The uniform recurrent network specifically considered in this design method is the ladder type of Fig. 1 having identical series impedances

z_1 and identical shunt impedances z_2, each of which has a physically realizable structure. For purposes of illustration, at the left end is shown a mid-series section whose two series impedances $\frac{1}{2}z_1$ are each half a full series impedance element. On the right is a mid-shunt section, its two shunt admittances $\frac{1}{2z_2}$ each being half that of a full shunt admittance; its shunt impedances are therefore, $2z_2$. Corresponding to these two mid-point terminations are the mid-series and mid-shunt characteristic impedances K_1 and K_2, respectively.

When any ladder type design has been obtained its mid-series and mid-shunt sections, being respectively in the form of three star-connected (T) and three delta-connected (Π) impedances, may serve as the basis of transformations by ordinary means to determine the elements of other uniform types (such as the lattice type shown in Fig. 6) having equivalent properties. Generally such equivalent uniform types are not as economical as the ladder type either due to difficulties of construction or a larger number of elements per section. The theory of composite wave-filters is included in that of uniform types as here presented and so does not require a separate treatment.

Fundamental Formulæ

The mathematical formulae upon which the design rests follows, their derivation being given in Appendix I.

$$\left.\begin{aligned}
\cosh\Gamma &= 1+\tfrac{1}{2}\frac{z_1}{z_2} = 1+\tfrac{1}{2}\gamma^2,\\
K_1 &= \sqrt{z_1z_2+\tfrac{1}{4}z_1^2} = \sqrt{1+\tfrac{1}{4}\gamma^2}\,k,\\
K_2 &= \frac{z_1z_2}{\sqrt{z_1z_2+\tfrac{1}{4}z_1^2}} = \frac{k}{\sqrt{1+\tfrac{1}{4}\gamma^2}} = \frac{k^2}{K_1},\\
e^{-\Gamma} &= \frac{2K_1-z_1}{2K_1+z_1} = \frac{2z_2-K_2}{2z_2+K_2},
\end{aligned}\right\} \qquad (1)$$

in which

z_1, z_2 = series and shunt impedances per section,
$\Gamma = A+iB$ = propagation constant per section,
K_1, K_2 = mid-series and mid-shunt characteristic impedances.

$$\gamma = \alpha + i\beta = \sqrt{\frac{z_1}{z_2}},$$

and $$k = \sqrt{z_1z_2},$$

wherein γ and k have the significance of being the propagation constant and characteristic impedance of the corresponding smooth line, *i.e.*, a line having series and shunt impedances z_1 and z_2, respectively, per unit length uniformly distributed along the line.

When z_1 and z_2 are dissimilar reactances, as in a non-dissipative wave-filter, *currents of frequencies within continuous frequency bands can be transmitted without attenuation* and the location of these bands on the frequency scale may be found from the conditions which must there be satisfied. As derived from the first equation of (1) the latter are

$$A=0,$$

and

$$\cos B=1+\tfrac{1}{2}\frac{z_1}{z_2}. \tag{2}$$

Since the cosine limits are $\pm$ 1 this shows that free transmission may occur at all frequencies corresponding to impedance ratio values satisfying the relation

$$-1\leqq\frac{z_1}{4z_2}\leqq 0, \tag{3}$$

a result which may be stated as follows:

The transmitting bands in a ladder type wave-filter having series and shunt impedances z_1 and z_2, respectively, include all frequencies at which these impedances are of opposite signs and the absolute value of z_1 is not greater than that of $4z_2$. This statement is useful in roughly determining the relative positions of such bands on an impedance diagram where z_1 and $4z_2$ have been plotted as functions of frequency.

In the attenuating bands corresponding to the remainder of the frequency range we have for the non-dissipative case

$$\cosh A=\pm\left(1+\tfrac{1}{2}\frac{z_1}{z_2}\right),$$

and

$$\sin B=0, \tag{4}$$

the sign above being taken such as to make the right member positive.

While the above formulae contain the necessary relations for wave-filter action they do not specify the physical structure of the reactance network. Hence we need to combine with them certain properties of physical reactances to arrive at a structure having desired characteristics. In wave-filter design all resistances in the physical reactance elements, although being unavoidable in construction, are neglected as they produce but secondary effects. When allowed for later their most pronounced effect is the introduction of small attenuation in the transmitting bands.

Reactance Theorems

The properties of physical reactances which are to be utilized may be stated in the following theorems:

1. *The reactance of any non-dissipative reactance network always has a positive slope with frequency, as well as abrupt changes from positive to negative infinity at anti-resonant frequencies, and may be represented identically (among others) either by a number of simple (series L and C) resonant components in parallel, or simple (parallel L and C) anti-resonant components in series.*

2. *To any non-dissipative reactance network there corresponds an inverse reactance network which is so related that the product of their impedances is a constant, independent of frequency.*

The proofs of these theorems are given in Appendix I, where with reactances which are known to be any series and parallel combinations of inductances and capacities the method of induction is readily applied. In the first theorem the simple component resonant at zero frequency is a single inductance and the one at infinite frequency a single capacity. Similarly the simple anti-resonant components corresponding to these limiting frequencies are single capacity and single inductance, respectively. In the second theorem, if we have given one reactance consisting of a number of simple anti-resonant components, all in series, the inverse network may be made up of the same number of simple resonant components all in parallel, each one of the latter corresponding to a particular one of the former. Moreover, any pair of these corresponding components are resonant and anti-resonant, respectively, at the same frequency and the ratio of inductance in one to capacity in the other is equal to the constant product of the two total impedances.

Phase Constant Theorem

The phase constant will not play any part in the present theory of design but it has this property: *The phase constant in a wave-filter always increases with frequency thruout each transmitting band.* As shown in Appendix I, this follows as a consequence of the positive slope of reactances. Consideration of this theorem will later be touched upon when discussing composite wave-filters.

1. "Constant k" Wave-Filter Having Any Preassigned Transmitting and Attenuating Bands

The "constant k" wave-filter belonging to any class[1] *is defined as that ladder type wave-filter whose product of series and shunt impedances, and therefore characteristic impedance, k, of the corresponding smooth line, is constant independent of frequency.*

The reasons for seeking the "constant k" wave-filter of any class are, briefly:

(1) Its physical structure is readily found which will give any preassigned transmitting and attenuating bands.

(2) Each of its two mid-point characteristic impedances passes thru the *same* values, different in the two cases, in *all* transmitting bands.

(3) Its design is preliminary to and furnishes a logical basis for the derivation of general wave-filters possessing desirable attenuation and impedance characteristics.

Letting the two impedances of the "constant k" wave-filter be denoted with extra suffixes as z_{1k} and z_{2k}, we have seen that if there is a relation between these impedances such that

$$z_{1k}\ z_{2k} = k^2 = \text{Constant}, \qquad (5)$$

the series and shunt impedances of the "constant k" wave-filter must be inverse networks to each other. Only one of them, say z_{1k}, need then be temporarily considered, the ratio $\frac{z_{1k}}{4z_{2k}}$, becoming $\left(\frac{z_{1k}}{2k}\right)^2$ which by (3) shows that free transmission occurs wherever the series impedance passes with increasing frequency thru the values from $z_{1k} = -i2k$ to $z_{1k} = o$, and $z_{1k} = o$ to $z_{1k} = +i2k$. At each critical frequency separating a transmitting from an attenuating band z_{1k} has the value $z_{1k} = \pm i2k$. By (4) an attenuating band includes a frequency at which z_{1k} is anti-resonant. Hence, *in a "constant k" wave-filter the transmitting and attenuating bands include the frequencies at which the series impedance is resonant and anti-resonant, respectively. The*

[1] The class of a wave-filter, as defined in the present paper, is determined by the number of its transmitting bands and their general locations on the frequency scale; the type by its general structure. Thus, the low-band-and-high pass class transmits in a band including zero frequency, in one internal band, and in a band including infinite frequency. A class is complementary to another if its transmitting and attenuating bands correspond in order to the attenuating and transmitting bands, respectively, of the other. One class is higher, or lower, than another if it has in addition to those of the latter at least one more, or one less, transmitting or attenuating band.

critical frequencies separating these bands are the frequencies at which the series impedance equals $\pm i2k$.

With these known facts and the properties of reactance networks, the determination of the physical structure and design of any "constant k" wave-filter is a relatively simple matter. At the series-resonant frequency of any transmitting band both characteristic impedances K_{1k} and K_{2k}, using the same notation as above, have by (1) the value k. This indicates that if k has been chosen equal to the impedance of the line (assumed as a constant resistance) with which the wave-filter is to be associated there will be no impedance irregularity at the junction of the mid-terminated wave-filter and the line for any of these series-resonant frequencies. We shall put, therefore,

$$k=\sqrt{z_{1k}z_{2k}}=\text{Mean Line Resistance}=R, \tag{6}$$

which is assumed given, and R will have this meaning thruout the remainder of this paper. At the *critical frequencies* we then have to satisfy the conditions

$$z_{1k}=\pm i\,2R, \tag{7}$$

where also

$$K_{1k}=0 \text{ and } K_{2k}=\infty.$$

If there are to be n transmitting bands z_{1k} may be designed out of n simple resonant components, *all in parallel*, wherein each component accounts for only one band. For example, with resonant components $z_{r1}\ldots z_{rn}$, we have

$$z_{1k}=\frac{1}{\dfrac{1}{z_{r1}}+\ldots+\dfrac{1}{z_{rj}}+\ldots+\dfrac{1}{z_{rn}}}.$$

This is sufficient since, owning to the positive slope of reactance, there is bound to be but one anti-resonant frequency and attenuating band between every adjacent pair of resonant frequencies. It is obvious that the component corresponding to the zero frequency transmitting band is an inductance, l_{1k}^{1}; the component corresponding to any (j) internal transmitting band is an inductance, l_{1k}, and capacity, c_{1k}^{j} in series; and the component corresponding to the infinite frequency transmitting band is a capacity, c_{1k}^{n}.

The magnitudes of the inductances and capacities will be uniquely determined by satisfying the relations (7) at all the critical frequencies. For at the critical frequency of the zero frequency transmitting band $z_{1k}=+i2R$; at the lower critical frequency of any internal transmitting band $z_{1k}=-i2R$ and at the higher critical frequency $z_{1k}=+i2R$; at the critical frequency of the infinite frequency transmitting band $z_{1k}=-i2R$. Hence, no matter what "constant k"

wave-filter is considered, the number of restrictions imposed on z_{1k} at the critical frequencies will always equal the total number of inductances and capacities involved, whose magnitudes are therefore given by the solution of the simultaneous equations (7).

By the second reactance theorem a corresponding value of z_{2k} may be obtained by designing it out of n components, *all in series*, wherein each component is the inverse network of a component in the series impedance, the product of their impedances being equal to R^2 to satisfy (6). The component in z_{2k} corresponding to the zero frequency transmitting band is a capacity, c^1_{2k}; that to any (j) internal transmitting band is a simple anti-resonant component of inductance, l^j_{2k}, and capacity c^j_{2k}, in parallel; and that to the infinite frequency transmitting band is an inductance, l^n_{2k}. The relations between inductances and capacities of the corresponding components are given by

$$\frac{l^1_{1k}}{c^1_{2k}} = \ldots = \frac{l^j_{1k}}{c^j_{2k}} = \frac{l^j_{2k}}{c^j_{1k}} = \ldots = \frac{l^n_{2k}}{c^n_{1k}} = R^2, \tag{8}$$

which determine the elements of z_{2k} as soon as those of z_{1k} are found.

An alternative method is to focus our attention upon the attenuation requirements. To give n attenuating bands, z_{1k} may be designed out of n simple anti-resonant components, *all in series*, each component accounting for only one band. Representing these anti-resonant components by $z_{a1} \ldots z_{an}$, the series impedance is

$$z_{1k} = z_{a1} + \ldots + z_{aj} + \ldots + z_{an}.$$

The component corresponding to the zero frequency attenuating band is a capacity, C^1_{1k}; that to any (j) internal attenuating band is a simple anti-resonant component of inductance, L^j_{1k}, and capacity C^j_{1k}, in parallel; and that to the infinite frequency attenuating band is an inductance, L^n_{1k}. As in the previous case z_{1k} must satisfy (7) at all the critical frequencies, which determines its elements. The corresponding shunt impedance, z_{2k}, may be designed out of n components, *all in parallel*, wherein each component is the inverse network of a component in the series impedance, their impedance product being R^2. The components in z_{2k} for the three typical attenuating bands above considered in the discussion of z_{1k} are in the same order, an inductance, L^1_{2k}, a simple resonant component of inductance, L^j_{2k}, in series with a capacity, C^j_{2k}, and a capacity, C^n_{2k} We have here

$$\frac{L^1_{2k}}{C^1_{1k}} = \ldots = \frac{L^j_{1k}}{C^j_{2k}} = \frac{L_{2k}}{C^j_{1k}} = \ldots = \frac{L^n_{1k}}{C^n_{2k}} = R^2. \tag{9}$$

A general comparison of these two methods of designing a "constant k" wave-filter shows that the series impedances in the two cases have the same number of inductances and the same number of capacities. Since the total number of these elements is the same in both and the two impedances are made equal at a number of critical frequencies equal to this total number, these impedances are identical at all frequencies. Similarly for the shunt impedances; all of which agrees with the first reactance theorem and leads to the following conclusion.

As regards propagation constant and impedance characteristics, only one "constant k" wave-filter exists in each class, and the magnitudes of its series and shunt impedances, each of which contains elements equal in number to the critical frequencies, are uniquely determined by the preassigned critical frequencies and the magnitude of k. Its physical structure, however, is in general not unique.

The structure of these impedances may in all but the lowest classes be given a variety of different forms, the number of inductances remaining fixed as well as the number of capacities. In the low pass, high pass, low-and-high pass, and band pass classes, the above two modes of derivation give the same designs for their respective series and shunt impedances. In those of a higher class the designs so obtained are different and for more than three elements per impedance may be put in even other forms.

Taking the "constant k", low-band-and-high pass wave-filter with critical frequencies f_o, f_1, f_2, and f_3, as an example, the first method gives the series impedance as an inductance in parallel with both a resonant component and a capacity, and the shunt impedance as a capacity in series with both an anti-resonant component and an inductance. The second method gives the structure shown in Fig. 2. Two other equivalent structures for the series impedance are possible; one is an inductance in parallel with the series combination of a capacity and an anti-resonant component, the other is a capacity in parallel with the series combination of an inductance and an anti-resonant component. Similarly the shunt impedance may have two other structures; one is a capacity in series with the parallel combination of an inductance and a resonant component, the other is an inductance in series with the parallel combination of a capacity and a resonant component. Relations between the element magnitudes are given in Appendix III, which contains general equivalent impedances. There being four equivalent structures for each of the series and shunt impedances this would mean a total of *sixteen possible structures* for this one "constant k" wave-filter. The impedance and attenuation diagrams in Fig. 2 illustrate some of its properties. Especially is it to be noted that the infinite attenuations, occurring where the series impedance is anti-resonant, take place at frequencies f_{a1} and f_{a2} which are not arbitrary but depend entirely upon the critical frequencies f_0, f_1, f_2 and f_3.

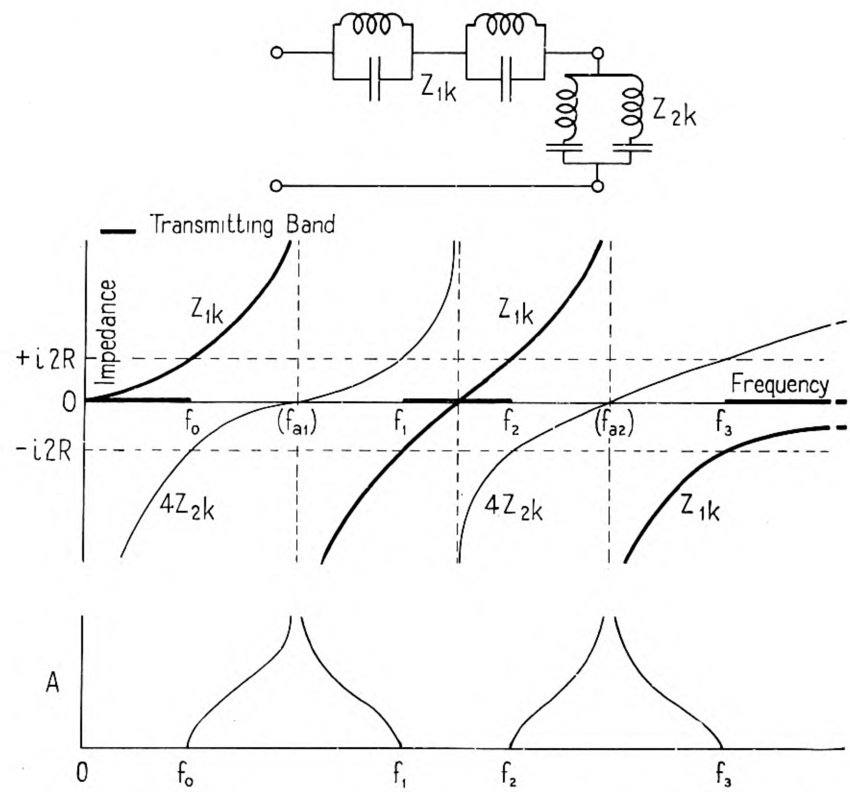


Fig. 2—"Constant k" Low-Band-and-High Pass Wave-Filter

It may be added that in "constant k" wave-filters every internal transmitting band is a confluent band formed by the junction of two bands occurring separately in a wave-filter of higher class but with the same configuration of elements.

Summarized, the above procedure for "constant k" wave-filter design is:

(1) Obtain synthetically a structural form for the series impedance, z_{1k}, from either the transmission or attenuation requirements;

(2) Determine the magnitudes of all inductances and capacities in z_{1k} from the conditions, $z_{1k} = \pm i2R$ at all preassigned critical frequencies, where R $(=k)$ is the given mean line resistance;

(3) Derive a structure, in addition to the inductance and capacity magnitudes, of the shunt impedance, z_{2k}, considering the latter as an inverse network to z_{1k}, where $z_{1k}z_{2k} = R^2$.

2. General Wave-Filters Having Any Preassigned Transmitting and Attenuating Bands and Propagation Constants Adjustable Without Changing One Mid-Point Characteristic Impedance.

It was shown above how a " constant k " wave-filter may always be designed so as to have any preassigned transmitting and attenuating bands. A method will now be given for deriving the two most general ladder types, each having one mid-point characteristic impedance equivalent at all frequencies to the corresponding mid-point characteristic impedance of the known " constant k " wave-filter; one of them has such equivalence at mid-series, and the other at mid-shunt. Because of this equivalence, these general wave-filters must necessarily have the same transmitting and attenuating bands as the " constant k " wave-filter which they include as a special case. Their propagation constants will be found to be adjustable over a wide range.

Mid-Series Equivalent Wave-Filter

Assume the known " constant k " wave-filter has n attenuating bands and that its series impedance derived by the second method has the form of n simple anti-resonant components in series, represented as

$$z_{1k} = z_{a1} + z_{a2} + \ldots z_{an}. \tag{10}$$

Its mid-series characteristic impedance is

$$K_{1k} = \sqrt{R^2 + \tfrac{1}{4}z_{1k}^2}. \tag{11}$$

Let the series and shunt impedances of the desired general wave-filter be z_{11} and z_{21}, respectively, where the second subscript $_1$ indicates that these impedances belong to the wave-filter which is to have mid-series equivalence with the " constant k " wave-filter. Then

$$K_{11} = \sqrt{z_{11}z_{21} + \tfrac{1}{4}z_{11}^2}, \tag{12}$$

and the *fundamental relation* is that

$$K_{11} = K_{1k}. \tag{13}$$

Certain inferences may be drawn as to the nature of the impedances z_{11} and z_{21}.

a. The series impedance z_{11} is similar in form to the series impedance z_{1k} and is anti-resonant at the same frequencies as z_{1k}. This follows directly from a comparison of formulae (11) and (12). For whenever z_{1k} is anti-resonant, corresponding to an attenuating band, K_{1k} is infinite, and to make K_{11} also infinite z_{11} must be anti-resonant irrespective of z_{21} in order to maintain an attenuating band at these frequencies.

b. The shunt impedance z_{21} corresponding to the series impedance z_{11} and the given class of wave-filter may, in its most general form, be taken as a parallel combination of simple resonant components (series L and C) equal in number to the total number of inductances and capacities contained in z_{11}. This is a consequence of a general conclusion based upon formulae (2) and (4) and the properties of reactances, namely that in an attenuating band corresponding to each branch of the series impedance frequency curve, where the absolute value of z_{11} passes once continuously thru all values from zero to infinity, the shunt impedance z_{21} can be resonant no more than once. Since, however, the number of branches in the z_{11} frequency curve equals the number of elements which z_{11} contains, the above statement is proven.

c. Series resonance and shunt anti-resonance coincide if both are included in an internal transmitting band. Series and shunt anti-resonance coincide if both are included in an internal attenuating band. This is a necessary relation in either case to preserve band confluency.

To ensure the necessary similarity between z_{11} and z_{1k} it will be assumed that for every series component in z_{1k} as above expressed there is one of proportional magnitude in z_{11} which latter may be written,

$$z_{11}=m_1z_{a1}+m_2z_{a2}+\ .\ .\ .\ m_nz_{an}, \tag{14}$$

where the coefficients, $m_1, \ldots m_n$, are positive real numerics. From the formulae (11), (12), and (13) the shunt impedance becomes

$$z_{21}=\frac{R^2+\frac{1}{4}(z^2{}_k-z^2_{11})}{z_{11}}. \tag{15}$$

If in this formula the assumed form (14) for z_{11} corresponding to any particular z_{1k} is substituted, it will be found that the resulting expression for z_{21} has exactly the requisite form to be the most general shunt impedance which that wave-filter may have. This therefore, justifies the assumption regarding z_{11} and shows the latter to give *the general case* having the specified characteristic impedance.

The coefficients, $m_1, \ldots m_n$, may be evaluated by fixing any n physically realizable conditions such as n resonant frequencies of the shunt impedance, which are frequencies of infinite attenuation in the wave-filter. From the foregoing not more than two such frequencies may be included in any internal, and but one in any other, attenuating band. However, since the number of such conditions equals the number of attenuating bands it will be considered most useful to fix one resonant frequency in each attenuating band. If z_{11} has N elements, where $N=2n-2$, $2n-1$, or $2n$, the shunt impedance will have $2N$ which may then be found.

An evaluation process possible here is first to write the expression for z_{21} in (15) as the ratio of two polynomials with two variables, in which the assumed relation for z_{11} has been substituted and the variables are an arbitrarily chosen known inductive impedance, z_L, and capacitive impedance z_C, such, for example, as may occur in z_{1k}. Put each component of the desirable parallel resonant component form of z_{21} in terms of these same two variables and two undetermined coefficients, as az_L+bz_C, etc., and write the corresponding polynomial ratio expression for z_{21} which will involve the coefficients. A comparison of the two expressions for z_{21} which must be equivalent gives $2N$ relations between the coefficients $m_1, \ldots m_n$ of z_{11} and the $2N$ coefficients a, b, etc., of z_{21}. Next fix n resonant frequencies of z_{21}, satisfying the relation

$$z_{21}=0, \tag{16}$$

at frequencies $f_{1\infty}, \ldots f_{n\infty}$, one arbitrarily chosen in each attenuating band. These give n simple ratios $\frac{a}{b}$,etc., which with the other relations make a total of $2N+n$ simultaneous equations from which to determine the same number of coefficients. Their solution will give all coefficients explicitly in terms of the independent critical frequencies f , $f_1 \ldots$, and frequencies of infinite attenuation $f_{1\infty}, \ldots f_{n\infty}$. It is more practical, however, to obtain such explicit solutions for the coefficients $m_1, \ldots m_n$ only, and to express the coefficients a, b, etc., as functions of the frequencies and the m's combined.

That the n additional conditions in (16) are the maximum number which can be imposed may be illustrated in the case of $n=2$ by the general low-band-and-high pass wave-filter of Fig. 3 corresponding to the " constant k " wave-filter of Fig. 2. This has a total of twelve elements per section which it will be seen are fully determined by the following twelve conditions: four at the critical frequencies f_o, f_1, f_2 and f_3, where $\frac{z_{11}}{4z_{21}} = -1$; four at frequencies f_{a1} and f_{a2} where both z_{11} and z_{21} are anti-resonant; one at a variable frequency in the internal transmitting band where z_{11} is resonant and z_{21} anti-resonant; one at

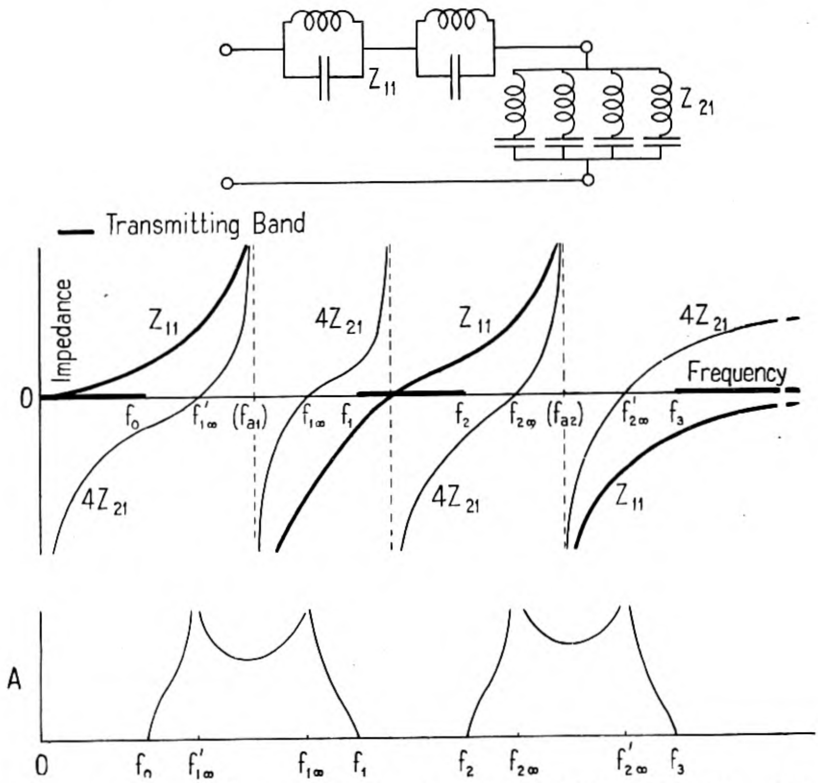


Fig. 3—General Mid-Series Equivalent Low-Band-and-High Pass Wave-Filter

one other frequency where the absolute value of the characteristic impedance is fixed; and two added conditions at the adjustable frequencies of infinite attenuation $f_{1\infty}$ and $f_{2\infty}$, bringing the total up to twelve.

In brief, the procedure for designing the general mid-series equivalent wave-filter is:

(1) Write down from the known "constant k" wave-filter having n preassigned attenuating bands the form of the series impedance with undetermined coefficients $m_1 \ldots m_n$ as in (14).

(2) Obtain two expressions for the shunt impedance, one derived thru the characteristic impedance of the "constant k" wave-filter and containing the coefficients $m_1 \ldots m_n$; the other from a consideration of its possible most general form corresponding to the series impedance, with coefficients a, b, etc. Equate these expressions at all frequencies and thus obtain a set of relations between the coefficients $m_1 \ldots m_n$ and a, b, etc., equal to the latter in number.

(3) Fix one resonant frequency of the shunt impedance, a frequency of infinite attenuation, in each attenuating band, using the second expression above which will determine n simple ratios $\frac{a}{b}$, etc.

(4) Solve these simultaneous equations by obtaining an explicit solution for the coefficients $m_1, \ldots m_n$ in terms of the critical frequencies $f_0, f_1 \ldots$ and frequencies of infinite attenuation $f_{1\infty} \ldots f_{n\infty}$, and a solution for the coefficients a, b, etc., in terms of these frequencies and the coefficients $m_1, \ldots m_n$.

This method will later be applied to the design of the low-and-band pass wave-filter.

Mid-Shunt Equivalent Wave-Filter

The general wave-filter whose mid-shunt characteristic impedance is equivalent to that of the "constant k" wave-filter can be obtained in a manner somewhat similar to the one above. However, it is possible to derive the mid-shunt equivalent directly from the mid-series equivalent wave-filter by a simple process wherein these two are assumed to have equivalent propagation constants.

Let the series and shunt impedances of this wave-filter be z_{12} and z_{22}, and its mid-series and mid-shunt characteristic impedances K_{12} and K_{22}, respectively. The fundamental condition here is that

$$K_{22} = K_{2k}. \tag{17}$$

Under the assumption that the wave-filter has a propagation constant equivalent to that of the general mid-series wave-filter, where $K_{11} = K_{1k}$, we may write from (1)

$$\frac{z_{11}}{z_{21}} = \frac{z_{12}}{z_{22}},$$

and

$$e^{-\Gamma} = \frac{2K_{1k} - z_{11}}{2K_{1k} + z_{11}} = \frac{2z_{22} - K_{2k}}{2z_{22} + K_{2k}}.$$

These relations and (1) give

$$z_{11}z_{22} = z_{12}z_{21} = K_{1k}K_{2k} = z_{1k}z_{2k} = R^2. \tag{18}$$

Hence, *the general mid-shunt equivalent wave-filter can be obtained by designing its series and shunt impedances as inverse networks, of impedance product R^2, to the shunt and series impedances, respectively, of the general mid-series equivalent wave-filter, under which conditions the two wave-filters have equivalent propagation constants.*

To illustrate, a structure for the general mid-shunt equivalent low-band-and-high pass wave-filter corresponding to Figs. 2 and 3 is

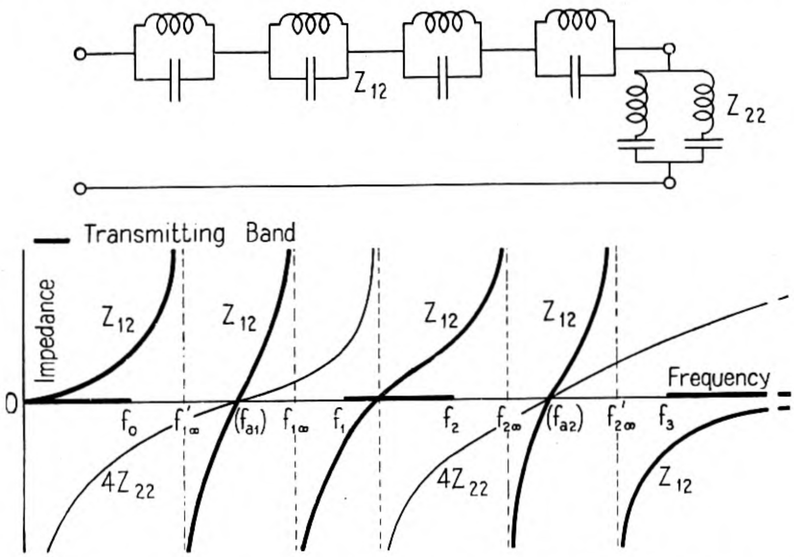


Fig. 4—General Mid-Shunt Equivalent Low-Band-and-High Pass Wave-Filter

shown in Fig. 4. The impedance diagram indicates how the transmitting and attenuating bands are produced. Here each anti-resonant component in the series impedance is responsible for one of the infinite attenuations shown in the equivalent attenuation diagram of Fig. 3. It may be seen also that when in practice it is necessary to balance the two sides of the line, the wave-filter of Fig. 4 requires more series balanced inductances and capacities than that of Fig. 3 to give an equivalent propagation constant. For this reason the mid-shunt equivalent wave-filter is usually not as economical as the mid-series equivalent wave-filter.

3. M-Type Wave-Filters.

The term M-type will be applied to that case in each of the above general wave-filters in which the coefficients $m_1 \ldots m_n$ coalesce to the single value $m_1 = \ldots = m_n = m$, leaving but one degree of freedom. They are of special interest because in wave-filters having many elements the impedances can be determined more directly than by the general methods above and, of greater importance, because the mid-shunt characteristic impedance, $K_{21}(m)$, of the mid-series equivalent M-type and the mid-series characteristic impedance, $K_{12}(m)$, of the mid-shunt equivalent M-type, both functions of m, can be made approximately a constant resistance over the greater part of every transmitting band, a desirable property.

In the mid-series equivalent M-type it follows from (14) and (15) that, since $z_{1k}z_{2k} = R^2$,

$$z_{11} = mz_{1k},$$

and

$$z_{21} = \frac{1-m^2}{4m} z_{1k} + \frac{1}{m} z_{2k}, \tag{19}$$

showing the shunt impedance to be expressible as a *series* combination of different proportions of the " constant k " series and shunt impedances. This structure is usually different from but equivalent to the mid-series equivalent wave-filter obtainable by the first method in which the m-coefficients are all equal to m. The value of the coefficient m is determined by fixing a resonant frequency of z_{21}, that is, any one frequency of infinite attenuation, f_∞. From (19), for $(z_{21})_{f_\infty} = 0$,

$$m = \sqrt{1 + \left(\frac{4z_{2k}}{z_{1k}}\right)_{f_\infty}}. \tag{20}$$

The corresponding mid-shunt equivalent M-type having the same propagation constant follows from (18) with impedances

$$z_{12} = \frac{1}{\dfrac{1}{mz_{1k}} + \dfrac{1}{\dfrac{4m}{1-m^2} z_{2k}}},$$

and

$$z_{22} = \frac{1}{m} z_{2k}. \tag{21}$$

Here the series impedance is expressible as a *parallel* combination of different proportions of the " constant k " impedances.[2]

[2] It is worth while to point out that from the nature of (19) and (21) these same relations result if z_{1k} and z_{2k} are the series and shunt impedances z_1 and z_2 of *any* ladder type recurrent network whatever. In order that there be a physically realizable structure corresponding to such general relations it is sufficient that $o < m \leq 1$. A change of m will change the propagation constant without changing the mid-series characteristic impedance of the first network, and mid-shunt of the second.

The characteristic impedances, $K_{21}(m)$ and $K_{12}(m)$, follow from the substitution of (19) and (21) in (1), and are given by the relations

$$\frac{R}{K_{21}(m)}=\frac{K_{12}(m)}{R}=\frac{\sqrt{1+\dfrac{z_{1k}}{4z_{2k}}}}{1+\dfrac{(1-m^2)}{4}\dfrac{z_{1k}}{z_{2k}}}. \tag{22}$$

Fig. 5 shows graphically how this impedance ratio, neglecting dis-

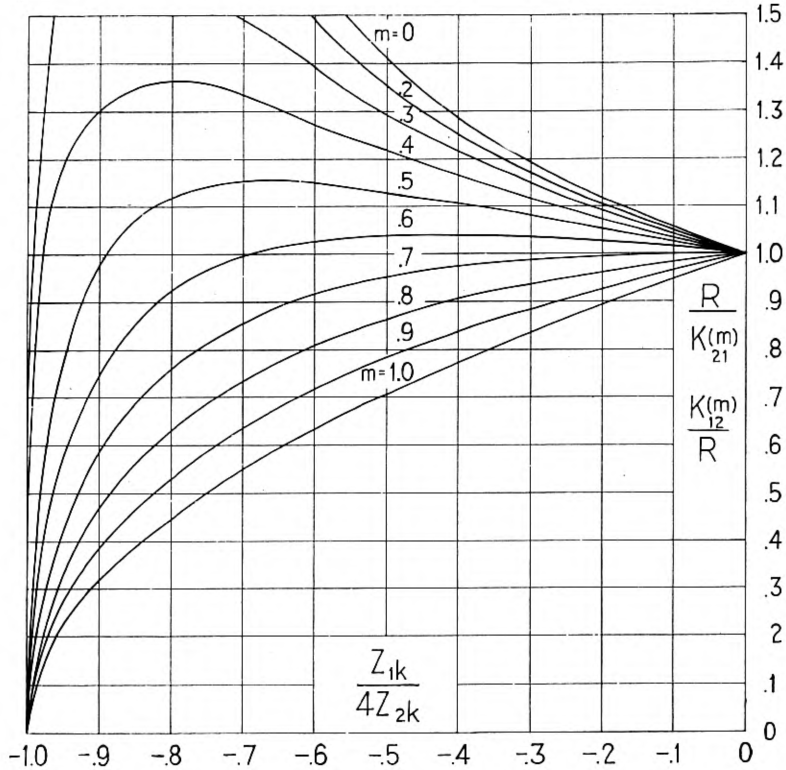


Fig. 5—M-Type Characteristic Impedances in Transmitting Band

sipation, depends upon m in any transmitting band. For the limiting values $m=1$ and $m=0$ it corresponds to $\frac{K_{1k}}{R}$ and $\frac{K_{2k}}{R}$, respectively.

For the intermediate value $m=.6$ it, and hence $K_{21}(m)$ and $K_{12}(m)$, is approximately constant over the greater part of the transmitting band thereby approaching the ideal sought. A wave-filter network having these latter terminations could then be connected between constant resistance terminal impedances without introducing appreciable reflection losses at the important frequencies to be transmitted. It may also be added that where a number of wave-filters transmitting in different bands are to be joined in series or in parallel the usual terminations correspond to $K_{12}(m)$ and $K_{21}(m)$, respectively (where m is about .6), with the omission of the terminal half-series impedance in the first case and terminal double-shunt impedance in the second. In the transmitting band of any one of these wave-filters the rôle of the omitted impedance is approximately fulfilled

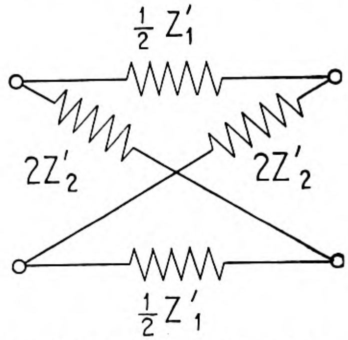


Fig. 6—Lattice Type Recurrent Network

by the resultant impedance of the other wave-filters. The approximation is very close when such connections are made with two complementary wave-filters having the same critical frequencies.

4. Equivalent Lattice Type Wave-Filters.

The lattice type of recurrent network shown in Fig. 6 offers a simple example of a uniform type which can physically be made to have properties equivalent to those of the ladder type. Its formulae for propagation constant and characteristic impedance in terms of the series and lattice impedances, $\frac{1}{2}z_1'$ and $2z_2'$, are known to be

$$\cosh \Gamma' = 1 + \frac{2z_1'}{4z_2' - z_1'},$$

and

$$K' = \sqrt{z_1' z_2'}. \tag{23}$$

A comparison of these formulae with those of the ladder type in (1) shows that when $\Gamma' = \Gamma$, and $K' = K_1$,

$$z_1' = z_1,$$

and

$$z_2' = \tfrac{1}{4}z_1 + z_2; \tag{24}$$

and that when $\Gamma' = \Gamma$, and $K' = K_2$,

$$z_1' = \frac{1}{\frac{1}{z_1} + \frac{1}{4z_2}},$$

and

$$z_2' = z_2. \tag{25}$$

In both cases it is apparent that for equivalent results the lattice type requires more elements than the ladder type and is, therefore, not as economical.

Part II. Design of Low-and-Band Pass Wave-Filters and Reduction to Wave-Filters of Lower Class

The foregoing theory of design can be applied separately to the design of wave-filters of each class in general use, which classes are the low pass, high pass, low-and-high pass, and band pass. However,

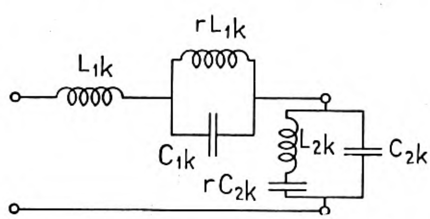


Fig. 7—"Constant k" Low-and-Band Pass Wave-Filter

instead of such individual treatment designs will first be derived for low-and-band pass wave-filters which are wave-filters of higher class than these four classes and include the latter as particular cases. The simplifications in structure and formulae which result upon their reduction to the lower classes will be considered later.

Low-and-Band Pass Wave-Filters

The structure of the "constant k" low-and-band pass wave-filter as derived from the attenuation requirements has the form of Fig. 7. Since this form may be obtained from that given in Fig. 2 by assuming the critical frequency, f_3, in the latter to be infinite, we may under this assumption refer to Fig. 2 for the impedance and attenuation characteristics corresponding to Fig. 7.

The series impedance z_1, expressed as a function of frequency is

$$z_{1k} = i2\pi f L_{1k}\left(1 + \frac{r}{1 - 4\pi^2 f^2 r L_{1k} C_{1k}}\right), \tag{26}$$

where r is the ratio between the two inductances. The magnitudes of L_{1k}, C_{1k} and r are found from the conditions (7) which z_{1k} must satisfy at the critical frequencies f_0, f_1, and f_2; namely, $z_{1k} = +i2R$, $-i2R$, and $+i2R$. The resulting simultaneous equations become

$$f_0 w + f_0^2 x - f_0^3 y = +1,$$

$$f_1 w - f_1^2 x - f_1^3 y = -1, \tag{27}$$

and

$$f_2 w + f_2^2 x - f_2^3 y = +1,$$

where $L_{1k} = \frac{yR}{\pi x}$; $C_{1k} = \frac{x^2}{4\pi R(wx - y)}$; and $r = \frac{wx}{y} - 1$.

The solution of (27) gives

$$L_{1k} = \frac{R}{\pi(f_0 - f_1 + f_2)},$$

$$C_{1k} = \frac{(f_0 - f_1 + f_2)^2}{4\pi[(f_0 f_1 - f_0 f_2 + f_1 f_2)(f_0 - f_1 + f_2) - f_0 f_1 f_2]R}, \tag{28}$$

and

$$r = (f_0 - f_1 + f_2)\left(\frac{1}{f_0} - \frac{1}{f_1} + \frac{1}{f_2}\right) - 1.$$

The corresponding shunt elements are obtained from the series elements by the inverse network relations, $\frac{L_{2k}}{C_{1k}} = \frac{L_{1k}}{C_{2k}} = R^2$, so that

$$L_{2k} = R^2 C_{1k},$$

$$C_{2k} = \frac{L_{1k}}{R^2}. \tag{29}$$

and

With the "constant k" wave-filter elements so determined we shall now derive the series and shunt impedances, z_{11} and z_{21}, of the general mid-series equivalent wave-filter. Putting for convenience

$$z_L = i2\pi f L_{1k}, \text{ and } z_C = \frac{1}{i2\pi f C_{1k}},$$

formula (26) becomes

$$z_{1k} = z_L + \frac{r z_L z_C}{r z_L + z_C},$$

and

$$z_L\, z_C = r s R^2, \tag{30}$$

where

$$s = \frac{4 f_0 f_1 f_2}{(f_0 - f_1 + f_2)^3}.$$

By (14) we may write for the general series impedance

$$z_{11}=m_1 z_L+\frac{m_2 r z_L z_C}{r z_L+z_C}, \tag{31}$$

in which the coefficients m_1 and m_2 are to be determined. Substitution of these relations in (15) gives one expression for the shunt impedance

$$z_{21}=\frac{\left\{\begin{array}{l}\frac{r^3 s}{4}(1-m_1^2)z_L^3+r^2\left[1+\frac{s}{2}\Big(1+r-m_1(m_1+m_2 r)\Big)\right]z_L^2 z_C\\ +r\left[2+\frac{s}{4}\Big((1+r)^2-(m_1+m_2 r)^2\Big)\right]z_L z_C^2+z_C^3\end{array}\right.}{m_1 r^3 s z_L^2+r^2 s(2m_1+m_2 r)z_L z_C+rs(m_1+m_2 r)z_C^2}. \tag{32}$$

Also, since the series impedance has three elements, the most general structure for z_{21} is three resonant components in parallel. Letting these components be az_L+bz_C, cz_L+dz_C, and ez_L+fz_C, as in Fig. 8, the corresponding total impedance expression is

$$z_{21}=\frac{\frac{ace}{bdf}z_L^3+\left(\frac{ac}{bd}+\frac{ae}{bf}+\frac{ce}{df}\right)z_L^2 z_C+\left(\frac{a}{b}+\frac{c}{d}+\frac{e}{f}\right)z_L z_C^2+z_C^3}{\frac{ac+ae+ce}{bdf}z_L^2+\frac{a(d+f)+c(b+f)+e(b+d)}{bdf}z_L z_C+\left(\frac{1}{b}+\frac{1}{d}+\frac{1}{f}\right)z_C^2}. \tag{33}$$

Equality between (32) and (33) at all frequencies requires that the following relations be satisfied:

$$\begin{aligned}
&\frac{ace}{bdf}=\frac{r^3 s}{4}(1-m_1^2),\\
&\frac{ac}{bd}+\frac{ae}{bf}+\frac{ce}{df}=r^2\left[1+\frac{s}{2}\Big(1+r-m_1(m_1+m_2 r)\Big)\right],\\
&\frac{a}{b}+\frac{c}{d}+\frac{e}{f}=r\left[2+\frac{s}{4}\Big((1+r)^2-(m_1+m_2 r)^2\Big)\right],\\
&\left(\frac{ce}{df}\right)\frac{1}{b}+\left(\frac{ae}{bf}\right)\frac{1}{d}+\left(\frac{ac}{bd}\right)\frac{1}{f}=m_1 r^3 s,\\
&\left(\frac{c}{d}+\frac{e}{f}\right)\frac{1}{b}+\left(\frac{a}{b}+\frac{e}{f}\right)\frac{1}{d}+\left(\frac{a}{b}+\frac{c}{d}\right)\frac{1}{f}=r^2 s(2m_1+m_2 r),\\
\text{and}\quad &\frac{1}{b}+\frac{1}{d}+\frac{1}{f}=rs(m_1+m_2 r),
\end{aligned} \tag{34}$$

where r and s are given in (28) and (30).

To fix one resonant frequency of z_{21} in each of the two attenuating bands, at $f_{1\infty}$ and $f_{2\infty}$, we may put

$$(az_L+bz_C)_{f_{1\infty}}=0,$$

and

$$(cz_L+dz_C)_{f_{2\infty}}=0,$$

which give finally

$$\frac{a}{b}=\frac{f_0f_1f_2}{(f_0-f_1+f_2)f_{1\infty}^2}r,$$

and

$$\frac{c}{d}=\frac{f_0f_1f_2}{(f_0-f_1+f_2)f_{2\infty}^2}r. \quad (35)$$

These eight simultaneous equations in (34) and (35) are sufficient to determine all the coefficients m_1, m_2, a, b, c, d, e, and f in terms of the critical frequencies f_0, f_1, and f_2, and frequencies of infinite at-

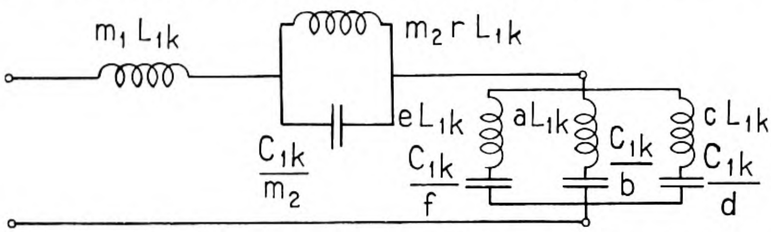


Fig. 8—General Mid-Series Equivalent Low-and-Band Pass Wave-Filter

tenuation, $f_{1\infty}$ and $f_{2\infty}$. The method of solution here used will be indicated only and the final results given in Appendix II. The combination of (35) and the first three equations of (34), makes it possible to eliminate all coefficients but m_1 and m_2 and to obtain formulae for the latter explicitly in terms of the frequencies. From (35) and the first equation and last three equations of (34), b, d, and f are calculable in terms of m_1, m_2, and the frequencies. These combined with (35) and the first equation of (34) furnish the values of a, c, and e. The formula for the dependent frequency of infinite attenuation, $f'_{1\infty}$, results from putting $(ez_L+fz_C)_{f'_{1\infty}}=0$.

The general mid-shunt equivalent wave-filter, having impedances z_{12} and z_{22}, will be derived from the general mid-series equivalent wave-filter above through the inverse network relations of (18); namely, $z_{11}z_{22}=z_{12}z_{21}=R^2$. For the series impedance we have upon the substitution of z_{21}

$$z_{12}=\frac{R^2}{z_{21}}=\frac{R^2}{az_L+bz_C}+\frac{R^2}{cz_L+dz_C}+\frac{R^2}{ez_L+fz_C}. \quad (36)$$

Taking the first term of the right member as typical, it may be transformed through (29) to the form

$$\frac{R^2}{az_L+bz_C}=\frac{1}{\frac{az_L}{R^2}+\frac{bz_C}{R^2}}=\frac{1}{i2\pi faC_{2k}+\frac{1}{i2\pi f\frac{L_{2k}}{b}}}, \tag{37}$$

which is the impedance of an anti-resonant component having an inductance, $\frac{L_{2k}}{b}$, and a capacity, aC_{2k}. Similarly each of the other two terms of (36) represents the impedance due to an anti-resonant component, in one case of elements $\frac{L_{2k}}{d}$ and cC_{2k}, and in the other of elements $\frac{L_{2k}}{f}$ and eC_{2k}.

The shunt impedance may by (29) and (31) be put in the form

$$z_{22}=\frac{R^2}{z_{11}}=\frac{R^2}{m_1z_L+\frac{m_2rz_Lz_C}{rz_L+z_C}}, \tag{38}$$

$$=\frac{1}{i2\pi fm_1C_{2k}+\frac{1}{i2\pi f\frac{L_{2k}}{m_2}+\frac{1}{i2\pi fm_2rC_{2k}}}},$$

and is the impedance of a capacity $m_1\ C_{2k}$ in parallel with a resonant component of inductance $\frac{L_{2k}}{m_2}$ and capacity m_2rC_{2k}. The structure corresponding to z_{12} and z_{22} is shown in Fig. 9.

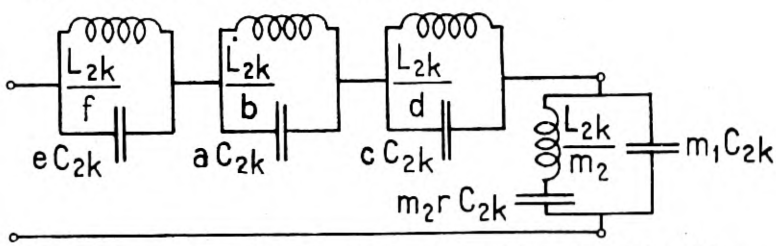


Fig. 9—General Mid-Shunt Equivalent Low-and-Band Pass Wave-Filter

The method of reducing these general wave-filters to the desired lower class wave-filters will now be taken up briefly. The resulting structures and formulae are given in Appendix II, where the two wave-filters having identical propagation constants are considered together and are numbered. The subscripts $_1$ and $_2$ on these numbers refer, respectively, to the mid-series and mid-shunt equivalent wave-filters. The quantities with brackets occurring in some of the formulae are included merely to indicate the origin of their equivalents from the low-and-band pass wave-filters.

Low Pass Wave-Filters

These are some of the simplest wave-filters and are here obtained by considering

$$f_0 = f_{1\infty} = f_1 = 0. \tag{39}$$

In general these wave-filters have three elements per section and are identical with the M-types since there is but a single coefficient $m_1 = m$. The "constant k" structure, series inductance and shunt capacity, results when $f_{2\infty} = \infty$.

High Pass Wave-Filters

These wave-filters which are complementary to the low pass wave-filters also have simple structures, in general three elements per section, the M-types. To derive them assume in the general formulae

$$f_0 = 0,$$

and

$$f_2 = f_{2\infty} = \infty. \tag{40}$$

The additional condition, $f_{1\infty} = 0$, gives the "constant k" wave-filter of series capacity and shunt inductance.

Low-and-High Pass Wave-Filters

For the low-and-high pass transmission characteristic put

$$f_2 = f_{2\infty} = \infty. \tag{41}$$

Here some simplifications in notation may be made, as is indicated in the formulae by the quantities in brackets. The general structures, M-types, require six elements per section. A limiting case, the "constant k" wave-filter, having four elements per section, results when $f_{1\infty} = \sqrt{f_0 f_1} = f'_{1\infty}$.

Band Pass Wave-Filters

With the condition

$$f_0 = 0 \tag{42}$$

an internal transmitting band is retained and also the two independent frequencies of infinite attenuation, $f_{1\infty}$ and $f_{2\infty}$. Depending upon the values of these frequencies, the wave-filter structures may have from three to six elements per section.

In the six element pair $f_{1\infty}$ and $f_{2\infty}$ are unrestricted except that they must lie within their respective attenuating bands. These wave-filters are the general ones including the others. A relation found to exist here is

$$\frac{1-m_1^2}{1-m_2^2}=\frac{f_1^2 f_2^2}{f_{1\infty}^2 f_{2\infty}^2}$$

which has been incorporated in the formulae. The three element structures, of which there are two pairs, come from putting $f_{1\infty}=0$ and $f_{2\infty}=f_2$ in one case, $f_{1\infty}=f_1$ and $f_{2\infty}=\infty$ in the other. Those having four elements are the "constant k," where $f_{1\infty}=0$ and $f_{2\infty}=\infty$, and the two similar appearing pairs in one of which $f_{2\infty}=f_2$, and in the other $f_{1\infty}=f_1$. Two pairs of five element structures exist, one with $f_{1\infty}=0$ and the other with $f_{2\infty}=\infty$. It is of interest to point out that $m_2=1$ in all of the band pass wave-filters where $f_{1\infty}=0$, and $m_1=1$ where $f_{2\infty}=\infty$, showing that in these cases certain of the elements will be like those of the "constant k" wave-filter. Also, in the limiting cases where a frequency of infinite attenuation coincides with a critical frequency, the attenuation constant increases from zero at this frequency to a finite limiting value at the other extreme of the attenuating band.

The M-type band pass wave-filters are given by putting $m_1=m_2=m$. Choosing $f_{2\infty}$ as the independent frequency, the formulae simplify to

$$m=\frac{\sqrt{\left(1-\frac{f_1^2}{f_{2\infty}^2}\right)\left(1-\frac{f_2^2}{f_{2\infty}^2}\right)}}{1-\frac{f_1 f_2}{f_{2\infty}^2}},$$

$$a=d=\frac{1-m^2}{4m}\left(1+\frac{f_{2\infty}^2}{f_1 f_2}\right), \tag{43}$$

$$b=c=\frac{1-m^2}{4m}\left(1+\frac{f_1 f_2}{f_{2\infty}^2}\right),$$

and

$$f_{1\infty}=\frac{f_1 f_2}{f_{2\infty}}.$$

Part III. Composite Wave-Filters

The preceding parts of this paper have considered wave-filters as made up of a series of uniform sections. We know, however, from this discussion that the propagation constants of certain general wave-filter sections can be changed without changing one of their mid-point characteristic impedances. Obviously then it should be possible to combine such sections so as to give a non-uniform network which introduces a number of different propagation constants.

The composite wave-filter is a network of serially connected wave-filter sections some or all of which are different in propagation constants, but adjacent sections of which are equivalent in characteristic impedance at their junction. The latter condition ensures the absence of impedance irregularities within the network. Consequently the composite wave-filter is specified by the sum of the propagation constants of the individual sections and the characteristic impedances of the end sections.

The advantage of composite over uniform wave-filters is in their flexibility of design by means of which it is easier and more economical to meet the attenuation and impedance requirements in many wave-filter networks. For example, to utilize the frequency range as completely as possible the attenuation of the network should in general rise rapidly upon entering the attenuating bands and remain high. It is also often desirable that the network have an approximately constant resistance terminal impedance in the transmitting bands. No uniform wave-filter possesses all these properties as it was found that the attenuation constant of any section varies markedly with frequency over the attenuating bands, being much higher in some parts then in others; then, too, the impedances of most wave-filters are not the best available. To give high attenuation at frequencies where the attenuation constant of a section is low requires a relatively large number of uniform sections and this means a surplus of attenuation at other frequencies. Aside from economic considerations this number is practically limited by the amount of attenuation introduced in the transmitting bands due to dissipation in the elements. In a composite wave-filter, however, it is possible to distribute the low and high attenuations of the individual sections over the frequency bands so that an efficient use is made of these attenuation properties and a more uniform high attenuation is produced; a desirable impedance characteristic is obtainable by M-type section terminations.

In the case of ladder types, for example, we may look upon the composite wave-filter as having been originally a number of sections of the general mid-series or mid-shunt equivalent wave-filters wherein now the propagation constants of the sections have been changed without changing their characteristic impedances. The mid-series and mid-shunt wave-filters may also both be included since their junction can be made through the intermediate use of the "constant

k" wave-filter, a half-section being the minimum. Again, mid-series and mid-shunt sections derived from prototypes other than the "constant k" wave-filter, such as have already been indicated in connection with the generalized M-type formulae, are other possible units. The two different half-series impedances which join where two mid-series sections are connected together can always be merged into one impedance having the same impedance structure but in general different magnitudes for all elements; a similar merging of shunt impedances can be effected at the junction of two mid-shunt sections. It is here from a structural standpoint that the ladder type is much superior to other types, such as the lattice type over which it has the additional advantage of a smaller number of elements per section. For if one or more sections of the lattice type are included in the composite network each section must be completely constructed since there is no possibility of merging adjacent impedances.

It is known that among band pass wave-filters having equivalent mid-point characteristic impedances some have positive phase constants and others negative at the same frequencies in the transmitting band. The question may be raised as to whether such sections can not be combined in a manner which will give zero phase in addition to zero attenuation throughout the transmitting band. The impossibility of this follows directly from the phase constant theorem previously given, namely, that the phase constant increases with frequency throughout the transmitting band, irrespective of its sign. Combining sections increases the rate of total phase change with frequency.

Equivalent Substitutions

There are equivalent structures for certain wave-filter sections as well as for many of their impedances and impedance combinations. This is of practical importance in design where it is sometimes advantageous to use one form in preference to another. The number of elements, their magnitudes, or both, are some of the determining factors in this choice.

The wave-filter sections here considered are of the band pass class and their equivalence relations, both as regards current propagation and impedance, are given by the following tabluation in which these wave-filters are referred to by number as in Appendix II. The subscripts $_1$ and $_2$ are omitted since it is to be understood that the relations apply on the one hand to mid-series sections having those numbers with a subscript $_1$ and on the other to mid-shunt sections numbered correspondingly with a subscript $_2$. We have then for mid-series or mid-shunt sections:

$$\begin{aligned} &(a) \quad IV \equiv VIII + IX, \\ &(b) \quad VII \equiv V + VI, \\ &(c) \quad X \equiv V + IX, \\ &(d) \quad XI \equiv VI + VIII, \end{aligned} \tag{44}$$

whence it follows that

$$(e) \quad IV + VII \equiv X + XI,$$

etc. To verify these identities we need to consider the propagation constants only since impedance equivalence is known to exist. This is most easily accomplished in either the mid-series or mid-shunt cases by using the formula for $e^{-\Gamma}$ in (1) to show the sufficient relation for propagation constant equivalence,

$$e^{-\Gamma} = e^{-\Gamma'} e^{-\Gamma''}. \tag{45}$$

Here Γ represents the propagation constant of the section in the left-hand member of (44) a, b, c, or d; Γ' and Γ'' those of the corresponding right-hand member sections. It can likewise be verified that these identities hold even when dissipation is present if in both structures all inductances have the same time constants and if a similar relation holds for all capacities. A comparison shows that the numbers of elements in the two structures corresponding to the left- and right-hand members of (44) are, respectively, 8 and 10 in (a), 6 and 8 in (b), 7 and 9 in both (c) and (d), and 12 and 12 in (e).

Equivalent impedance structures involving two inductances and two capacities have already been mentioned in the discussion of the " constant k " low-band-and-high pass wave-filter in Part I. These also include equivalent three element structures. The formulae which hold when a transformation is made from one structure to an equivalent one follow directly from those for certain combinations of two different general impedance components, as given inAppendix III. Because of this generality of the components, equivalence exists even when there is dissipation provided the inductances and capacities have time constants which are, respectively, the same in all. Moreover, since the two structures are identical from an impedance standpoint at all frequencies of the steady periodic state, they will be identical similarly under any conditions of the transient state. The method of deriving the formulae consists in first forming for the two corresponding networks their general impedance expressions which are found to have the same functional form in the two components and differ only in the constant factors involving the network parameters. These corresponding factors in the two expressions are then equated to make the two impedances identical at all frequencies and it is this set of equations which leads to the relations between the parameters of the two networks. The list of structures given in Appendix III covers the usual transformations in practice and could be extended by adding more and more elements.

Among other types of possible substitutions are obviously those involving a change from three star-connected (T) to three delta-connected (Π) similar impedances, or vice versa, and from three star- or delta-connected inductances to a transformer with mutual impedance. As a simple illustration consider the mid-series band pass wave-filter VI_1 having series inductance and capacity and shunt capacity which can be put in the form of series inductances connecting a series of three star-connected capacities. Changing these capacities into the delta form gives a recurrent structure in which inductances alternate with capacities for the series impedances and capacities form the shunt impedances. Similarly V_1 may be changed to a structure in which inductances alternate with capacities for the series impedances and inductances form the shunt impedances. Another structure for the latter is a series of transformers connected by series capacities.

Composite Band Pass Wave-Filter Illustration

A band pass wave-filter has been chosen to show what can be accomplished by means of a composite structure towards realizing the ideal of attenuation and impedance characteristics. The transmitting band and impedance are specified by

$$f_1=4{,}000\sim,$$
$$f_2=7{,}000\sim,$$
and
$$R=600 \text{ ohms}.$$

The sections arbitrarily taken to make up the structure are one each of the following:

$$IV_1, \text{ M-type}, m=.6, (f_{1\infty}=3739\sim, f_{2\infty}=7489\sim),$$
$$X_1, f_{2\infty}=8300\sim,$$
and
$$XI_1, f_{1\infty}=3300\sim,$$

where a half section of the M-type is placed at each end so as to give the network a symmetrical terminal characteristic impedance of

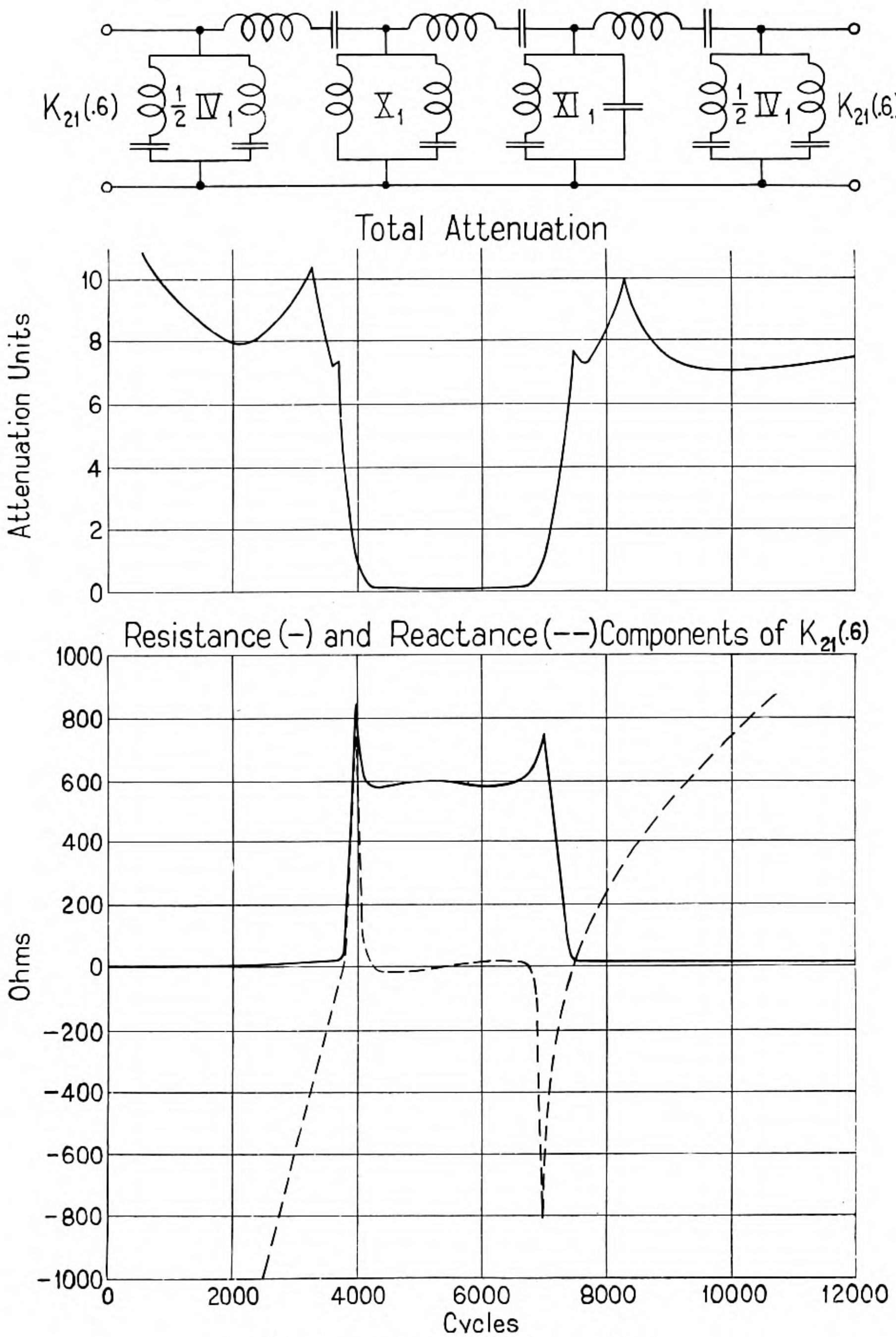


Fig. 10—Composite Band Pass Wave-Filter.

$K_{21}(m=.6)$, as in Fig. 10. Dissipation in the inductances is included by assuming effective coil resistance $= \frac{1}{100}$ coil reactance; it has the effect of eliminating abrupt changes in the attenuation and impedance characteristics. Computations made on this basis give the sum of the three attenuation constants and the impedance $K_{21}(.6)$ as shown in the figure.

The attenuation over a range of 2500 cycles about the center of the transmitting band is less than .19 attenuation units and for frequencies in the attenuating band is high, remaining after the first maximum on either side of the transmitting band above a value 7.30 in the lower frequency attenuating band and a value 7.10 in the upper. The characteristic impedance $K_{21}(.6)$ over the 2500 cycle range is everywhere within 3% of the desired resistance value, 600 ohms, and has here a negligible reactance. Its resistance component has maxima at the critical frequencies and decreases rapidly to small values in both attenuating bands. The reactance component is negative like a capacity reactance at very low frequencies, has a positive maximum at the lower critical frequency and negative minimum at the upper critical frequency, and is positive like an inductive reactance at very high frequencies. This demonstrates the possibilities of the composite structure method.

APPENDIX I

Derivation of Fundamental Formulae

Although formulae for the propagation constant and characteristic impedances of the 'adder type of recurrent network are well known and follow readily from a consideration of the current and voltage relations shown in Fig. 1, it is perhaps of interest to derive them as a special case of general formulae which involve admittances[3] and which are directly applicable to any type of recurrent passive structure including loaded lines.

Let the periodic section of the recurrent structure be defined by the one-point and two-point admittances A_{aa}, A_{bb}, and A_{ab}, where the subscripts $_a$ and $_b$, respectively, refer to its two pairs of terminals. Then the current at the junction, q, in terms of the voltages at the junctions $q-1$, q, and $q+1$, is

$$I_q = A_{ab} V_{q-1} - A_{bb} V_q = A_{aa} V_q - A_{ab} V_{q+1}, \qquad (1)$$

[3] The solution by Difference Equations in terms of the admittances was suggested by J. R. Carson and is a convenient form for expressing the general results.

whence

$$(A_{aa}+A_{bb})V_q-A_{ab}(V_{q-1}+V_{q+1})=0 \tag{2}$$

which is the Difference Equation of Propagation.

Letting

$$V_q=Me^{-q\Gamma}+Ne^{q\Gamma}, \tag{3}$$

equation (2) becomes

$$(A_{aa}+A_{bb}-2A_{ab}\cosh\Gamma)V_q=0$$

which gives, for all values of V_q,

$$\cosh\Gamma=\frac{A_{aa}+A_{bb}}{2A_{ab}}.$$

Since equations (1) when combined give

$$I_q=\tfrac{1}{2}(A_{aa}-A_{bb})V_q+\tfrac{1}{2}A_{ab}(V_{q-1}-V_{q+1}),$$

we have upon the substitution of (3)

$$I_q=\frac{1}{K_a}Me^{-q\Gamma}-\frac{1}{K_b}Ne^{q\Gamma},$$

wherein the characteristic impedances K_a and K_b, as defined by the equation, are

$$\left.\begin{matrix}\dfrac{1}{K_a}\\ \dfrac{1}{K_b}\end{matrix}\right\}=A_{ab}\sinh\Gamma\pm\tfrac{1}{2}(A_{aa}-A_{bb}).$$

In terms of the admittances then

$$\cosh\Gamma=\frac{A_{aa}+A_{bb}}{2A_{ab}},$$

and (4)

$$\left.\begin{matrix}K_a\\ K_b\end{matrix}\right\}=\tfrac{1}{2}\left(\frac{A_{aa}+A_{bb}}{A_{aa}A_{bb}-A_{ab}^2}\right)\left\{\sqrt{1-\left(\frac{2A_{ab}}{A_{aa}+A_{bb}}\right)^2}\mp\left(\frac{A_{aa}-A_{bb}}{A_{aa}+A_{bb}}\right)\right\}.$$

These formulae can readily be expressed in terms of the impedances Z_{aa}, Z_{bb}, and Z_{ab}; or in terms of the three star-connected (T) or three delta-connected (Π) impedances which may represent the section.

Another general formula for the propagation constant which is sometimes convenient may be derived as follows. Assume that the recurrent structure is open-circuited at the junction q; then in (1) $I_q=0$, so that

$$\frac{V_{q-1}}{V_q}=\frac{A_{bb}}{A_{ab}}=\frac{1}{v_{ab}},$$

and

$$\frac{V_{q+1}}{V_q}=\frac{A_{aa}}{A_{ab}}=\frac{1}{v_{ba}},$$

in which v_{ab} and v_{ba} represent the transfer voltage ratios, taken in the two directions, of an open-circuited section. By (4) we find that

$$\cosh \Gamma = \tfrac{1}{2}\left(\frac{1}{v_{ab}} + \frac{1}{v_{ba}}\right). \tag{5}$$

Hence, *the hyperbolic cosine of the propagation constant in a section of any recurrent network is the arithmetic mean of the reciprocals of the two transfer voltage ratios of an open-circuited section.*

For a symmetrically terminated section

$$A_{aa} = A_{bb} = A_o,$$

$$A_{ab} = A_T,$$

$$K_a = K_b = K,$$

and

$$v_{ab} = v_{ba} = v_T.$$

Hence,

$$\cosh \Gamma = \frac{A_o}{A_T} = \frac{1}{v_T},$$

and

$$K = \frac{1}{A_o\sqrt{1-\left(\frac{A_T}{A_o}\right)^2}} = \frac{1}{A_o\sqrt{1-v_T^2}}. \tag{6}$$

In the ladder type of Fig. 1 consider first a mid-series section. For this

$$A_o = \frac{\frac{1}{2}z_1 + z_2}{z_1 z_2 + \frac{1}{4}z_1^2},$$

$$A_T = \frac{z_2}{z_1 z_2 + \frac{1}{4}z_1^2}.$$

Then

$$\cosh \Gamma = 1 + \tfrac{1}{2}\frac{z_1}{z_2},$$

and

$$K_1 = \sqrt{z_1 z_2 + \tfrac{1}{4}z_1^2}. \tag{7}$$

For a mid-shunt section

$$A_o = \frac{z_1 + 2z_2}{2z_1 z_2},$$

and

$$A_T = \frac{1}{z_1},$$

giving necessarily the same propagation constant formula as in (7) and

$$K_2=\frac{z_1z_2}{\sqrt{z_1z_2+\frac{1}{4}z_1^2}}=\frac{z_1z_2}{K_1}. \tag{8}$$

The two formulae

$$e^{-\Gamma}=\frac{2K_1-z_1}{2K_1+z_1}=\frac{2z_2-K_2}{2z_2+K_2} \tag{9}$$

may be verified by substitution in (7) and (8).

In the lattice type of Fig. 6 the admittances are

$$A_o=\frac{z_1'+4z_2'}{4z_1'z_2'},$$

and

$$A_T=\frac{4z_2'-z_1'}{4z_1'z_2'},$$

which lead simply to formulae

$$\cosh\Gamma'=1+\frac{2z_1'}{4z_2'-z_1'},$$

and

$$K'=\sqrt{z_1'z_2'}. \tag{10}$$

Properties of Reactances

The first half of Theorem 1 on non-dissipative reactance networks, stated in Part I and relating to the positive slope of reactance with frequency, can be shown easily by the method of induction where the reactance network is the usual case of series and parallel combinations of inductances and capacities, considered as non-dissipative. Let z' and z'' be two impedances, and let z_S and z_P be the impedances of their combinations in series and in parallel, respectively. It follows that their derivatives with respect to frequency have the relations

$$\frac{dz_S}{df}=\frac{dz'}{df}+\frac{dz''}{df},$$

$$\frac{dz_P}{df}=\frac{1}{\left(1+\frac{z'}{z''}\right)^2}\frac{dz'}{df}+\frac{1}{\left(1+\frac{z''}{z'}\right)^2}\frac{dz''}{df}. \tag{11}$$

These show that if z' and z'' are reactances having positive slopes with frequency, z_S and z_P will also have positive slopes. Beginning then with the two simplest elements known to have positive reactance slopes, a single inductance and a single capacity, we may combine them and add others in any series and parallel combinations with the result of a positive total reactance slope in every case, due to the above relations. That this property is not limited to such combinations is seen from the general impedance expression for a non-dissipative reactance network,[4]

$$z = iM\frac{f(f_2^2-f^2)\ .\ .\ .\ (f_{2n}^2-f^2)^u}{(f_1^2-f^2)\ .\ .\ .\ (f_{2n-1}^2-f^2)}. \tag{12}$$

Here M is a positive real, and the resonant and anti-resonant frequencies, $f_1 \ldots f_{2n}$, alternate and are in the order of increasing magnitude. The exponent, u, is unity or zero according as a resonant or an anti-resonant frequency is the last of the series. Assuming without loss of generality that f_1 is not zero, the reactance increases with frequency from zero frequency up to $f=f_1$, since all the factors are positive. As f passes thru this anti-resonant frequency the reactance changes abruptly from positive to negative infinity and when f increases to the resonant frequency f_2 the negative reactance increases to zero. As f increases beyond the value f_2 the reactance is again positive and the cycle of reactance changes with frequency begins over again.

The possibility of representing such a general reactance identically at all frequencies by a network constructed of either a number of simple resonant components in parallel, or simple anti-resonant components in series, follows from the fact that in any particular case the number of inductances and capacities involved is always equal to the total number of conditions which this network must satisfy to obtain such equality. Thus, its reactance must be zero and infinite at the given resonant and anti-resonant frequencies, respectively, and must have a definite magnitude at some one other frequency, which conditions are sufficient to determine all the impedance elements. In general, other equivalent combinations of inductances and capacities are also possible.

Theorem 2, relating to inverse networks, will be proved by an inductive method in which the given reactance network is assumed to have the form of series and parallel combinations of inductances and capacities, a form which by the first theorem can be taken to represent the reactance of any non-dissipative reactance network. Let z_1' and z_2' be one pair of impedances which are inverse networks of impedance product D^2 to each other, and let z_1'' and z_2'' be another pair so that

$$z_1'z_2' = z_1''z_2'' = D^2 = \text{a constant positive real.}$$

Then z_1' and z_1'' *in series*, and z_2' and z_2'' in *parallel* are a pair of inverse

[4] See paper by G. A. Campbell, Vol. I, No. 2, p. 30, this Journal.

networks of impedance product D^2. This is readily shown, for here we have

$$(z_1'+z_1'')\left(\frac{z_2'z_2''}{z_2'+z_2''}\right)=\frac{(z_1''z_2'')z_2'+(z_1'z_2')z_2''}{z_2'+z_2''}=D^2.$$

Similarly z_1' and z_1'' *in parallel*, and z_2' and z_2'' *in series* are another pair of impedance product D^2.

The simplest pair of inverse networks in the case of reactances is an inductance and a capacity. If in an elementary application of the above relations the element L_1' corresponds to z_1', C_2' to z_2', C_1'' to z_1'', and L_2'' to z_2'', where then

$$\frac{L_1'}{C_2'}=\frac{L_2''}{C_1''}=D^2, \tag{13}$$

it follows that L_1' and C_1'' *in series* or *in parallel*, and L_2'' and C_2' *in parallel* or *in series*, respectively, are inverse networks of impedance product D^2. By successive applications of these relations we may construct any given reactance and its inverse network.

It should be mentioned that these inverse network relations are even more general than has been considered above, for an elemental pair of inverse networks, besides an inductance and a capacity, is two resistances.

Phase Constant

To show that the phase constant increases with frequency throughout each transmitting band of a wave-filter, we may proceed as follows, basing the proof primarily upon the fact that the slopes with frequency of non-dissipative reactances are positive. Consider a mid-series section of the ladder type z_1, z_2. The impedances as measured across one pair of terminals when the other pair is open or short-circuited are, respectively,

$$Z_o=\tfrac{1}{2}z_1+z_2,$$

and

$$Z_s=\tfrac{1}{2}z_1+\frac{z_1z_2}{z_1+2z_2},$$

whose derivatives with respect to frequency may be written

$$\frac{dZ_o}{df}=i\,s^2, \text{ and } \frac{dZ_s}{df}=i\,t^2,$$

where s^2 and t^2 represent essentially positive quantities in accordance with the above underlying fact.

The general propagation constant formula is

$$\cosh(A+i\,B)=\cosh A\,\cos B+i\sinh A\,\sin \text{B},$$

$$=1+\tfrac{1}{2}\frac{z_1}{z_2}=1+\tfrac{1}{2}\frac{r_1+i\,x_1}{r_2+i\,x_2}, \qquad (14)$$

$$=\frac{r_2\,(r_1+2r_2)+x_2\,(x_1+2x_2)}{2(r_2^2+x_2^2)}+i\frac{(r_2x_1-r_1x_2)}{2(r_2^2+x_2^2)};$$

r_1, r_2 and x_1, x_2 being the resistance and reactance components of the two impedances, z_1 and z_2.

In the transmitting bands of non-dissipative wave-filters where $r_1=r_2=o$, the formula becomes

$$\cos B=1+\tfrac{1}{2}\,\frac{x_1}{x_2}.$$

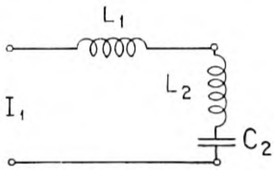
Differentiating this equation and introducing the above facts, we obtain for the rate of change of the phase constant with frequency

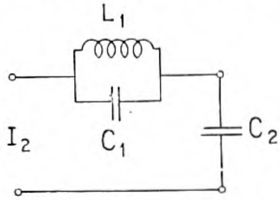
$$\frac{dB}{df}=\frac{(1-\cos B)}{x_1\sin B}\,(s^2\sin^2 B+t^2\cos^2 B). \qquad (15)$$

This rate obviously has the same sign as the denominator which we shall show is positive. For, the non-dissipative wave-filter being considered as the limiting case of one having small positive dissipation, we may temporarily return to the general propagation constant relations (14) in which r_1 and r_2 are assumed to be positive infinitesimals. Then in the limit when $r_1=r_2=0$, since also x_1 and x_2 are of opposite signs in any transmitting band, it follows that x_1 and $\sin B$ are of the same sign and that their product is positive.

APPENDIX II.

I.—General Low Pass Wave-Filters of Ladder Type



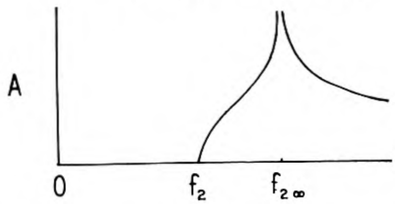


$L_1 = mL_{1k}, \quad L_2 = cL_{1k}, \quad C_2 = mC_{2k}.$

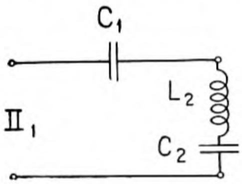
$L_1 = mL_{1k}, \quad C_2 = mC_{2k}, \quad C_1 = cC_{2k}.$

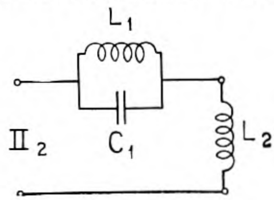
$$L_{1k} = \frac{R}{\pi f_2}, \qquad C_{2k} = \frac{1}{\pi f_2 R},$$

$$m \equiv [m_1] = \sqrt{1 - \frac{f_2^2}{f_{2\infty}^2}}, \qquad c = \frac{1-m^2}{4m}.$$



II.—General High Pass Wave-Filters of Ladder Type



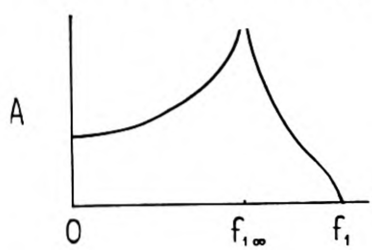


$C_1 = \frac{C_{1k}}{m}, \quad L_2 = \frac{L_{2k}}{m}, \quad C_2 = \frac{C_{1k}}{b}.$

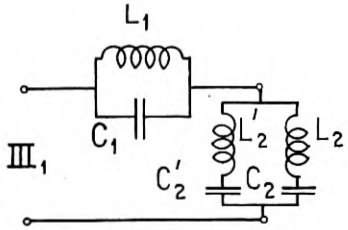
$L_1 = \frac{L_{2k}}{b}, \quad L_2 = \frac{L_{2k}}{m}, \quad C_1 = \frac{C_{1k}}{m}.$

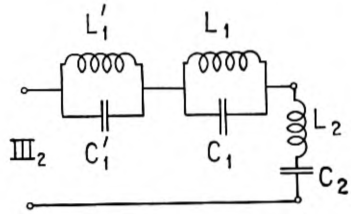
$$C_{1k} = \frac{1}{4\pi f_1 R}, \qquad L_{2k} = \frac{R}{4\pi f_1},$$

$$m \equiv [m_2] = \sqrt{1 - \frac{f_{1\infty}^2}{f_1^2}}, \qquad b = \frac{1-m^2}{4m}.$$



III. General Low-and-High Pass Wave-Filters of Ladder Type





$$L_1 = mL'_{1k}, \quad L_2 = a'L_{2k},$$

$$C_1 = \frac{C_{1k}}{m}, \quad C_2 = \frac{C'_{2k}}{b'},$$

$$L_2' = e'L_{2k},$$

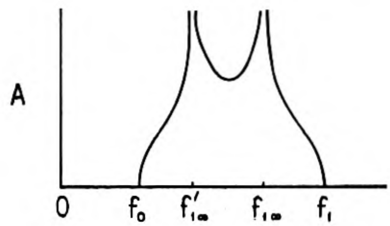
$$C_2' = \frac{C'_{2k}}{f'}.$$

$$L_1 = \frac{L_{1k}}{b'}, \quad L_2 = \frac{L_{2k}}{m},$$

$$C_1 = a'C_{1k}, \quad C_2 = mC'_{2k},$$

$$L_1' = \frac{L_{1k}}{f'},$$

$$C_1' = e'C_{1k}.$$



$$L'_{1k} \equiv [rL_{1k}] = \frac{(f_1 - f_0)R}{\pi f_0 f_1}, \qquad L_{2k} = \frac{R}{4\pi(f_1 - f_0)},$$

$$C_{1k} = \frac{1}{4\pi(f_1 - f_0)R}, \qquad C'_{2k} \equiv [rC_{2k}] = \frac{f_1 - f_0}{\pi f_0 f_1 R},$$

$$m \equiv [m_2] = \frac{\sqrt{\left(1 - \frac{f_0^2}{f_{1\infty}^2}\right)\left(1 - \frac{f_{1\infty}^2}{f_1^2}\right)}}{1 - \frac{f_0}{f_1}},$$

$$a' \equiv \left[\frac{4a(f_1 - f_0)^2}{rf_0 f_1}\right] = \frac{1}{m}\left(1 + \frac{f_0 f_1}{f_{1\infty}^2}\right) = \left[\frac{4f(f_1 - f_0)^2}{f_0 f_1}\right] \equiv f',$$

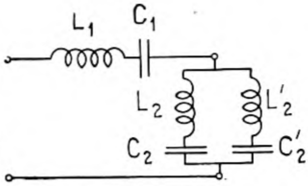
$$b' \equiv \left[\frac{4b(f_1 - f_0)^2}{f_0 f_1}\right] = \frac{1}{m}\left(1 + \frac{f_{1\infty}^2}{f_0 f_1}\right) = \left[\frac{4e(f_1 - f_0)^2}{rf_0 f_1}\right] \equiv e',$$

$$f'_{1\infty} = \frac{f_0 f_1}{f_{1\infty}}.$$

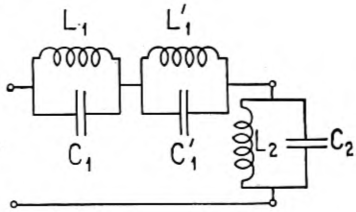
IV. General Band Pass Wave-Filters of Ladder Type

Six Element Structures

IV₁



IV₂



$$L_1=m_1L_{1k},\qquad L_2=aL_{1k},$$

$$C_1=\frac{C_{1k}}{m_2},\qquad C_2=\frac{C_{1k}}{b},$$

$$L_2'=cL_{1k},$$

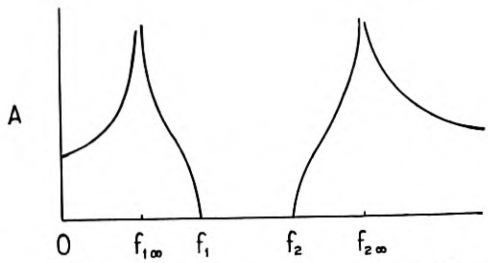
$$C_2'=\frac{C_{1k}}{d}.$$

$$L_1=\frac{L_{2k}}{b},\qquad L_2=\frac{L_{2k}}{m_2},$$

$$C_1=aC_{2k},\qquad C_2=m_1C_{2k},$$

$$L_1'=\frac{L_{2k}}{d},$$

$$C_1'=cC_{2k}.$$



$$L_{1k}=\frac{R}{\pi(f_2-f_1)},\qquad L_{2k}=\frac{(f_2-f_1)R}{4\pi f_1f_2},$$

$$C_{1k}=\frac{f_2-f_1}{4\pi f_1f_2R},\qquad C_{2k}=\frac{1}{\pi(f_2-f_1)R},$$

$$g=\sqrt{\left(1-\frac{f_{1\infty}^2}{f_1^2}\right)\left(1-\frac{f_{1\infty}^2}{f_2^2}\right)},\qquad h=\sqrt{\left(1-\frac{f_1^2}{f_{2\infty}^2}\right)\left(1-\frac{f_2^2}{f_{2\infty}^2}\right)},$$

$$m_1=\frac{\dfrac{f_1f_2}{f_{2\infty}^2}g+h}{1-\dfrac{f_{1\infty}^2}{f_{2\infty}^2}},\qquad m_2=\frac{g+\dfrac{f_{1\infty}^2}{f_1f_2}h}{1-\dfrac{f_{1\infty}^2}{f_{2\infty}^2}},$$

$$a=\frac{(1-m_1^2)\,f_{2\infty}^2}{4gf_1f_2}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right)=\frac{(1-m_2^2)\,f_1f_2}{4gf_{1\infty}^2}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right),$$

$$b=\frac{(1-m_2^2)}{4g}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right),$$

$$c=\frac{(1-m_1^2)}{4h}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right),$$

$$d=\frac{(1-m_1^2)\,f_{2\infty}^2}{4hf_1f_2}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right)=\frac{(1-m_2^2)\,f_1f_2}{4h\;f_{1\infty}^2}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right).$$

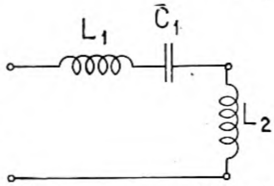
$M-$types: $\quad m=m_1=m_2=\dfrac{h}{1-\dfrac{f_1f_2}{f_{2\infty}^2}},\qquad f_{1\infty}=\dfrac{f_1f_2}{f_{2\infty}},\qquad g=h.$

Band Pass Wave-Filters of Ladder Type

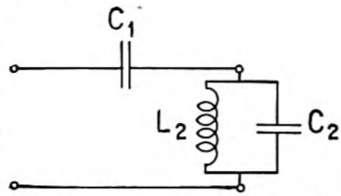
Three Element Structures

V. $f_{1\infty}=0, \quad f_{2\infty}=f_2.$

V₁



V₂

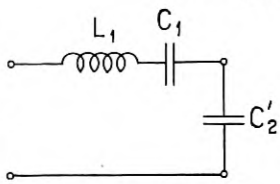


$$L_1=\frac{f_1R}{\pi f_2(f_2-f_1)}, \quad L_2=\frac{(f_1+f_2)R}{4\pi f_1f_2}, \qquad C_1=\frac{f_1+f_2}{4\pi f_1f_2R}, \quad L_2=L_{2k}=\frac{(f_2-f_1)R}{4\pi f_1f_2},$$

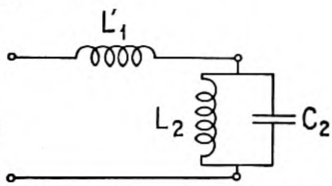
$$C_1=C_{1k}=\frac{f_2-f_1}{4\pi f_1f_2R}. \qquad C_2=\frac{f_1}{\pi f_2(f_2-f_1)R}.$$

VI. $f_{1\infty}=f_1, \quad f_{2\infty}=\infty.$

VI₁



VI₂



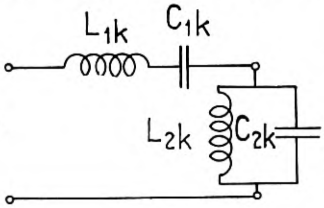
$$L_1=L_{1k}=\frac{R}{\pi(f_2-f_1)}, \quad C_2'=\frac{1}{\pi(f_1+f_2)R}, \quad L_1'=\frac{R}{\pi(f_1+f_2)}, \quad L_2=\frac{(f_2-f_1)R}{4\pi f_1^2},$$

$$C_1=\frac{f_2-f_1}{4\pi f_1^2R}. \qquad C_2=C_{2k}=\frac{1}{\pi(f_2-f_1)R}.$$

Four Element Structures

VII. "Constant k," $f_{1\infty}=0, \quad f_{2\infty}=\infty, \quad k=R.$

VII₁ = VII₂

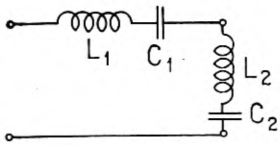


$$L_{1k}=\frac{R}{\pi(f_2-f_1)}, \quad L_{2k}=\frac{(f_2-f_1)R}{4\pi f_1f_2},$$

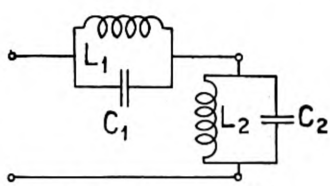
$$C_{1k}=\frac{f_2-f_1}{4\pi f_1f_2R}, \quad C_{2k}=\frac{1}{\pi(f_2-f_1)R}.$$

VIII. $f_{2\infty}=f_2.$

VIII₁



VIII₂



$$L_1=m_1L_{1k}, \quad L_2=\frac{(1-m_1^2)}{4m_1}L_{1k}, \qquad L_1=\frac{4m_2}{1-m_2^2}L_{2k}, \quad L_2=\frac{L_{2k}}{m_2},$$

$$C_1=\frac{C_{1k}}{m_2}, \quad C_2=\frac{4\,m_2}{1-m_2^2}C_{1k}. \qquad C_1=\frac{(1-m_1^2)}{4\,m_1}C_{2k}, \quad C_2=m_1C_{2k}.$$

$$m_1=\frac{f_1}{f_2}m_2, \qquad m_2=\sqrt{\frac{1-\frac{f_{1\infty}}{f_1^2}}{1-\frac{f_{1\infty}^2}{f_2^2}}}.$$

Four Element Structures—(Continued)

IX. $f_{1\infty} = f_1$.

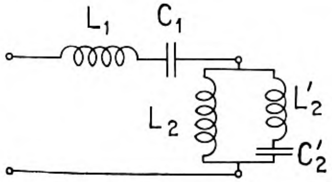
Same Structural Forms and L. C. Formulae as in $VIII_1$ and $VIII_2$.

$$m_1 = \sqrt{\frac{1 - \frac{f_2^2}{f_{2\infty}^2}}{1 - \frac{f_1^2}{f_{2\infty}^2}}}, \qquad m_2 = \frac{f_1}{f_2} m_1.$$

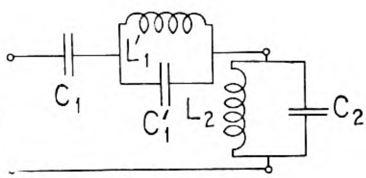
Five Element Structures

X. $f_{1\infty} = 0$.

X_1



X_2



$L_1 = m_1 L_{1k}$, $L_2 = a L_{1k}$,

$C_1 = C_{1k}$, $L_2' = \frac{(1-m_1^2)}{4h} L_{1k}$,

$C_2' = \frac{h}{a} C_{1k}$.

$C_1 = a C_{2k}$, $L_2 = L_{2k}$,

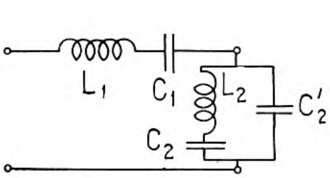
$L_1' = \frac{h}{a} L_{2k}$, $C_2 = m_1 C_{2k}$,

$C_1' = \frac{(1-m_1^2)}{4h} C_{2k}$.

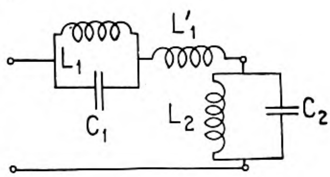
$$h = \sqrt{\left(1 - \frac{f_1^2}{f_{2\infty}^2}\right)\left(1 - \frac{f_2^2}{f_{2\infty}^2}\right)}, \qquad m_1 = \frac{f_1 f_2}{f_{2\infty}^2} + h \qquad a = \frac{(1-m_1^2) f_{2\infty}^2}{4 f_1 f_2}.$$

XI. $f_{2\infty} = \infty$.

XI_1



XI_2



$L_1 = L_{1k}$, $L_2 = \frac{d}{g} L_{1k}$,

$C_1 = \frac{C_{1k}}{m_2}$, $C_2 = \frac{4g}{1-m_2^2} C_{1k}$,

$C_2' = \frac{C_{1k}}{d}$.

$L_1 = \frac{4g}{1-m_2^2} L_{2k}$, $L_2 = \frac{L_{2k}}{m_2}$,

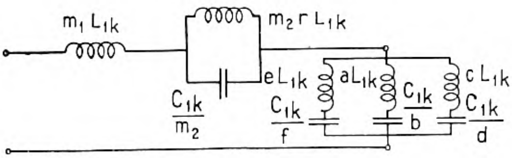
$C_1 = \frac{d}{g} C_{2k}$, $C_2 = C_{2k}$,

$L_1' = \frac{L_{2k}}{d}$.

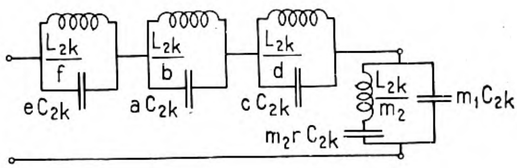
$$g = \sqrt{\left(1 - \frac{f_{1\infty}^2}{f_1^2}\right)\left(1 - \frac{f_{1\infty}^2}{f_2^2}\right)}, \qquad m_2 = g + \frac{f_{1\infty}^2}{f_1 f_2}, \qquad d = \frac{(1-m_2^2) f_1 f_2}{4 f_{1\infty}^2}.$$

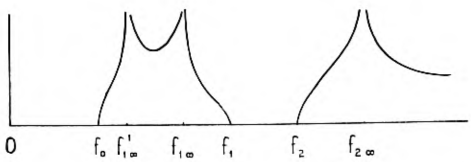
XII.—General Low-and-Band Pass Wave-Filters of Ladder Type

XII₁



XII₂





$$L_{1k}=\frac{R}{\pi(f_0-f_1+f_2)},\qquad L_{2k}=\frac{(f_0-f_1+f_2)^2R}{4\pi\left[(f_0f_1-f_0f_2+f_1f_2)(f_0-f_1+f_2)-f_0f_1f_2\right]},$$

$$C_{1k}=\frac{(f_0-f_1+f_2)^2}{4\pi[(f_0f_1-f_0f_2+f_1f_2)(f_0-f_1+f_2)-f_0f_1f_2]R},\quad C_{2k}=\frac{1}{\pi(f_0-f_1+f_2)R},$$

$$r=(f_0-f_1+f_2)\left(\frac{1}{f_0}-\frac{1}{f_1}+\frac{1}{f_2}\right)-1,$$

$$g=\sqrt{\left(1-\frac{f_0^2}{f_{1\infty}^2}\right)\left(1-\frac{f_{1\infty}^2}{f_1^2}\right)\left(1-\frac{f_{1\infty}^2}{f_2^2}\right)},\quad h=\sqrt{\left(1-\frac{f_0^2}{f_{2\infty}^2}\right)\left(1-\frac{f_1^2}{f_{2\infty}^2}\right)\left(1-\frac{f_2^2}{f_{2\infty}^2}\right)},$$

$$m_1=\frac{\dfrac{f_1f_2}{f_{2\infty}^2}g+h}{1-\dfrac{f_{1\infty}^2}{f_{2\infty}^2}},\qquad m_2=\frac{\left(\dfrac{f_0-f_1+f_2}{f_{2\infty}}-\dfrac{f_0f_1f_2}{f_{2\infty}^3}\right)g+\left(\dfrac{(f_0-f_1+f_2)f_{1\infty}^2}{f_1f_2f_{2\infty}}-\dfrac{f_0}{f_{2\infty}}\right)h}{\left[\left(1-\dfrac{f_0}{f_1}+\dfrac{f_0}{f_2}\right)\left(\dfrac{f_0-f_1+f_2}{f_{2\infty}}\right)-\dfrac{f_0}{f_{2\infty}}\right]\left[1-\dfrac{f_{1\infty}^2}{f_{2\infty}^2}\right]},$$

$$a=\frac{f_0f_1f_2r}{(f_0-f_1+f_2)f_{1\infty}^2}b,$$

$$b=\frac{(f_0-f_1+f_2)^2[(1-m_1^2)f_{1\infty}^4f_{2\infty}^2-f_0^2f_1^2f_2^2]}{4grf_0f_1^2f_2^2[(f_0-f_1+f_2)f_{1\infty}^2-f_0f_1f_2]}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right),$$

$$c=\frac{f_0f_1f_2r}{(f_0-f_1+f_2)f_{2\infty}^2}d,$$

$$d=\frac{(f_0-f_1+f_2)^2[(1-m_1^2)f_{1\infty}^2f_{2\infty}^4-f_0^2f_1^2f_2^2]}{4hrf_0f_1f_2f_{1\infty}^2[(f_0-f_1+f_2)f_{2\infty}^2-f_0f_1f_2]}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right),$$

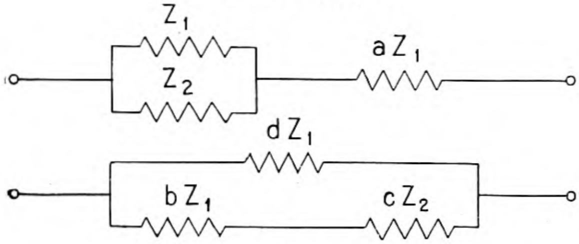
$$e=\frac{(1-m_1^2)f_{1\infty}^2f_{2\infty}^2r}{(f_0-f_1+f_2)f_0f_1f_2}f,\qquad f'_{1\infty}=\frac{f_0f_1f_2}{\sqrt{1-m_1^2}\,f_{1\infty}f_{2\infty}},$$

$$f=\frac{(f_0-f_1+f_2)^2}{4rf_{1\infty}^2f_{2\infty}^2}\cdot\frac{[(1-m_1^2)f_{1\infty}^4f_{2\infty}^2-f_0^2f_1^2f_2^2][(1-m_1^2)f_{1\infty}^2f_{2\infty}^4-f_0^2f_1^2f_2^2]}{[(1-m_1^2)(m_1-h)f_{1\infty}^2f_{2\infty}^4-m_1f_0^2f_1^2f_2^2]\,[(1-m_1^2)f_{1\infty}^2f_{2\infty}^2-(f_0-f_1+f_2)f_0f_1f_2]}.$$

APPENDIX III

Equivalent Networks and Transformation Formulae

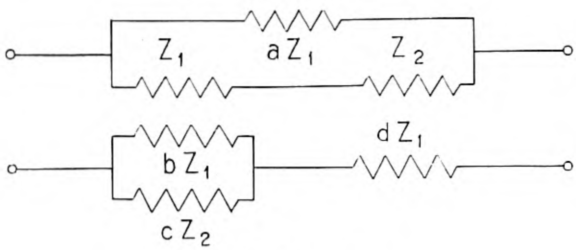
Transformation A



Equivalent when

$$b=a(1+a),\qquad c=(1+a)^2,\qquad d=1+a.$$

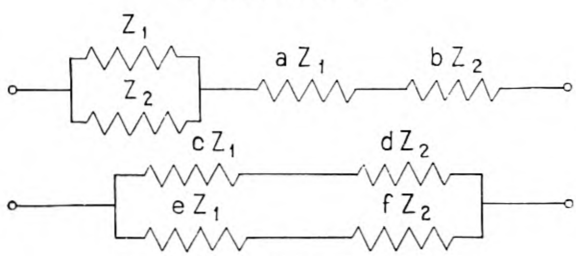
Transformation B



Equivalent when

$$b=\frac{a^2}{1+a},\qquad c=\left(\frac{a}{1+a}\right)^2,\qquad d=\frac{a}{1+a}.$$

Transformation C



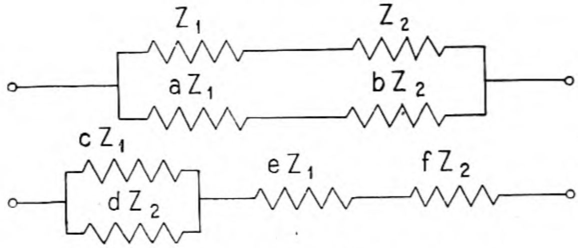
Equivalent when

$$c=\frac{N(M+N)}{M+N-2b},\qquad d=\frac{2bN}{M+N-2b},$$

$$e=\frac{N(M-N)}{N-M+2b},\qquad f=\frac{2bN}{N-M+2b},$$

$$M=1+a+b,\qquad N=\sqrt{(1+a+b)^2-4ab.}$$

Transformation D

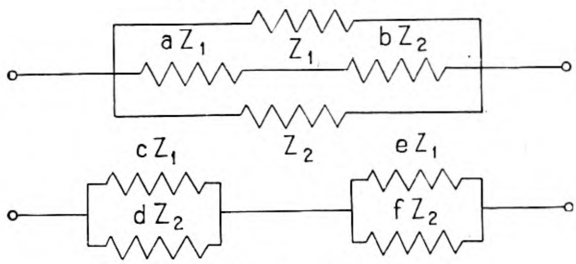


Equivalent when

$$c=\frac{(a-b)^2}{(1+a)(1+b)^2}, \qquad d=\frac{(a-b)^2}{(1+a)^2(1+b)},$$

$$e=\frac{a}{1+a}, \qquad f=\frac{b}{1+b}.$$

Transformation E



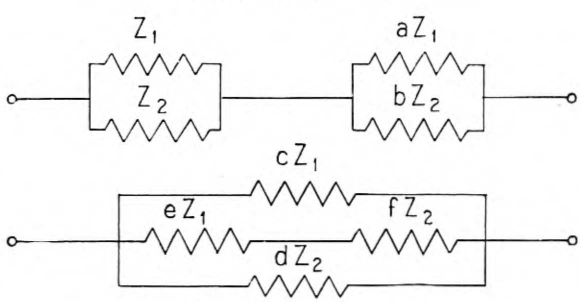
Equivalent when

$$c=\frac{(2b-M+N)(M+N)}{4bN}, \qquad d=\frac{2b-M+N}{2N},$$

$$e=\frac{(M+N-2b)(M-N)}{4bN}, \qquad f=\frac{M+N-2b}{2N},$$

$$M=1+a+b, \qquad N=\sqrt{(1+a+b)^2-4ab}.$$

Transformation F



Equivalent when

$$c=1+a, \qquad d=1+b,$$

$$e=\frac{a(1+a)(1+b)^2}{(a-b)^2}, \qquad f=\frac{b(1+a)^2(1+b)}{(a-b)^2}.$$

Specializing Transportation Equipment in Order to Adapt it Most Economically to Telephone Construction and Maintenance Work

By J. N. KIRK

Introduction

IN this paper is described in a general way the interesting application of motor vehicles and their associated apparatus in connection with outside plant construction and maintenance work, outlining through the successive stages of development what has been accomplished in this respect up to the present time. In order to present a comprehensive picture covering this field of activity, the more primitive types of equipment, together with the manual methods of doing work, are shown in comparison with representative instances of higher development during the past few years in which this phase of the work has been given particular consideration.

The telephone system is different from most public utilities in that it is responsible for a universal service throughout the United States. Wherever the highways and byways may lead, and in many instances where no traveled way could well exist, will be found the familiar and indispensable telephone, with the wire and cable on pole line and in underground conduit. Irrespective of the remoteness of the territory, of the subsurface or the climatic conditions involved, there must always be found a way to construct and maintain the telephone plant. To install this widely distributed plant and continuously safeguard the service in response to the ever increasing public demands, it is essential that facilities be provided for the prompt and safe transportation of quantities of heavy, bulky materials and gangs of men to any point in the telephone system during emergencies as well as under normal conditions, and that provision also be made to supplement the necessary manual operations in every way possible by the proper adaptation of mechanical apparatus.

It might be helpful in this consideration to compare the construction problems of the Telephone Companies with the production problems of any large manufacturing concern. The transportation of raw materials, of the products during manufacture, and of the finished products, together with the application of labor saving machinery in this connection, unquestionably constitute a very real problem to the manufacturer. In this case, however, all of the activities are so completely concentrated and under his control to such an extent as to greatly simplify the efficient and economical operation of all units involved. Let us consider this large, self-contained manufacturing plant completely dismembered, with the various machines and manufacturing processes widely scattered over distances of many miles instead of a few feet, and we have a very fair comparative picture of the relative importance of the Telephone Companies' transportation and construction apparatus problems in providing and maintaining efficient service. Because of this fundamental condition which obtains in the telephone industry, all outside plant machinery units must be portable, of comparatively small capacity and yet of high efficiency.

To meet these exacting requirements the Bell System is ever on the alert to avail itself of every possible advantage in the development, adaptation and application of transportation equipment, machinery and methods. By means of this mechanical equipment the heavy units of material are handled with ease, safety and dispatch by the gangs, leaving them fresh for the lighter detail work requiring dexterity but practically no heavy, straining effort.

When one speaks of automotive and construction apparatus or machinery developments as applied to the telephone business, such developments must naturally appeal to many as being foreign to and rather difficult to closely associate with the furnishing of telephone service. We are, however, in the midst of a truly mechanical age and the more we study and experiment with the adaptation of mechanical equipment to the new lines of telephone activity, the broader seem to be the fields of applicability and the more evident becomes the necessity of closely coordinating the various phases of adapting commercial equipment and developing new types of apparatus for telephone use.

It is the intention in the following to outline a number of the more important developments associated with the adaptation of mechanical methods to outside plant construction and maintenance work. In presenting the picture contrasting the construction methods of today with the earlier practices, one cannot but note the remarkable developments and improvements which have come about.

Transportation Equipment

It is reported that some forty years ago, after deliberating for an entire day the directors of one of the now large Associated Companies decided that the volume and nature of the company's business warranted the purchase of a horse and buggy.

Figure 1 represents such an outfit as was probably purchased and which, in connection with the telephone business of today, is about as rare as the motor vehicle is common-place.

Fig. 1—Horse-Drawn Vehicle in Telephone Service—Courtesy *Telephone Review*

As representative of some fifteen years later we have illustrated in figure 2 the one-horse, light construction wagon, the predecessor of the three-quarter and one ton motor vehicles which now handle light

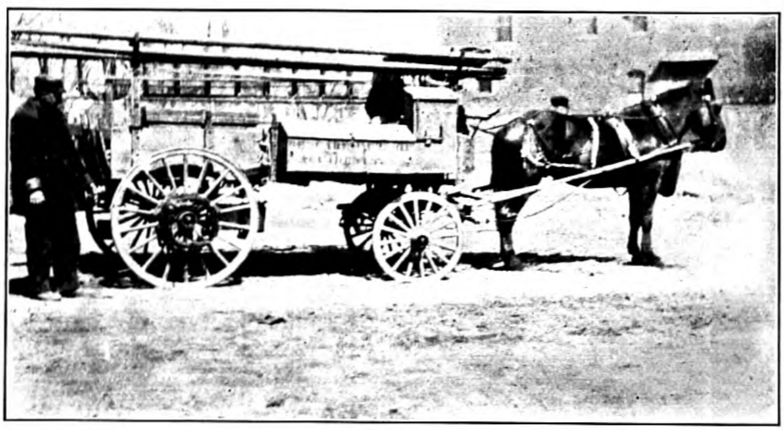


Fig. 2—Light Construction Truck of about 1896

construction, certain classes of station installation work, section line work, etc. It is interesting to note the improvised reel on the rear wheel of the wagon and also the warning "BE CAREFUL OF ACCIDENTS" which is printed on the side of the body. These features are indicative of the fact that the labor saving equipment and "safety first" movements which have now reached such broad proportions in many other industries were recognized as important factors in the System at least as far back as 1896.

Fig. 3—Heavy Construction Truck Carrying Gang, Tools and Materials, 1896

The heavy construction gang unit of 1896 shown in Figure 3, brings to mind the original method of employing large gangs which, with practically no labor saving equipment available, necessarily had to handle the heavy features of outside construction work by "main strength".

In the interval between the advent of the horse-drawn vehicle and that of the motor vehicle into the telephone business, bicycles were used to some extent. These comparatively slow, energy consuming vehicles, however, soon were superseded by the motorcycles which for a few years, principally during the period between 1914 and 1920, were considered a very necessary factor and played an important part in connection with the maintenance and, to a lesser extent, the construction of the telephone plant.

Several hundred machines of these types were at one time used by the various companies, but experience has indicated that their use results in high maintenance, that they present many features hazard ous to the employees and general public, and that they are more or less detrimental to the health of those who use them to any great extent.

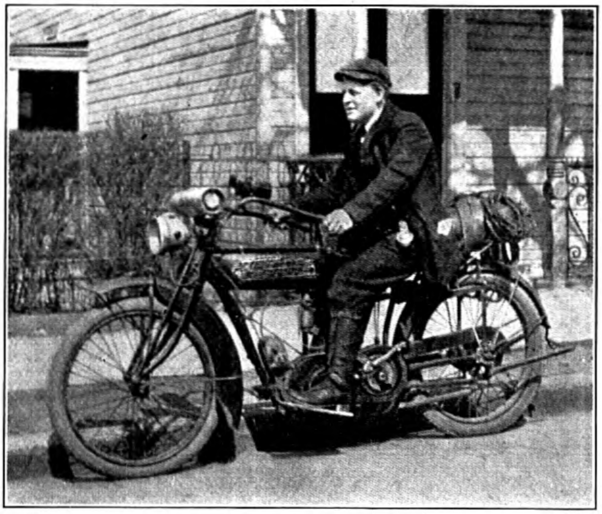


Fig. 4—Motorcycle which was Used for Maintenance Work—Courtesy *Telephone Review*

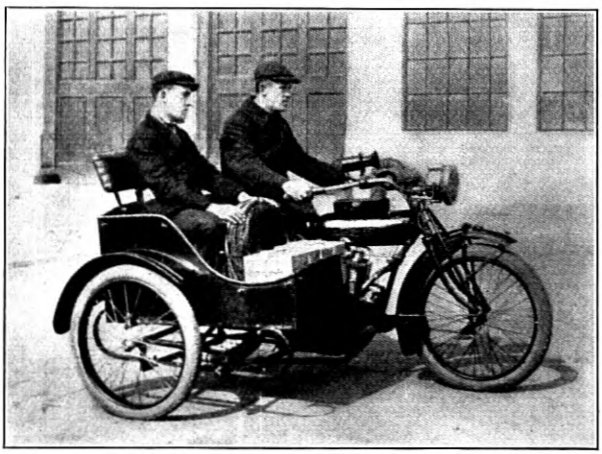


Fig. 5—Motorcycle with Sidecar Used for Instrument Installation Work—Courtesy *Telephone Review*

While the motorcycles have the advantage that they can generally worm their way through traffic more readily than an automobile, this advantage is completely overbalanced by the universal tendency to speed in riding motorcycles, by the many serious accidents from skidding on wet pavements, the d'fficulty in riding over roads having deep wheel tracks, the entire lack of weather protection for the rider, and the instability of the sidecar outfits when turning corners. The use of motorcycles by the Telephone Companies is now practically, if not entirely, obsolete.

The many adaptations of the Ford car have proven in over the motorcycle by a large margin from practically every viewpoint. There are now more than 5,000 Fords in the service of the Associated Companies. This group of cars is often referred to in telephone parlance as the "mosquito fleet" and it is interesting to note that the building up of this fleet had its inception as late as about 1914.

Approximately 80 per cent of these Fords are equipped with various types of boxes and specially designed bodies which permit the carrying of light loads of materials and tools. On account of their large numbers, low operating costs and remarkable ability to negotiate almost impassible roads, they go far toward coordinating the operation of the widely scattered units of the Telephone System.

Fig. 6—A Seriously Overloaded Ford Carrying Splicers and Their Supplies

In telephone work the Ford runabouts average approximately 9,000 miles per car per year. Normally, their net loads vary from 150 pounds to 750 pounds, although in emergencies they are sometimes seriously overloaded.

Fig. 6 shows a telephone company Ford seriously overloaded while transporting splicers' equipment. In fact, the net load carried by this particular car, including the four men, was about 1,300 pounds. This illustrates a case where the service for which the vehicle was originally supplied, has outgrown the load-carrying and space capacity of the unit. Of course, if this practice were permitted to continue or become general, it would be expensive, both from a motor vehicle operating and gang service viewpoint, not to mention the hazard presented in carrying two of the men in such a precarious position.

It is apparent that in order to find a particular item of tools or material on this car it might be necessary to completely unload. As regards the effect upon the car, the tires frequently blow out, the front construction requires constant attention to keep it tight, the springs depress to the extent that the fenders are permitted to ride upon the tires, the steering is difficult, etc.

Fig. 7—Ford Truck Equipped with Modern Side Box Body

As soon as it was recognized that this particular service was outgrowing the transportation unit, a special side box body upon a high speed one-ton Ford truck was developed and is now undergoing service trials in order to properly provide a unit having ample space and load-carrying capacity. Fig. 7 shows some of the latest ideas in the design of such an outfit. Note the ample kerosene tank slung under the rear end of the body with a convenient filler pipe on the rear end of the left side box and a faucet under the tank with hose connection for filling the splicers' furnaces with kerosene.

As an illustration of a Ford runabout especially adapted for the work of serving an installer and helper in placing telephone sets together with the inside wiring and the drop wires from pole to house, Fig. 8 is presented.

Fig. 8—Ford Runabout of the Latest Type in Installation Service

It will be noted that in this design the body extension back of the seat is limited in order that only a small weight over-hang back of the rear axle is possible. This is important in order not to over-strain the rear spring. The body design is made as light in weight as practicable in order to provide ample net load carrying capacity.

There are now on the market innumerable Ford accessories which are claimed to correct all of the ills to which the Ford is subject. Careful studies and field trials, however, indicate that by far the greater portion of these devices are of no advantage and many are actually detrimental to the efficiency and safety of operation. However, through careful selection and in some cases modification of certain of these accessories to meet specific telephone service requirements it now seems probable that somewhat more efficient, economical and safer operation will be realized.

Fig. 9—Heavy Construction Gang Truck, 1910. One of the First in Telephone Service

About 1910, carefully prepared studies indicated the practicability and economy of utilizing gasoline driven motor trucks for the transportation of men, supplies and construction equipment of various kinds.

The first automobile trucks were proven in over horse drawn vehicles on the basis of using the trucks as purely transportation units. However, it soon developed that there were many possible economical applications of the motor truck in connection with the placing, moving and removal of pole lines, aerial cables, underground cables, wires, etc., bringing into use the many accessory devices such as winches, derricks, earth boring machines, various types of trailers, pumps and other safety and labor saving apparatus. The importance of some of these devices in telephone construction work will later be described.

The motorizing of the Bell System has been very rapid since 1910. Because of the widely scattered distribution of outside telephone plant it is necessary, in transporting the workmen, together with their tools and materials, to employ in the Bell System approximately

3,000 trucks and tractor-trucks of from ½ ton to 15 tons capacity. These together with the "Mosquito Fleet" and the relatively small number of supervisory passenger cars of a better class, make a total motor vehicle strength of over 8,000 units in the Bell System. In addition to this Company owned equipment, there are employed annually by the Associated Companies several hundred hired motor vehicles.

In the neighborhood of 25,000 employees depend upon the System's transportation equipment as an indispensable part of their daily work, that is, in its capacity of labor saving machinery as well as in moving the men, together with their tools and materials, from their bases of operation to the job and back, and also between jobs. The annual cost of providing this transportation service for the Bell System is in the neighborhood of twelve to fourteen million dollars. Although this total is a sizable amount, it is actually small when compared with the service rendered and when considered upon the basis of slightly less than $6 average cost per car per day used, including all units from 750 to 30,000 pounds net carrying capacities.

Studies are constantly being made in connection with the opportunities presented along the line of increasing the mechanical efficiencies and lowering the maintenance costs of the various units. As the result of this work much is being accomplished in conserving the working time and energy of the men by employing proper labor saving facilities with the motor vehicles in order to do practically all of the slow, heavy work by proper application of power from the motor vehicle engines. The continuation of this field of study should tend toward offsetting the constantly increasing construction costs.

The realization of the most important savings in the motor vehicle field, that is by making the truck units serve the gangs as labor saving machines in addition to their use as transportation equipment, involves the use of winches driven from the truck engines, derricks for all kinds of pole work, for handling loading pots, etc., suitable trailers for transporting poles, reels of cable and other materials, the use of quick acting safe drawbars for trailing loads behind the trucks, the use of the truck equipment for pulling the proper tension into aerial cable strand and for pulling in the aerial cable, the use of the power equipment with suitable accessory appliances for pulling in or removing underground cable, of power driven collapsible reels for quickly pulling down and coiling up open wire, employing improved methods with the assistance of the power equipment for the handling of all heavy loads (such as reels of cable on and off trucks), and for numerous other uses. In addition to these savings, important economies can be realized by equipping the construction units with suitable bodies to meet the various construction requirements.

In reviewing the progress in the use of motor vehicles it is interesting to note that in the first few years it became apparent that in order to properly utilize the units it would be necessary to equip them with special side box bodies, winches, derricks, etc. Designs for these various items of equipment were prepared in accordance with the best information at hand and the resulting units of about 1914-1916

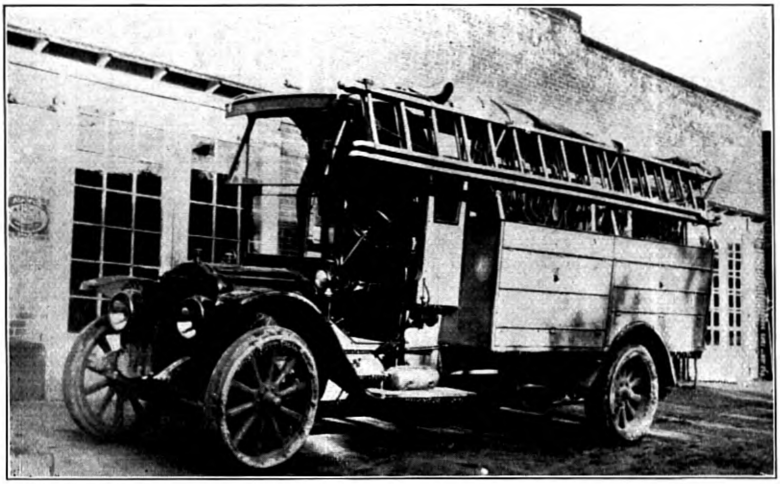


Fig. 10—An Old Type of Side Box Body Construction

were so heavy and bulky that some of the 3-ton trucks carrying this equipment were loaded practically to capacity exclusive of their complement of material, tools and men. This led to the introduction of 5-ton truck units in heavy construction work. Figure 10 shows an old type of body equipment in which the arrangement and size of the side box compartments was such that practically no free load capacity was available. A rear view of this outfit would more clearly indicate the absolute lack of space for carrying materials such as reels of strand and cable, etc.

In the past few years and at the present time, the developments are toward lighter weight, more efficient bodies and labor saving equipment as is illustrated in figure 11 and later in this paper under the discussion of winches.

The use of this equipment is permitting a material reduction in gang sizes which in itself further reduces the weight to be carried on the truck. The net result is that instead of a 3 or 5-ton unit weighing loaded 18,000 or 25,000 pounds, it is possible to handle the work more satisfactorily with 2 or 2½-ton units weighing in the neighborhood of 12,000 pounds.

The advantages gained by this reduction in truck size are large. Not only is the initial and operating cost of the equipment much less but the more important feature is that these 2-ton trucks can penetrate and economically operate in territories where a heavier

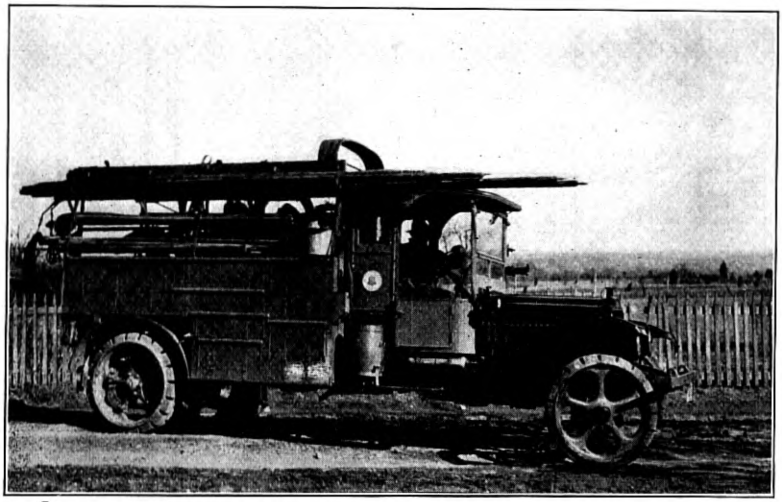


Fig. 11—Latest Type 2-Ton Heavy Construction Unit

unit could not negotiate the roads. Many country bridges will not carry more than 6 tons. Also, on narrow country roads, the comparatively shorter wheel base of the 2-ton truck permits easier turning around or maneuvering.

Figs. 11, 12, 13 and 18 present illustrations of some of the latest developments in line construction truck design and associated equipment. The particular type of body shown has been selected as an example from the various types employed by the Telephone Companies because of its broad use and because it so well illustrates the general development which is taking place. The outfits shown, except in Fig. 13, are of 2 to 2½-ton capacity and perhaps the most outstanding feature is that of the rugged and compact body arrangement, each detail of which has been specially designed to meet a particular construction need. The tool and machinery equipment is applicable to the most exacting requirements of the average outside construction job. The arrangement is such that all necessary tools and materials can be carried in a safe and orderly manner, and the truck power plant, through the introduction of suitable winch equipment, is available for the heaviest duty, slow speed work, as well as the lightest duty, high speed work which may be encountered.

A more complete description of some of the principal features embodied in this combination material distributing, tool and gang delivering unit, power plant and general work shop, may be of interest.

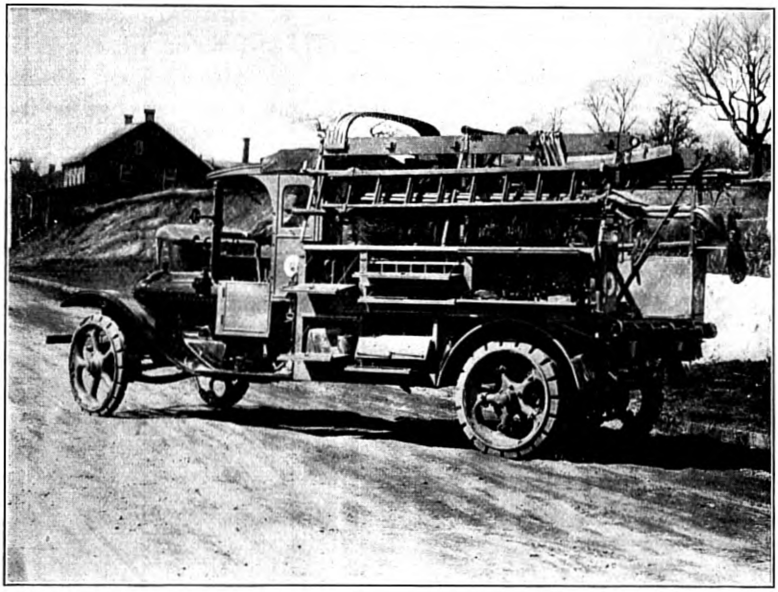


Fig. 12—Latest Type 2-Ton Heavy Construction Unit Showing Tool, Material and Locker Compartments Open

With regard first to some of the more important points incorporated in the body: Every construction crew must carry a large number of different comparatively small materials and tools. The old method of piling the mixed tools and materials in large boxes carried in the truck body led to much lost time on the job in looking for particular items as required in the course of the day's work. The foreman could never be quite sure as to just what he had on his truck, which resulted in two unsatisfactory and uneconomical conditions: First—otherwise unnecessary extra trips were made between the job and the storeroom to secure materials thought to be on the truck but which could not be located when needed. Second—due to the lack of orderly arrangement, much more material was generally carried than was actually needed, which resulted in excessive loads upon the trucks and in the aggregate an unnecessarily large material supply balance for the company.

The new type of body is the result of careful field study. In this particular one, of the several designs necessary to meet the requirements of the subdivisions into which the construction work naturally divides itself, it will be noted that side boxes are provided of such sizes as to satisfactorily house in an orderly manner the small tools and materials, suitable hangers and racks are arranged to carry the larger tools and materials, space is available for chauffeur's chains, tools, grease, etc., and compartments are also provided for the extra clothing and lunches of the men. Safe and readily accessible locations are provided for the heavier equipment, such as members of the pole derrick, digging bars, shovels, ladders, etc. Fig. 13 shows a close up view of the orderly and readily accessible arrangement of tools and materials.

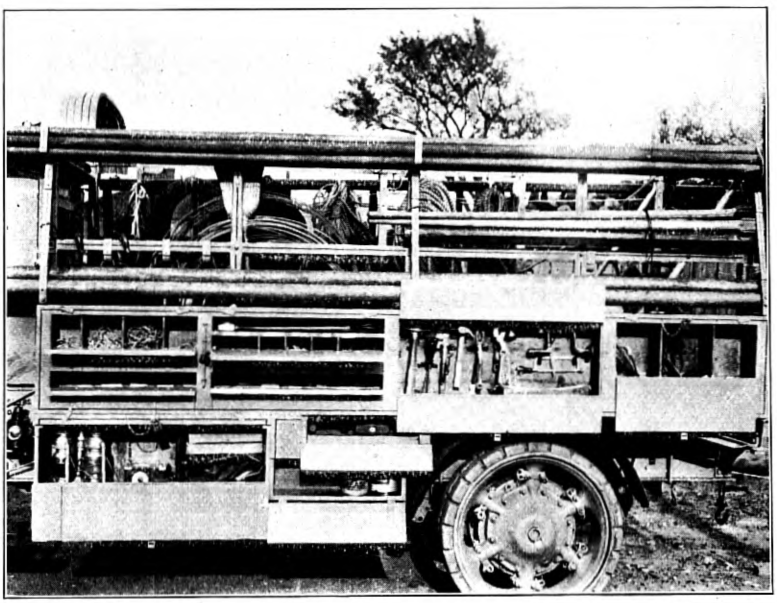


Fig. 13—Side Boxes Opened Showing Arrangement of Tools and Materials

It should be particularly noted in this connection that the truck body arrangement is such that with its full complement of tools and materials there is available a maximum of free load space.

With the further thought of conserving the health of the crew when operating in sections where suitable drinking water cannot be obtained, a sanitary keg is provided for carrying an ample supply of pure water. Paper drinking cups are used.

A safe, clean, dry, convenient location for the "Safety First" kit is built into the top of the cab.

A small vise for the use of the gang and chauffeur is attached to one of the running boards.

The cab also incorporates every possible feature of safety and protection to the driver, and a tarpaulin is so arranged as to provide maximum protection for the men in case of bad weather.

As may be noted, in Fig. 12 a spindle and sheave have been provided which can be mounted across the top body rails either at the rear end or the middle of the truck in order to permit the use of the winch rope for loading and unloading cable reels or reels of strand without the use of skids.

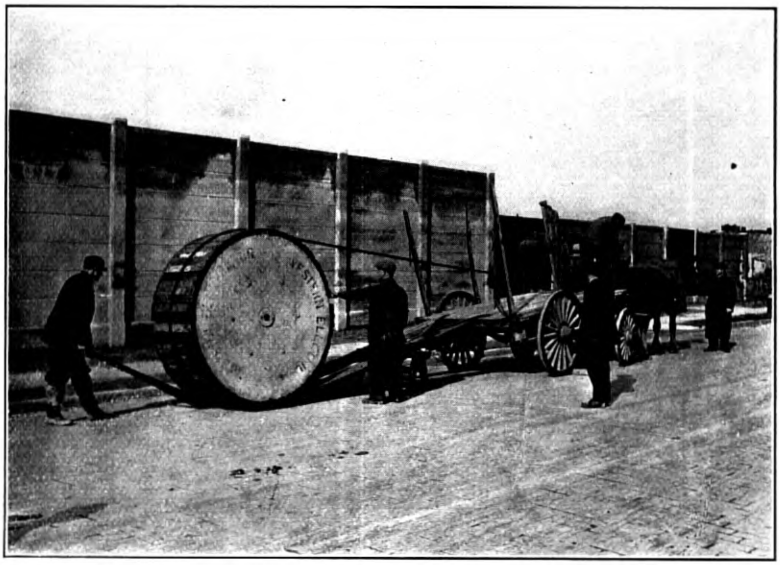


Fig. 14—Loading Cable Reel on Horse-drawn Truck by Means of Two-Man Power Hand Winch

Fig. 14 shows the old manual method of loading a reel of cable on a horse-drawn truck. This operation involves the slow and laborious method of rolling a two- or three-ton load up an inclined plane by means of a hand winch. It should be noted that six men are engaged in handling this reel and that at least two of them must of necessity occupy positions which present more or less hazard in the event that the winch rope should break or some part of the mechanism otherwise fail to hold the suspended load. This familiar method of winch operation by means of a manila rope laboriously wound upon a ratchet stop drum by two men, was limited entirely to loading and unloading heavy items of material from the truck platform. For this purpose it was, however, a great improvement upon former methods even though it was very slow and not entirely free from danger.

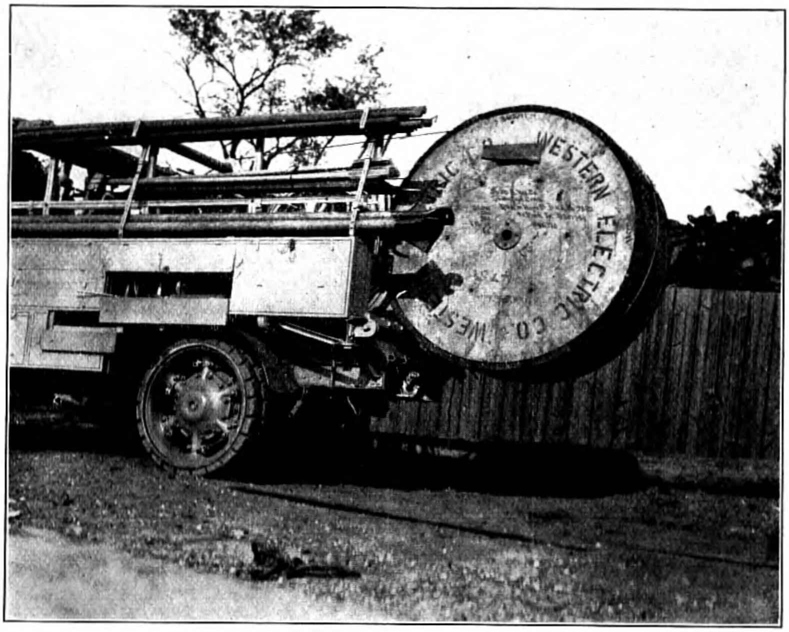


Fig. 15—Loading Cable Reel by Use of Sheave and Spindle with Rope Sling, Without Skids

In Fig. 15 a similar reel of cable is being loaded on a motor truck by means of the engine operated winch in conjunction with the sheave and spindle feature previously mentioned. In this case the possibility of hazard to the workman is completely removed. The reel is loaded in a fraction of the time required by the old manual method and the entire operaton, after adjusting the winch line, is completed by the chauffeur from his position in the cab. In the event that the winch rope or other parts of the mechanism should fail, the result would be a vertical drop of the reel of cable, perhaps slightly damaging the reel, but the employees are not required to take positions where they are in any danger.

Fig. 16 shows the first type of power winch application to telephone construction work. This unit consisted of a slow speed, heavy duty, single cylinder, gasoline engine unit permanently mounted on a-horse-drawn truck. It was used principally for pulling in underground cable and was a great improvement over the former method of pulling by means of horses. It will be noted that on this winch steel rope was used.

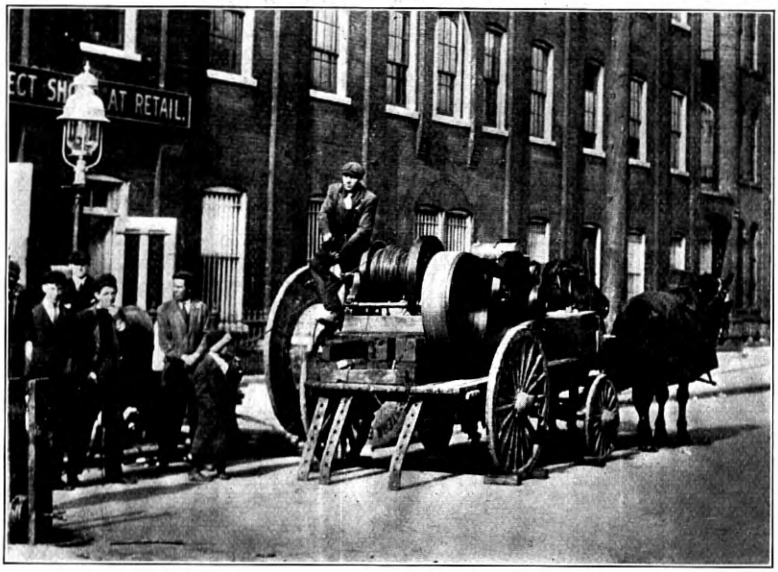


Fig. 16—Old Type Gasoline Engine Driven Winch on Horse-Drawn Truck

With the engine propelled truck came the possibility of utilizing the truck power through a special power take-off to drive a winch which would not only be more powerful, but also much more rapid in action and distinctly superior with regard to the important feature of control.

Fig. 17 illustrates one of the original types of power winch on a 5-ton truck chassis. This was an adaptation of one of the best hoisting winches then available and some ingenious controls were developed at the time in order to facilitate or in fact, even permit of its operation on the trucks.

While this unit was a wonderful labor saving device and opened up the possibilities of the broad field of usefulness for truck operated winches, its size and weight were such that it could not well be used on trucks of less than 5-tons capacity. It will be noted that the winch extends well up to the cab window and would practically fill the front end of the body. Its net weight exclusive of the truck power take-off was 2,300 pounds.

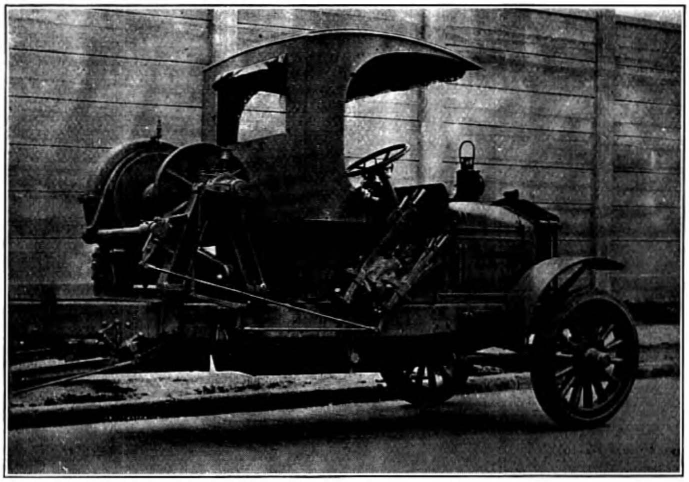


Fig. 17—Old Type Heavy Winch on 5-Ton Truck Chassis

The desirability and in fact the indispensable need of using winches on the smaller trucks has led to the development of a very compact light weight unit which will handle about 900 feet of 7/16″ steel rope and withstand a pull of 10,000 pounds on a single line. Experience indicates, that this winch is capable of meeting the maximum requirements generally encountered in construction work.

The compactness of this winch is illustrated by Fig. 18 which shows it below the cab window with only the upper half of the drum projecting above the floor line in order to give the rope proper clearance in winding and unwinding. This winch weighs slightly less than 500 pounds.

In closing this discussion of motor vehicle application to telephone work it might be of interest to examine the curve in Fig. 19, which shows the rate of growth of the motor vehicle fleet in the Bell System.

This curve prepared from such information as is now available presents a reasonably accurate picture of the motor vehicle development which began in the Bell System as early as 1904.

As explanatory of this curve it may be noted that previous to 1910 very few cars and no trucks were purchased. From 1910 to 1913 various types of equipment were placed in service largely upon an experimental basis. The results of these experimental installations were so favorable that from 1913 to 1919 the growth was very rapid

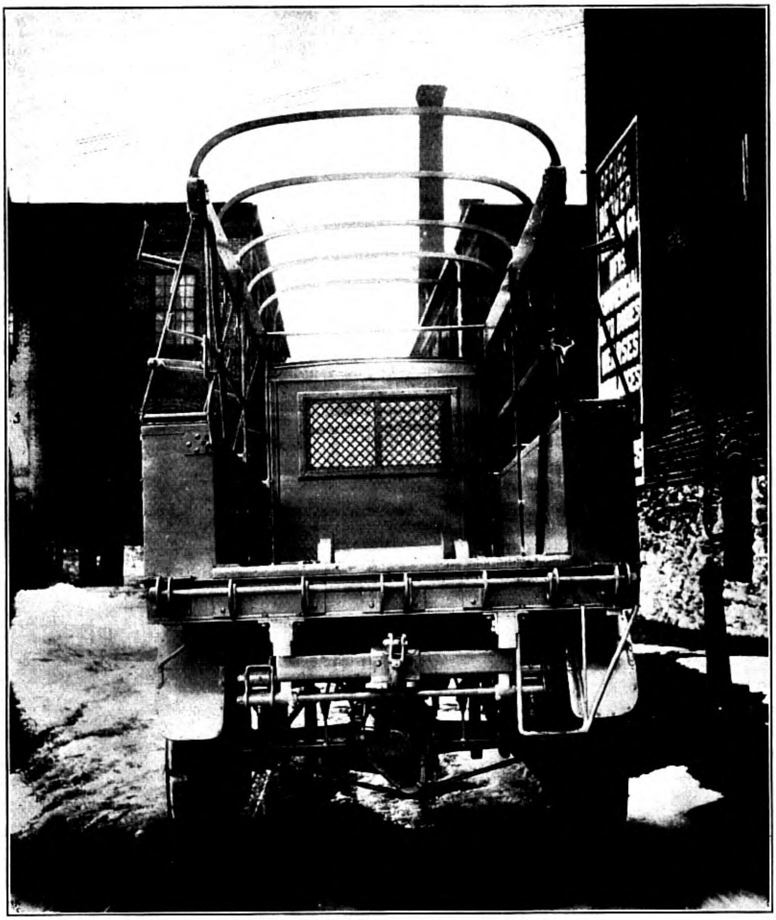


Fig. 18—Rear View of Latest Type 2-Ton Construction Truck Showing Winch Below Cab Window. Generous Clear Body Platform Space is Evident

due to superseding the large number of horse-drawn trucks with motor vehicles as well as providing additional motor vehicles to keep pace with the growth of the telephone industry. From 1919 to 1922 the slope of the curve indicates a slow, steady growth which corresponds with the growth in requirements of the telephone construction and maintenance organizations in handling their steadily increasing activities.

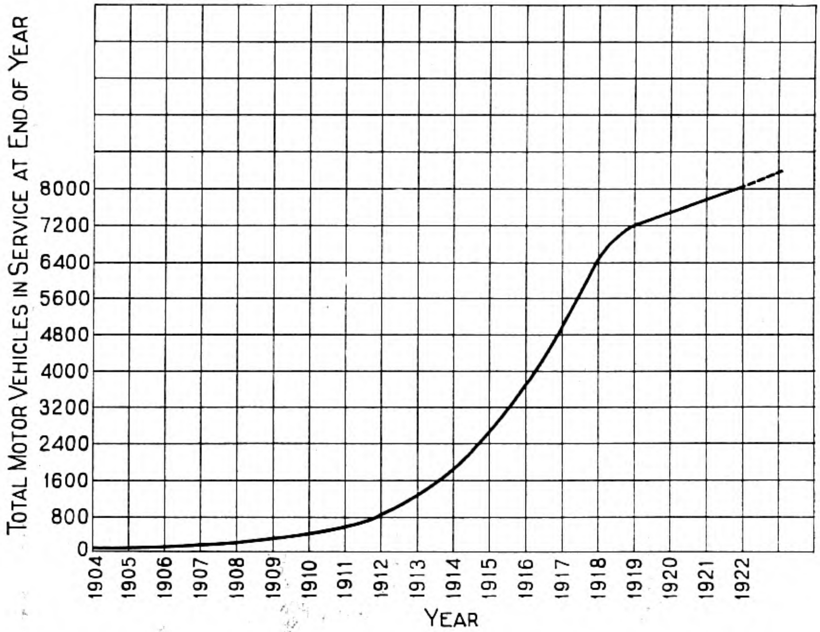


Fig. 19—Curve Showing Growth of Motor Vehicle Fleet in Bell System

From the foregoing it will be noted that a period of 40 years has witnessed a striking development in transportation and associated equipment as applied to telephone construction work, and studies now under way indicate that there is yet much to be accomplished.

In a future issue will be discussed the adaptation to telephone work of the more important items of labor saving machinery such as pole derricks, trailers of various types, earthboring machines, air compressors and compressed air tools, etc.

Telephone Transmission Over Long Cable Circuits

By A. B. CLARK

SYNOPSIS: The application of telephone repeaters has made it possible to use small gauge cable circuits to handle long distance telephone service over distances up to and exceeding 1,000 miles. A general picture of the long toll cable system which is being projected for use in the northeastern section of the United States was presented recently by Mr. Pilliod and published in the July number of the Technical Journal.

Many of the circuits in these toll cables are so long electrically that a number of effects, which are comparatively unimportant in ordinary telephone circuits, become of large and sometimes controlling importance. For example, the time required for voice energy to traverse the circuits becomes very appreciable so that reflections of the energy may produce "echo" effects very similar to echoes of sound. The behavior of the circuits under transient impulses, even when two-way operation is not involved so that "echoes" are not experienced, is very important. In order to keep within proper limits of variation of efficiency with frequency over the telephone range special corrective measures are necessary. Owing to the small sizes of the conductors, the attenuations in the longer circuits are very large. Special methods are, therefore, required to maintain the necessary stability of the transmission, including automatic means for adjustment of the repeater gains to compensate for changes in the resistance of the conductors caused by temperature changes.

THIS paper aims to present an idea of what is involved in the transmission of voice currents over long toll cable circuits. Because of the breadth of the subject covered, no attempt has been made to make the discussions of the various items complete, or to include many of the results of the experimental and theoretical work which contributed to a solution of the problems and which has involved the cooperative efforts of a large number of engineers and investigators. This paper should be considered merely as an introduction to the subject. It is hoped that subsequent papers will be presented dealing with these matters in more detail.

For the benefit of those who are not intimately in touch with telephone transmission work, the different types of circuits used in toll cables are first briefly reviewed. The important characteristics of the loading systems are then presented. Following this, various important effects encountered in long cable circuits are discussed and their reactions on the design of cable systems indicated.

In view of the discussion on telephone repeaters given in the Gherardi-Jewett paper,[1] which was presented before this Institute on October 1, 1919, it will be assumed that the reader of the present paper is familiar with the general features possessed by the various types of such devices and, accordingly, no descriptions of them are given, their overall performance only being of interest in the present connection.

[1] Transactions of A. I. E. E., Vol. XXXVIII Part 2—Page 1287.

I. Different Types of Circuits

The different types of circuits used in toll cables are illustrated in diagrammatic form in Figure 1. Circuit "b" is a two-wire telephone circuit employing a 21-type telephone repeater. This type of circuit is employed only for handling connections on which but one telephone repeater is involved. Circuit "c" is a typical two-wire circuit on which the familiar 22-type telephone repeaters are operated. Circuit "d" is of the four-wire type which employs two transmission paths, one for each direction. The function of the pilot wire circuits, "a," will be taken up later.

With the exception of circuit "b", which possesses the limitation that it cannot advantageously be connected to another circuit containing telephone repeaters, the circuits shown in the figure may be connected when required to circuits of the same or other types, such as open-wire circuits, to build up various telephone connections. In general, circuits such as "c", employing 22-type repeaters, are used for handling connections of moderate lengths, while circuits such as "d", of the four-wire type, are employed for the longer connections where the transmission requirements are more severe.

In addition to employing the cable conductors for furnishing telephone service, these may also be arranged to furnish D.C. telegraph service. Apparatus for compositing the circuits so as to permit this superposition of the D.C. telegraph is indicated on the drawing. In general, the method of compositing the small gauge cable circuits is the same as that employed for compositing open-wire lines. The telegraph circuits in cable, however, operate with a metallic instead of a grounded return and employ much weaker currents than those common on open wires. Telegraph currents employed in the cables are comparable in magnitude with the voice currents.

The two-wire circuits in toll cables employ conductors of No. 19 or No. 16 American wire gauge, while for the four-wire circuits, No. 19 gauge conductors are usually employed. (No. 19 gauge weighs 20½ pounds per wire mile or 5.8 kilograms per kilometer. No. 16 gauge weighs twice as much).

II. Loading Characteristics

Two weights of loading are usually employed. These are commonly known as "medium heavy loading" and "extra light loading" and in this paper they will be referred to for brevity as "M.H.L." and "X.L.L." respectively. The medium heavy loading employs coils having an inductance of about 0.175 henry in the side circuits,

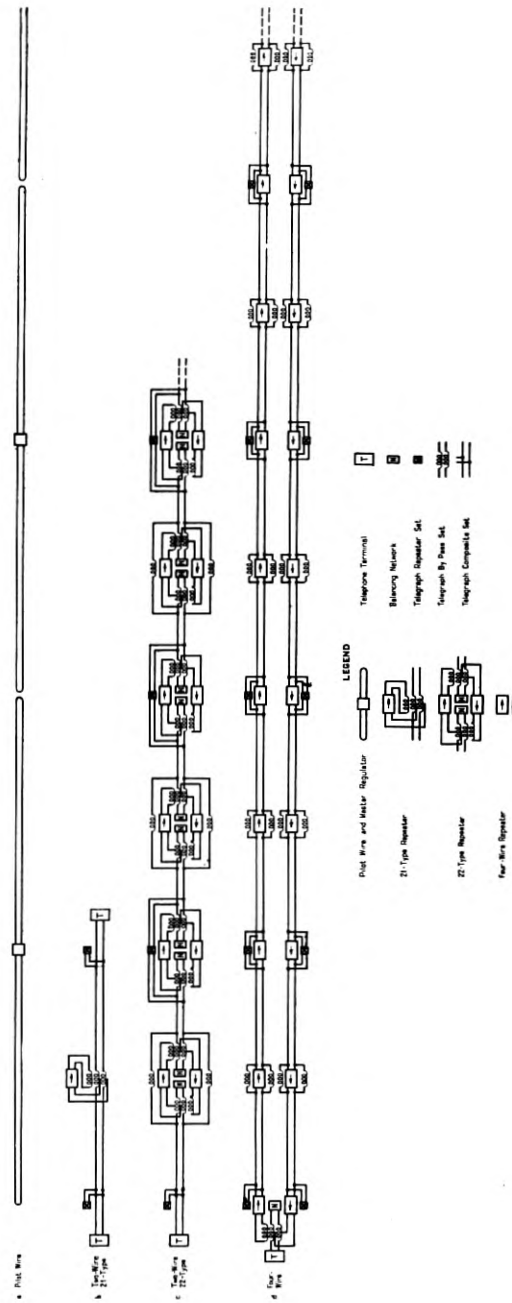


Fig. 1—Different types of cable circuits.

spaced 6,000 feet apart (approximately 1.8 kilometers); the extra light loading employs coils having an inductance of about 0.044 henry for the side circuits with the same spacing. The capacity per loading section for the side circuits is approximately 0.074 mf.

The medium heavy loaded side circuits have a characteristic impedance of about 1600 ohms, and a cutoff frequency of about 2800 cycles. The extra light loaded side circuits have an impedance of about 800 ohms and a cutoff frequency of about 5600 cycles.

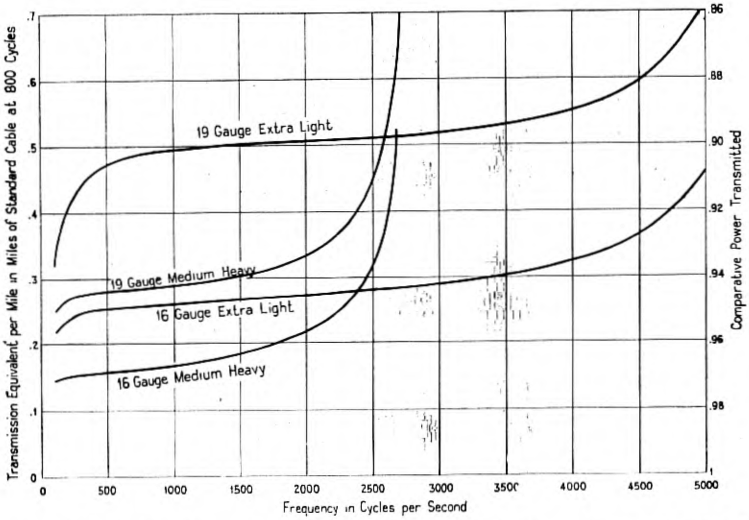


Fig. 2—Attenuation-frequency characteristics of loaded cable side circuits.

Figure 2 shows the attenuation-frequency characteristics of No. 19 and No. 16 gauge side circuits with the two types of loading. It will be observed that the M.H.L. circuits have lower attenuation for frequencies below about 2500 cycles, as should be expected from the fact that the inductance per mile introduced by the loading coils is greater. However, the attenuation is more nearly equal at different frequencies in the case of the X.L.L. circuits, this being particularly true at the higher voice frequencies.

Another important characteristic of loaded circuits when repeaters are involved is their velocity of propagation. Since the inductance per mile of X.L.L. circuits is only ¼ of that for M.H.L. circuits, the velocity of propagation is twice as great for the X.L.L. circuits as indicated by the well-known approximate formula—

$$V = (L\,C)^{-1/2}$$

Where V is the velocity in unit lengths per second, L is the inductance in henries per unit length and C is the capacity in farads per unit length, the unit of length for expressing velocity, inductance and capacity being the same.

The X.L.L. type of loading is best for the longer circuits, because of the more nearly equal attenuation of currents of different frequencies, its higher velocity of propagation which permits more efficient operation of telephone repeaters, and also its comparative freedom from transient effects, as will be explained in more detail later. For the shorter circuits where these effects are not so important, the M.H.L. type is satisfactory electrically and is therefore employed since fewer repeaters are required owing to the lower attenuation.

III. "Echoes"

As is well known, whenever points of discontinuity or unbalance occur in a telephone circuit, reflections of electrical energy take place. If the circuit is long so that the time for transmission is appreciable and if also the losses are not so great as to cause the reflected energy to become inappreciably small before it reaches the ear of a listener, echo effects will be experienced. While, in general, reflections take place in any telephone circuit actual echoes are never appreciable unless telephone repeaters are employed. In the case of circuits with repeaters, the electrical length is usually great enough so that an appreciable length of time is required for the voice currents to travel to some discontinuity and back again. Furthermore, the repeater gains keep the reflected voice currents large.

It should be understood that the echo effects which are experienced in long repeatered circuits are due to the same unbalances, which, on shorter circuits, bring in trouble due to "singing", or distortion of the voice waves due to "near-singing". On electrically long circuits, due to the comparatively great time lags involved, the echo effects become of controlling importance. Consequently, it is, in general, necessary on such circuits to work the repeaters at gains well below those at which "singing" or distortion due to "near-singing" is experienced.

The echo effects which occur in four-wire circuits will first be discussed, since the effects are simpler in this case than they are in the case of a two-wire circuit.

Figure 3-a shows a four-wire circuit in diagrammatic form, while Figure 3-b shows the echoes which are caused by the unbalances at the terminals. When someone at terminal A talks to a person at

terminal B, the heavy line in Figure 3-b shows the direct transmission, which takes place over the top pair of wires in Figure 3-a. When this current reaches the distant terminal, part of it goes to the listener while another part, due to the imperfections of balance between the line and network at that terminal, travels back through the pair of wires at the bottom of Figure 3-a toward terminal A. The talker at terminal A will hear this current as an echo if the four-wire circuit is long enough so that the time lag is appreciable. This first echo heard by the talker divides at terminal A in the same way as did the direct transmission at terminal B, part of it taking the upper path of figure 3-a back toward the listener. The listener will, therefore, first receive the direct tansmission and then a little later an echo. This process is repeated producing successive echoes which are received at both terminals A and B as indicated.

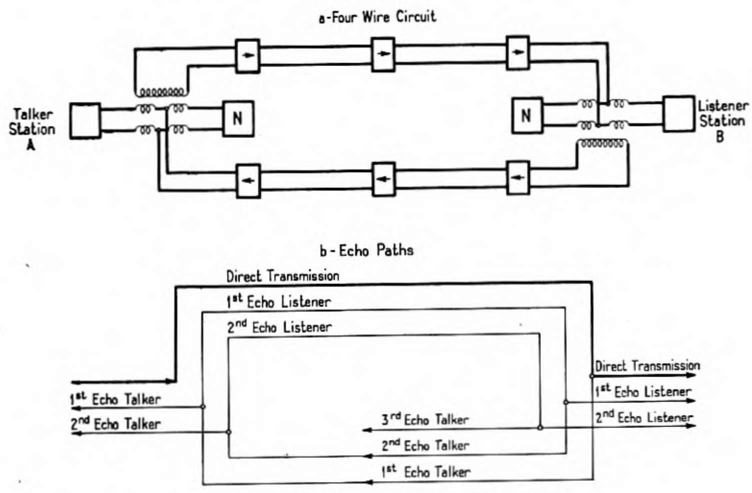


Fig. 3—Echo paths in four-wire circuit.

A four-wire circuit 1000 miles (1600 kilometers) long has been set up in which the balances at the two ends were deliberately made poor so as to exaggerate the effects. More than a dozen successive echoes could be heard before they became inaudible. Since for each echo the voice energy traveled 2000 miles (3200 kilometers) this energy must have travelled the distance around the world before becoming inaudible.

In order that a circuit will be satisfactory for regular telephone use, the echoes must be kept small as compared to the direct transmission. Evidently if the first echoes are small as compared to the direct transmission, the later echoes will be much smaller in magnitude. For example, if the power in the first echo, heard by the listener, is 1-10 as great as the directly transmitted power, the second echo will have only 1-100 as much power, the third echo 1-1000 etc.

The velocity of an X.L.L. circuit is approximately 20,000 miles (32,000 kilometers) per second, while the velocity with M.H.L. is only 10,000 miles (16,000 kilometers) per second. It is thus seen that the time required for voice energy to travel from one end to the other of an X.L.L. circuit 1,000 miles (1600 kilometers) long is 0.05

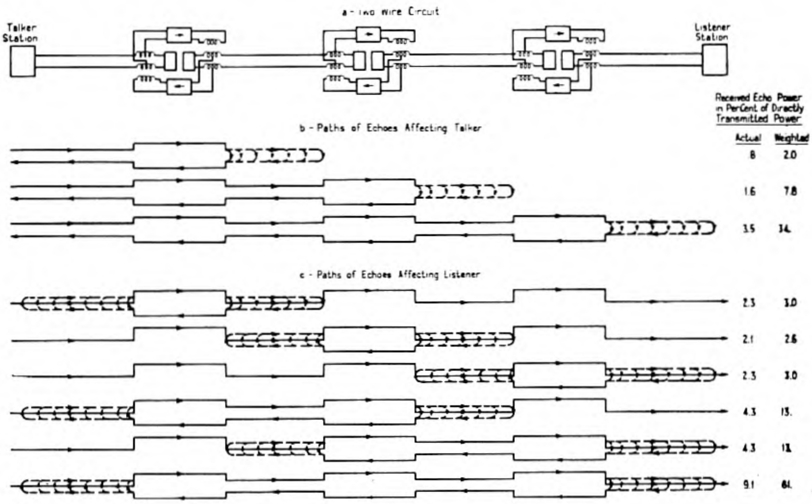


Fig. 4—Echo paths in two-wire repeatered circuit.

second. An echo traveling from one end of the circuit to the other and back again would, therefore, arrive 0.1 second behind the impulse which started the echo. With M.H.L. circuits these times are of course doubled.

Figure 4 illustrates the condition existing in a two-wire circuit. For simplicity, the first echoes only are shown, the later echoes being less important owing to their comparative weakness as explained above. In such a circuit reflections occur not only at the terminals, but at a number of intermediate points in the circuit, the condition of balance between the networks associated with the telephone repeaters and the corresponding lines being necessarily imperfect. This imperfection of balance is due in part to lack of perfect balance of the apparatus closely associated with the repeater, and in part to the small irregularities which exist in the make-up of any practical loaded line. A further cause is the reflection at the adjacent repeaters, due to the difference between the repeater impedance and the line impedance.

It will be noted that three sets of echoes are shown which affect the "talker". In addition to these which involve one or more repeaters, a comparatively small amount of power is reflected back to the "talker" from the various irregularities between the "talker" station and the nearest repeater. These reflections have not been indicated since their effects are of negligible importance. Six sets of echoes affect the "listener". Both for the echoes affecting the "talker" and the "listener", the dotted lines indicate reflections from a number of different points where irregularities exist as explained above.

In circuits containing a larger number of repeaters the numbers of sets of echoes affecting the talker and listener are, of course, greater. The number of sets of first echoes affecting the talker is equal to the number of repeaters. The number affecting the listener is equal to $\frac{N(N+1)}{2}$ where N is the number of repeaters.

It is, of course, obvious, that, for either four-wire or two-wire circuits, if the circulating energies are large, they will have an adverse effect on the ability of two people to carry on a conversation over a telephone circuit. Not only will the transmission received by the listener be adversely affected, but the talker will be considerably distracted, particularly when the time of the transmission over the circuit is so long that he hears a distinct echo of his words.

Experiments have shown that the effects of the echoes both on the listener and talker become more serious as their time lag is increased. This means that as telephone circuits are made longer it is necessary either to improve balances or to design the telephone circuits so that the velocity of propagation will be higher. This necessity for making the velocity of propagation high on long circuits was one of the principal reasons which led to the selection of extra light loading for the longer circuits.

Figure 5 shows very approximately how the effects of the echoes vary with the length of time by which they are delayed. One curve is given for the effect on the "talker", another for the effect on the "listener". Both curves indicate, for various time lags, the comparative magnitude of echoes which are small enough to be inappreciable when ordinary telephone conversations are carried on. The curve applying to the "listener" is referred to the direct power which he receives, while the curve for the "talker" is referred to the power which he puts into the circuit.

In Figure 4 showing the condition existing in a two-wire circuit, the comparative magnitudes of the power in each echo are indicated, a typical condition of the lines being assumed. For the listener the echo power is expressed as a percentage of the directly transmitted power which he receives. In the case of the talker, it is expressed as a percentage of the power which he puts into the circuit. In addition to the comparative amounts of power in each echo, "weighted" magnitudes are indicated. The "weighted" figures take account of the fact that the effects of a given amount of echo become more serious as the time lag is increased as indicated by the curves in Figure

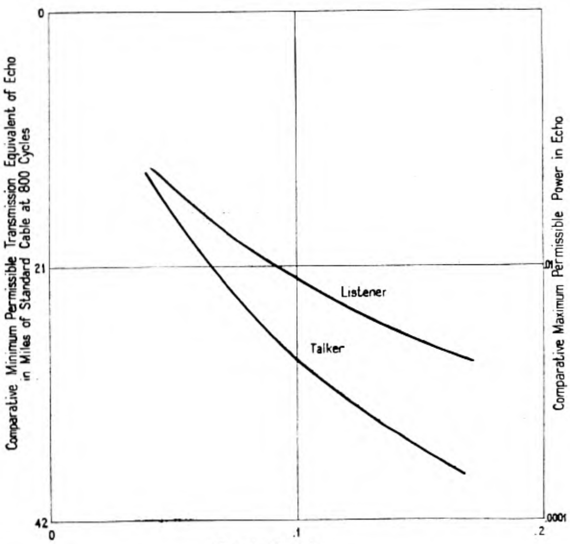


Fig. 5—Effect of echoes on talker and listener.

5. Referring to Figure 4, it will be noted that the "weighted" magnitudes of the power in the echoes are largest for the long paths. In general, this condition exists in the case of the majority of long two-wire repeatered circuits in cable.

In order to compare the behavior of a four-wire circuit with a two-wire circuit, consider again Figures 3 and 4. It will be observed that in Figure 4, showing the two-wire circuit, there is one echo received by the talker which travels from one end of the circuit to the other. Referring to Figure 3 showing a four-wire circuit, it will be seen that this echo corresponds to the one labelled "1st echo talker". Similarly for the echoes affecting the listener, the echo whose path is longest in the two-wire circuit corresponds to a similar echo in the four-wire circuit. Since many additional echo paths are present in the two-wire circuit, it is evident that, other things being equal, the overall transmission result obtainable from the two-wire circuit cannot be made as good as that obtainable from the four-wire circuit.

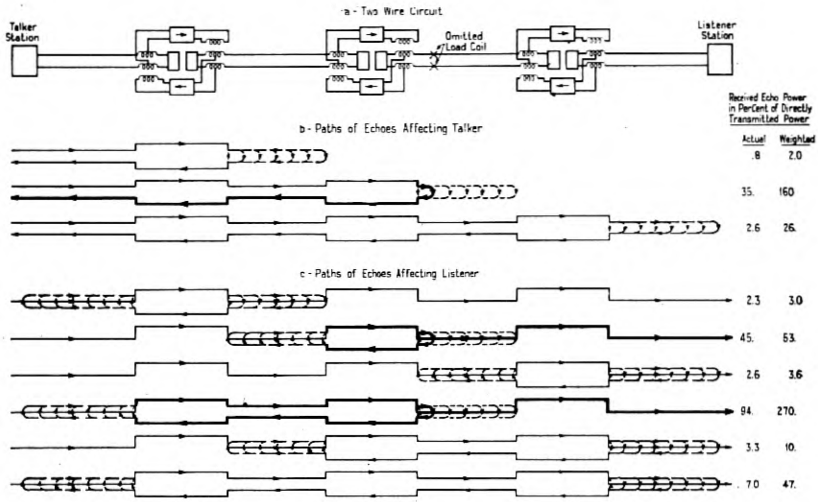


Fig. 6—Echo paths in two-wire repeatered circuit with omitted loading coil.

In a two-wire circuit it is, of course, obvious that any defect in the lines which will cause a large irregularity will result in a considerable impairment of the circuit. Figure 6 shows the effect of omitting a loading coil at an intermediate point in a circuit, the conditions in this circuit being assumed to be the same as those in Figure 4 with the exception of the omitted loading coil. The omitted loading coil introduces a large impedance irregularity which causes certain of the echoes to be made much greater in comparative magnitude as indicated. In order to reduce the echoes in the circuit with the omitted loading coil sufficiently to make the circuit satisfactory for telephone use, it is necessary to reduce the repeater gains. In this particular case it is necessary to lower the total gain about 4 miles, which increases the overall transmission equivalent of the circuit from about 10 miles for the normal condition to about 14 miles for the condition with the omitted loading coil.

Before leaving the subject of "echoes" it is believed that it will be of interest to point out some of the important characteristics of two-way repeatered circuits which result from these effects.

1. The minimum permissible net equivalent (total loss minus total repeater gain in one direction) of a four-wire circuit of a given length depends only on the velocity of propagation and the balance conditions at the terminals of the circuit. When conditions are such that the balance conditions cannot be improved, increasing the velocity of propagation will enable a lower net equivalent to be obtained.

2. In the case of a two-wire circuit with reasonably smooth lines, the exact location of the repeaters and the gains at which individual repeaters are worked have little effect on the overall result so far as echo effects are concerned. This follows from the fact that the echo paths from end to end of such a circuit are usually of more importance than the shorter echo paths. Evidently, moving the individual repeaters about or altering their gains has no effect on the longest paths, provided the total gain in each direction is kept constant.

3. In the case of a two-wire circuit of a given length, the velocity of propagation and smoothness of the lines are of most importance in limiting the possible net equivalent, the line attenuation being of secondary importance.

   For example, in the case of the transcontinental (New York-San Francisco) open-wire line, the original circuit was loaded. (Although this paper deals particularly with repeaters on cable circuits this example was selected because it so well illustrates this point.) The velocity of propagation was such that voice currents required about 0.07 second to travel from one end of the circuit to the other. The total line equivalent was equal to about 56 miles of standard cable. By applying repeaters to this circuit it was possible to obtain a working net equivalent of about 21 miles.

   The unloading of the circuit increased the velocity so that the time of transmission was reduced to 0.02 second, about 0.3 of the time required when the circuit was loaded. The attenuation was increased so that the total line equivalent without repeaters was equal to about 120 miles of standard cable, a little more than twice the equivalent of the loaded circuit. By applying repeaters of an improved type to this circuit so as to keep the quality good in spite of the increased attenuation and

> correspondingly increased gain required, it was possible to obtain a working equivalent of only 12 miles of standard cable as compared to the original figure of 21 miles. This means that with the same amount of speech power applied at one end, the power received over the non-loaded circuit is 7 times as large as that formerly received over the loaded circuit.[2]

The example of the transcontinental line, above, may well bring up the question as to why it is that cable circuits are loaded. This is done for two reasons: In the first place, it is in general cheaper to load cables than it is to make up the increased attenuation by means of more repeaters. In the second place the loading lessens the amount of distortion introduced by the cable circuits. In the case of open wire circuits, their series inductance is sufficient to keep the distortion small.

IV. Attenuations and Corresponding Amplifications—Power Levels

Owing to the fact that the weight of loading applied to the longest cable circuits is very light, the attenuation of such circuits is very great. A four-wire X.L.L. 19 gauge circuit 1,000 miles long has the enormous line equivalent of 500 miles of standard cable. The total power amplification applied to this circuit by the repeaters exceeds 10^{47}. This amount of amplification is more than enough to talk half way around the world at the equator using non-loaded No. 8 Birmingham Wire Gauge open-wire commonly employed for handling very long distance business (No. 8 B.W.G. copper weighs 435 pounds per wire mile, or 120 kilograms per kilometer).

In order to obtain an idea of how enormous this amplification is, assume that no repeaters were employed and an attempt were made to apply enough power at one end of the circuit to enable the normal amount of speech power to be received at the distant end. The power applied at the sending end would then have to be about 50 quadrillion times as great as the total power which it is estimated is radiated by the sun.

While the total amount of power amplification is very great, the amount of amplification put in at any one point is, of course, limited. The maximum amount of power at a repeater point is limited partly by the capacity of the vacuum tubes and partly by the power carrying capacity of the telephone circuit, including the loading coils. (By power carrying capacity is here meant the ability to carry voice waves

[2] A material improvement in the telephone quality was also effected by the unloading of the circuit.

without serious distortion.) It is also necessary to limit this power to avoid serious crosstalk into other circuits.

In addition to these limitations on the maximum power, it is necessary to insure that the power at any point in a circuit does not become too small. Otherwise, the normal voice power will not be sufficiently large as compared to the power of crosstalk from other circuits. It is, furthermore, evident that the ratio of power from extraneous sources, such as paralleling telegraph circuits and power supply circuits, to the voice power should be as small as practicable in order to keep the circuits free from noise.

Figure 7 will give an idea of how the telephone power attenuates and is amplified in a long circuit. The circuit shown is similar to those which it is proposed to employ between New York and Chicago, *i. e.*, it is a four-wire X.L.L. 19 gauge circuit largely in aerial cable, equipped with automatic means for compensating for the changes in attenuation caused by the effects of varying temperatures on the resistance of the conductors. (These automatic devices are described in a later section of this paper.) For simplicity, the power levels for transmission in one direction only are shown. The solid lines show the power levels when the temperature is a maximum so that the attenuations are greatest, while the dotted lines show the levels when the temperature is a minimum and the losses are, therefore, also a minimum. The shaded areas between the lines represent the changes which take place during the course of a year.

When the requirement is introduced that transmission must take place in both directions it is found that at the points in the circuits going in one direction where the power is a maximum, the power going in the opposite direction in other circuits is a minimum. This represents a very bad condition for crosstalk from one four-wire circuit into another. In order to overcome this the conductors carrying strong voice power are kept electrically separated or shielded from those carrying weak power as indicated schematically in Figure 8. The conductors which carry strong voice power are shown heavy, while those carrying weak power are shown light. In the cable proper the separation is effected by grouping the conductors in two bunches, one for transmission in one direction, the other for transmission in the opposite direction, taking care that these two bunches of conductors are separated electrically as far as possible. In the loading coil pots the coils employed on the circuits for transmission in the two directions are similarly kept separated. In the offices the separation is effected by arranging the repeaters and other apparatus as shown in the figure. It will be observed that no special separation

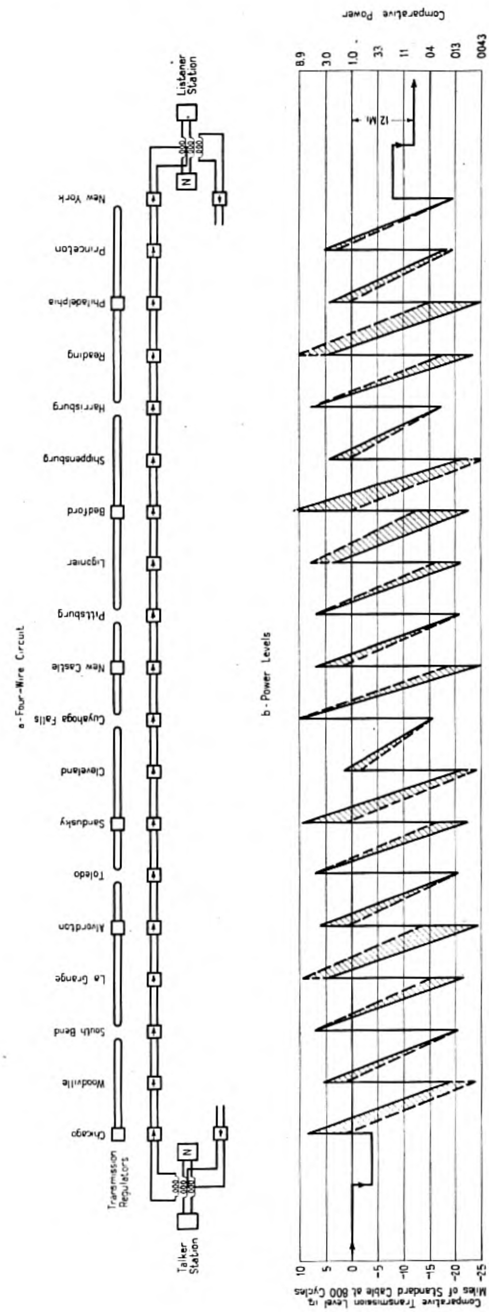


Fig. 7—Power levels in New York-Chicago extra light loaded four-wire circuit.

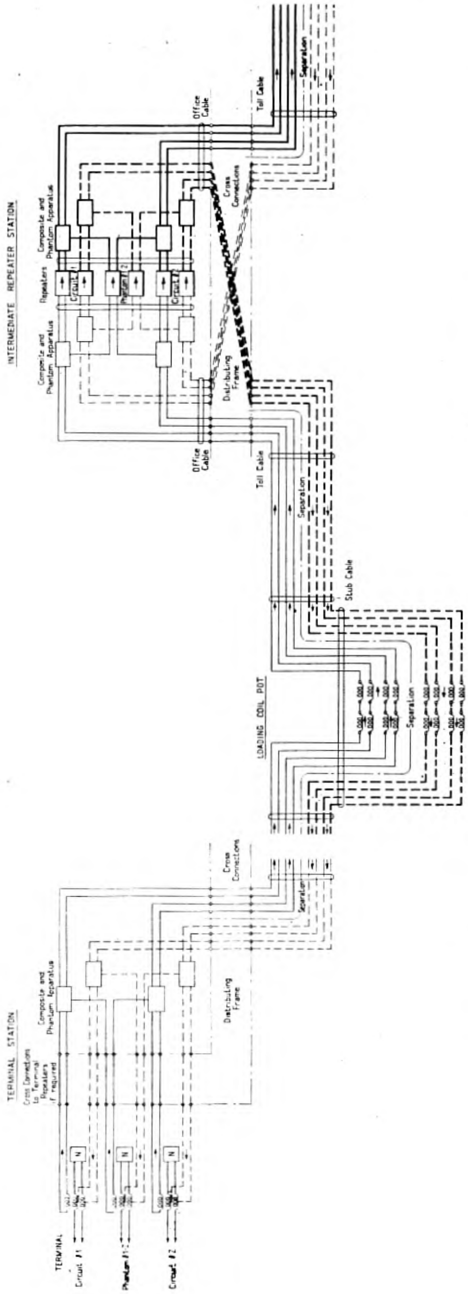


Fig. 8—Four-wire system—Segregation method to reduce cross-talk.

is shown between the repeaters transmitting in the two directions, since to keep the conductors carrying weak power separated from those carrying strong power, it is merely necessary to keep the apparatus and cabling connected to the inputs of the repeaters separated from the apparatus and the cabling connected to the repeater outputs.

V. Steady State Distortion

The possible sources of distortion may be divided broadly into (1) repeaters and auxiliary apparatus and (2) the lines.

With reference to the distortion introduced by the repeaters, the vacuum tube is fortunately very nearly perfect, at least in so far as concerns practical telephony. At one time, for purposes of test, a circuit was set up containing 32 vacuum tubes in tandem. On this circuit the distortion was so small that when listening to ordinary conversation it was difficult to detect any difference in the quality of transmission before and after traversing the 32 vacuum tubes.

It is beyond the limits of this paper to enter into the problems of design which were encountered in the development of the repeater circuits. For the present purpose of considering the overall performance of repeatered circuits in cable no serious error will be made if it is assumed that the complete repeater circuits meet the requirements for an ideal repeater as set up in the Gherardi-Jewett paper.

Considering next the lines, it is necessary to make the loading very regular so that balance difficulties will not cause an undue amount of trouble on two-wire circuits. Regularity of the loading is also essential in order to avoid irregular transmission of different frequencies. In order to secure this regularity of loading, it is necessary that the spacing between loading points be made very uniform and that the cable be so manufactured that the electrostatic capacity of its circuits be held within close limits. The loading coils themselves must be closely alike in their electrical properties and furthermore, the coils must be stable, *i. e.* these electrical properties must not change appreciably due to the passage of voice currents or other currents required for cable operation through them.

Next, it is necessary to design the repeaters and associated apparatus used on the longer circuits, particularly the four-wire circuits, so as to put in different amounts of gain at different frequencies, thereby making the overall transmission at different frequencies approximately constant in spite of the fact that the loss introduced by the cable circuits at different frequencies is not constant. Figure 9 shows the overall or net transmission equivalent plotted against frequency for an X.L.L. four-wire circuit 1080 miles long (1750 kilometers) which was set up for purposes of test. The heavy line in this figure shows the overall result which was actually obtained with repeaters and associated apparatus designed to equalize the transmission, while the dotted line shows what the characteristic would have been had the repeaters introduced exactly the same amount of gain at all frequencies.

VI. Transients

In comparatively short telephone circuits, good quality will usually be assured if the transmission, as measured at different single frequencies within the voice range, is kept approximately constant.

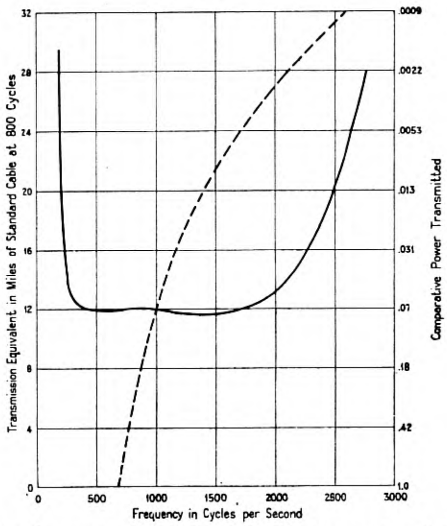


Fig. 9—Transmission frequency characteristic of long extra light loaded four-wire circuit.

For electrically long circuits, however, this is not sufficient. Not only must the "echo" effects be kept within proper limits, but consideration must be given to the fact that when electrical impulses are applied to such circuits, peculiar transient phenomena are experienced. These transient phenomena occur in equal degree in two-way circuits and in circuits arranged to transmit in one direction only, that is, they are not related to "echo" effects.

In order to give an idea of the nature of some of the transient effects, some oscillograms are shown in Figures 10, 11, 12 and 13. Figure 10 shows an 1800-cycle current before and after traversing a cable circuit of an earlier type 1050 miles (1700 kilometers) long. This particular circuit was No. 13 A.W.G. weighing 82 pounds per wire mile

(23 kilograms per kilometer) loaded with inductance coils of 0.2 henry spaced 1.4 miles (2.25 kilometers) apart and contained 6 one-way repeaters. It will be noted that the first sign of the arrival of the received current occurs about 0.1 second after the wave is put on at the sending end. This time checks with the formula for velocity given above. The time required after arrival of the first impulse (point "a") until the wave builds up to a practically steady-state condition at point "b" is about 0.055 second. The steady condition is interrupted at point "c" by the arrival of the break transient, the time interval between points "b" and "c", representing the period when the wave is in the steady-state, being only 0.01 second. The wave required about 0.055 second to die out—interval between points "c" and "d".

It is interesting to note the behavior of the current during the building-up and dying-out intervals. During the building-up process the frequency of the received current increases from a very low value at point "a" until at point "b" it becomes the same as that of the source. The magnitude of the received current also increases until at point "b" it reaches a value corresponding to the steady-state transmission equivalent of the line. The interval "a-b" is determined solely by the structure of the line and has nothing to do with the time during which the current is supplied at the sending end.

The dying-out process can be considered to be caused by the application at the time of break of a second current equal in value to the current originally applied but opposite in phase, so that the sum of the two currents will be zero. Hence, it is to be expected that the received current will disappear by adding to the steady-state a transient similar to the building-up transient in the interval "a-b". That this is true is indicated by the behavior during the interval "c-d". At first the low frequency current of the break transient produces a displacement of the axis of the steady current. As the frequency approaches a steady value a beating effect becomes noticeable which grows smaller until complete opposition of phase obtains and the received current disappears.

Figure 10 clearly indicates that a pulse of voice current having a frequency in the neighborhood of 1800 cycles, even though received in proper volume if steadily applied, would be badly distorted.

When carrying on a conversation over such a circuit as this, distortion of the voice waves makes understanding difficult while peculiar metallic ringing sounds are very noticeable.

Next consider a circuit of the same character with half the length. The effect of a circuit of this length on an 1800-cycle wave is shown in the oscillogram of Figure 11. It will be observed that the propagation time has been cut in half while the lengths of time for the received wave to build up and die out have also each been cut in two. This checks with theoretical work, indicating that the severity of this type of transient effect is directly proportional to the length of the circuit. This fact that the transient effect is proportional to the length of the circuit furnishes the reason why a short circuit may give tolerably good results, while a long circuit gives poor results.

Figure 12 is of interest as indicating what takes place when we apply a current at the sending end of the circuit whose frequency is so high that no appreciable amount of the steady current will pass through the circuit. In this case only transient oscillations appear at the receiving end of the circuit. This particular circuit was of the same type as the above, although it was only 350 miles long (570 kilometers).

A large number of oscillograms of this sort have been taken in connection with the study of these transient effects. From these and theoretical considerations[3] it has been proved that the effects in a given circuit are much worse at high frequencies than at low frequencies, the severity of the effects, within certain limits, being a function of the ratio of the frequency being transmitted to the frequency of cutoff of the loaded circuit. The gauge of the circuit has practically no effect.

Since in order to give good quality it is necessary to transmit fairly well all frequencies up to at least 2000 cycles, it is obvious that on long circuits in order to keep the transient effects small, the frequency of cutoff must be kept high. In order to do this, it is necessary either to make the loading coils of very low inductance or to space them very close together. This is another one of the reasons why extra light loading was adopted for the long cable circuits. (It will be remembered that the inductance of the side circuit loading coils is only 0.044 henry and the spacing 6000 feet).

The effect of lighter loading on the transient behavior of telephone currents, is shown in Figure 13, which shows a 2000-cycle wave transmitted over an X.L.L. circuit about 1050 miles (1700 kilometers) long. This circuit contained 23 one-way repeaters. It will be observed that both the building-up and dying-out transient periods are very much reduced, which means that all pulses of telephone currents up to at least 2000 cycles will pass through such a circuit with very little distortion.

[3] John R. Carson—"Theory of the Transient Oscillations of Electrical Networks and Transmission Systems". Transactions of A. I. E. E. Vol. XXXVIII, page 407.

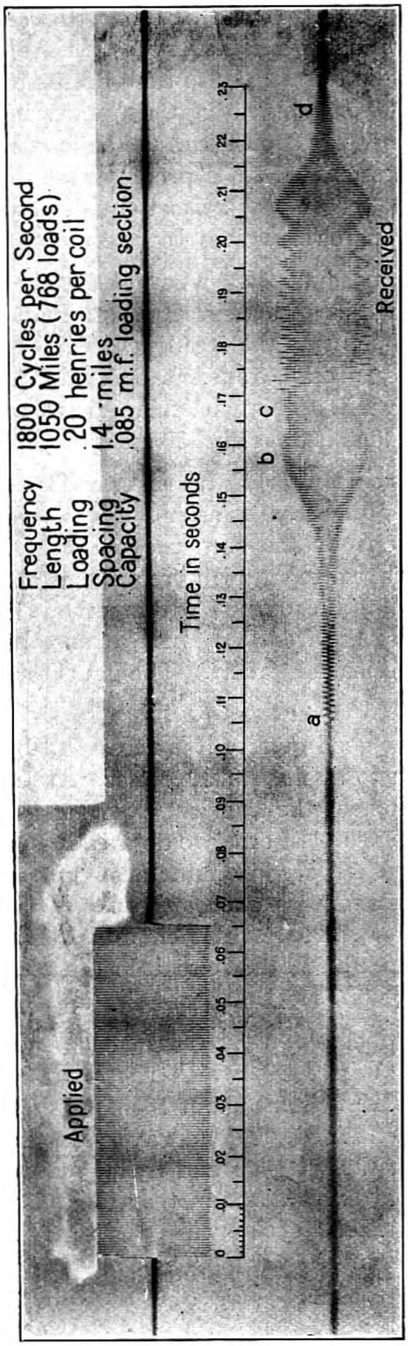


Fig. 10—Transients in 13-gauge medium heavy loaded cable.

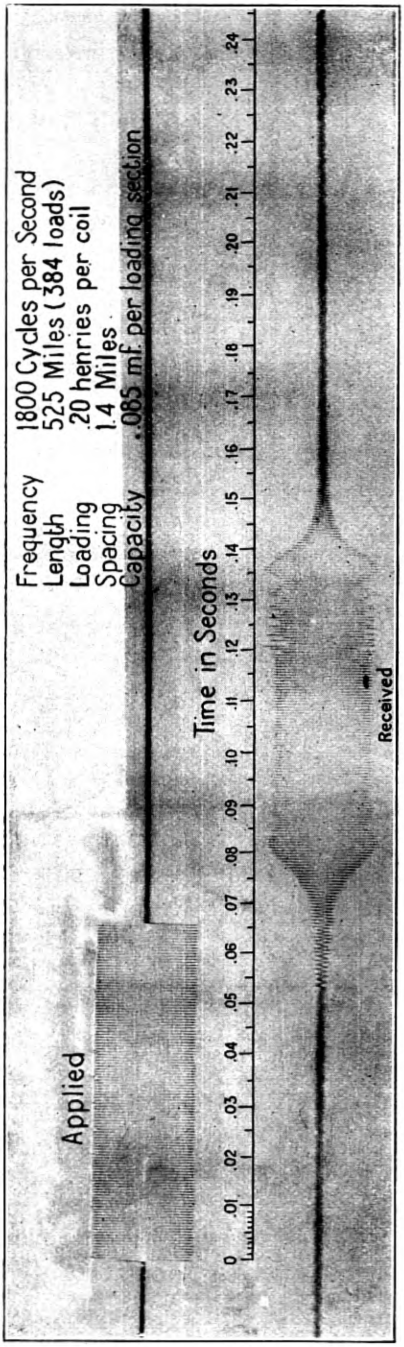


Fig. 11—Transients in 13-gauge medium heavy loaded cable.

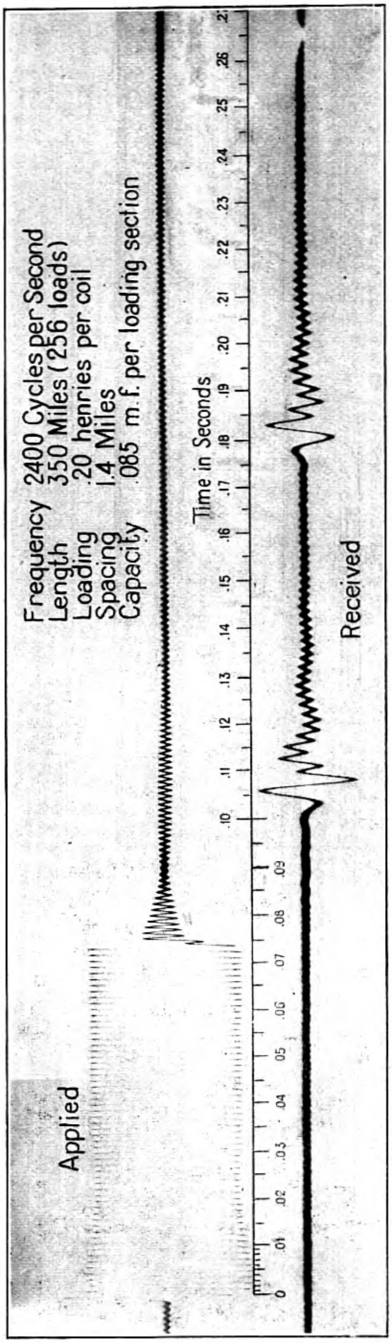


Fig. 12—Transients in 13-gauge medium heavy loaded cable.

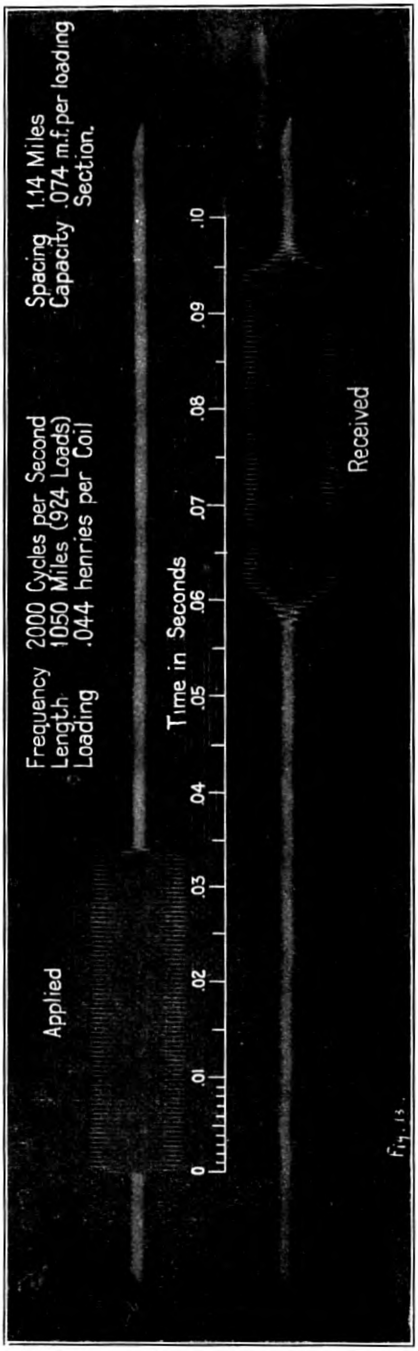


Fig. 13—Transients in 19-gauge extra light loaded cable.

VII. Stability

As has been pointed out, the magnitude of the line transmission loss in a repeatered circuit is of comparatively small importance in determining its possible transmission equivalent, whether the circuit be worked on a four-wire or two-wire basis. However, it is of extreme importance to be sure that the repeater gains are kept adjusted so as to compensate exactly for a large part of the transmission loss in the circuit, so that the difference between the total loss in the circuit and the total gain, which represents the net equivalent of the circuit, will be kept constant.

On certain of the long circuits this difference is very small as compared to the quantities which are subtracted. For example, in the case of a 1000-mile four-wire circuit using X.L.L. 19-gauge conductors, the total line transmission loss is about 500 miles. Not counting the gain required to make up for losses in apparatus and office cabling, the total gain is about 488 miles, the difference, 12 miles, representing the net equivalent. Evidently only a very small percentage change in either the transmission losses or the gains will have a large effect on the net equivalent. This represents about the most severe condition. Some examples of less severe conditions are—

2–Wire 19-gauge M.H.L. circuit 200 miles long (320 kilometers). Line equivalent 58 miles. Repeater gain exclusive of gain required to make up for loss in apparatus and office cabling 46 miles. Net equivalent 12 miles.

4–Wire 19-gauge M.H.L. circuit 500 miles long (800 kilometers). Line equivalent 145 miles. Repeater gain exclusive of gain required to make up for loss in apparatus and office cabling 133 miles. Net equivalent 12 miles.

In order to maintain the necessary constancy of the overall or net transmission equivalent of long repeatered circuits in cable, it is necessary first of all to maintain the gains of the individual repeaters within close limits. In addition, periodic transmission measurements are required over the complete circuits, supplemented by suitable adjustment of certain of the individual repeaters whenever the overall equivalent falls outside of the prescribed limits. Also, on the very long small gauge circuits, the changes in attenuation, due to the resistance changes caused by temperature variations, become so large that it is practically essential to provide automatic means for overcoming these effects.

The methods employed in maintaining the gains of the individual repeaters and of the overall transmission equivalents within proper limits will first be described, after which the automatic transmission regulators will be discussed.

VIII. Important Tests and Adjustments

In order to hold the repeater gains constant, close inspection limits are placed on the vacuum tubes during the course of manufacture to insure great uniformity of the product, as well as consistency of performance. In operating the repeaters, considerable care is taken to maintain constancy of the operating currents and voltages. The operating limits of currents and potentials together with the corresponding gain variations for one of the types of tube in common use are given in the following table:

Variable Quantity	Prescribed Limits	Gain Variation
Plate Potential	130±5 volts	±.2 mile
Grid Potential	9±1 volt	±.3 mile
Filament Current	1.25±.05 ampere	Very small for new tube—1 mile for tube just before replacement.

In addition to maintaining the tube currents and voltages within the required limits, the gains of the individual repeaters are checked periodically. Suitable adjustments are made when the repeater gains fall outside of the prescribed limits. When the filament emission of a tube becomes so low that the above specified variation in the filament current results in more than 1 mile gain variation the tube is replaced.

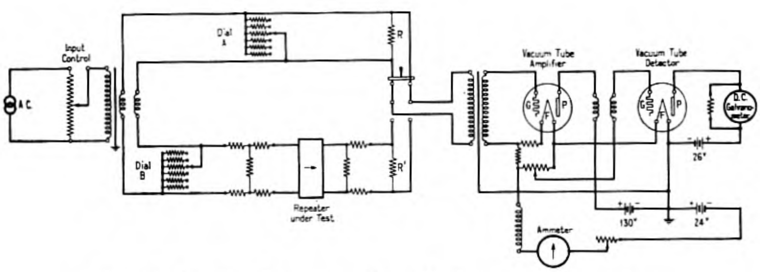


Fig. 14—Device for measuring telephone repeater gains.

A gain measuring device as indicated schematically in Figure 14 is employed for this purpose. The measurement of gain is effected by comparison of the voltages across two resistances, one of which forms part of a circuit which includes the repeater, the other being simply a reference circuit. An amplifier-detector combination amplifies the voltages across these resistances and then rectifies them so as to obtain an indication on a d.c. galvanometer. Equality of voltages across the two resistances, which are designated as R and R' in the figure, is thus indicated by equal deflections of the galvanometer. When this condition is secured, the repeater gain is read directly from the dials A and B. By means of this device, it is readily possible to measure the gain of a repeater within a few tenths of a mile. Owing to the fact that the measuring circuits are comprised entirely of resistances, the readings of the set are independent of frequency, so that gains can be measured at all important telephone frequencies.

As pointed out above, transmission measurements over the complete circuits including the telephone repeaters are required at periodic intervals in order to insure that proper transmission standards are being maintained. By means of such measurements, the variations in the overall equivalent of the circuits due to the cumulative effect of small gain variations, slight variations which remain after the automatic transmission regulators have compensated for the major variations in the conductors and variations from other causes including the effect of different conditions of humidity on the wiring in the offices, are determined and compensated for. These measurements are made by applying a known electromotive force through a known resistance to one end of the circuit and receiving the current at the distant end with a suitable calibrated arrangement employing an indicating meter. Since this type of measurement is similar in principle to the method employed for measuring the gains of the individual repeaters, it will not be described.

IX. Automatic Transmission Regulators

Since the resistance of long cable circuits employing small gauge conductors is comparatively large, it is, of course, evident that changes in this resistance caused by temperature changes to which the cable circuits are subject will have a large effect on transmission. For example, in the case of an X.L.L. 19-gauge 1000-mile circuit (1600 kilometers) in aerial cable, the total attenuation changes more than 110 transmission miles during the course of a year. This corresponds to a variation in the received power of more than 10^{10} or ten billion times.

It is, of course, essential to provide special means to counteract these effects. Furthermore, since the temperature changes which

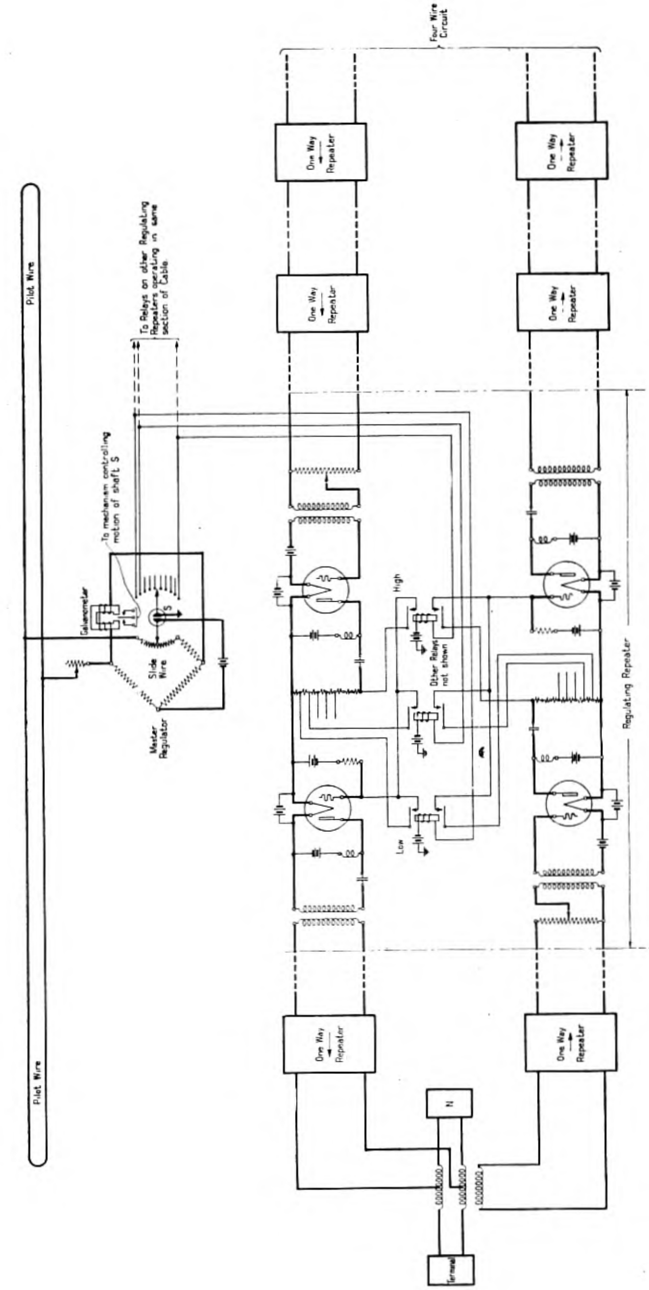


Fig. 15—Pilot wire automatic transmission regulator.

occur in an aerial cable are very rapid, it is practically essential to make these means automatic. In the case of X.L.L. 19-gauge circuits whose variation is greatest, it is necessary to locate the automatic regulators, in general, at every third or fourth repeater station in order to keep the transmission levels within proper limits. In Figure 1-a, a typical method of locating the regulating devices along a cable is indicated. In this sketch each square indicates a master automatic transmission controlling device while the loops extending in either direction from the squares indicate the cable circuits which control the functioning of these devices.

An automatic transmission regulator is shown schematically in Figure 15. The device comprises a Wheatstone bridge arrangement. In one arm of the bridge, pilot wire pairs, extending in either direction in the cable, are included as indicated in the figure. The Wheatstone bridge has associated with it certain apparatus which will not be described here in detail, which functions in such a manner as to automatically keep the bridge balanced at all times. In the process of maintaining balance of the bridge, angular motion is conveyed to a shaft which is proportional to the resistance variations which the cable circuits undergo. The movement of the shaft causes different contacts to be made and thus controls relays which in turn control the gains of the telephone repeaters, one way of doing this being indicated in the figure. The repeater gains are thus caused to be raised and lowered automatically, and thereby overcome the differences in attenuation caused by the temperature changes in the cable conductors.

Probability Curves Showing Poisson's Exponential Summation

By GEORGE A. CAMPBELL

IN many important practical operations the constant probability of an event happening in a single trial is extremely small, but the number of trials is so large that the event may actually occur a sufficient number of times to become a matter of importance. The curves of Figs. 1 and 2 show the probability P of such an event happening at least c times in a number of trials for which the average number of occurrences is a. The probability range shown is from 0.000001 to 0.999999 and the average extends from 0 to 15 in Fig. 1 and to 200 in Fig. 2. An open scale is obtained at both ends, even when the probability approaches to within one part in a million of the limits 0 and 1, by employing an ordinate scale corresponding to the normal probability integral.

In the practical use of these curves the first question which arises is—What number of trials is necessary to make the curves applicable? In practice an infinite number of trials, which is the case for which the curves are drawn, can never be attained; and if we had absolutely no knowledge of the relation between the probabilities for an infinite number and a finite number of trials, the curves would have a theoretical interest only. We do, however, know in a general way when a finite number of trials approximates to the limiting case; the more complete and precise our knowledge on this point, the more generally useful the curves will become. Without attempting to go into the question exhaustively, which would require most careful analysis, a general answer will be found to the question as to the number of trials required by plotting the simple functions $(a/c)^c$, $\frac{1}{2}(c-a-1)$, and $\frac{1}{2}c(c-a-1)$.

The characteristic of all probability curves when n is either finite or infinite, is shown by Fig. 3, where $P(c,n,a)$ denotes the probability of an event happening at least c times in n trials when the average number of occurrences is a. Any curve $P(c,n,a)$ is contained between the ordinates at $a=0$ and $a=n$ and is asymptotic to these ordinates; it cuts $P=\frac{1}{2}$ between $a=c-1$ and $c-0.3$. Thus as n decreases from infinity to c, the central portion of the c curve changes but little, but the curve is confined to the narrowing band to the left of $a=n$ and becomes steeper. On reducing n to $c-1$ the c curve disappears entirely, since c cases cannot occur; the number of trials

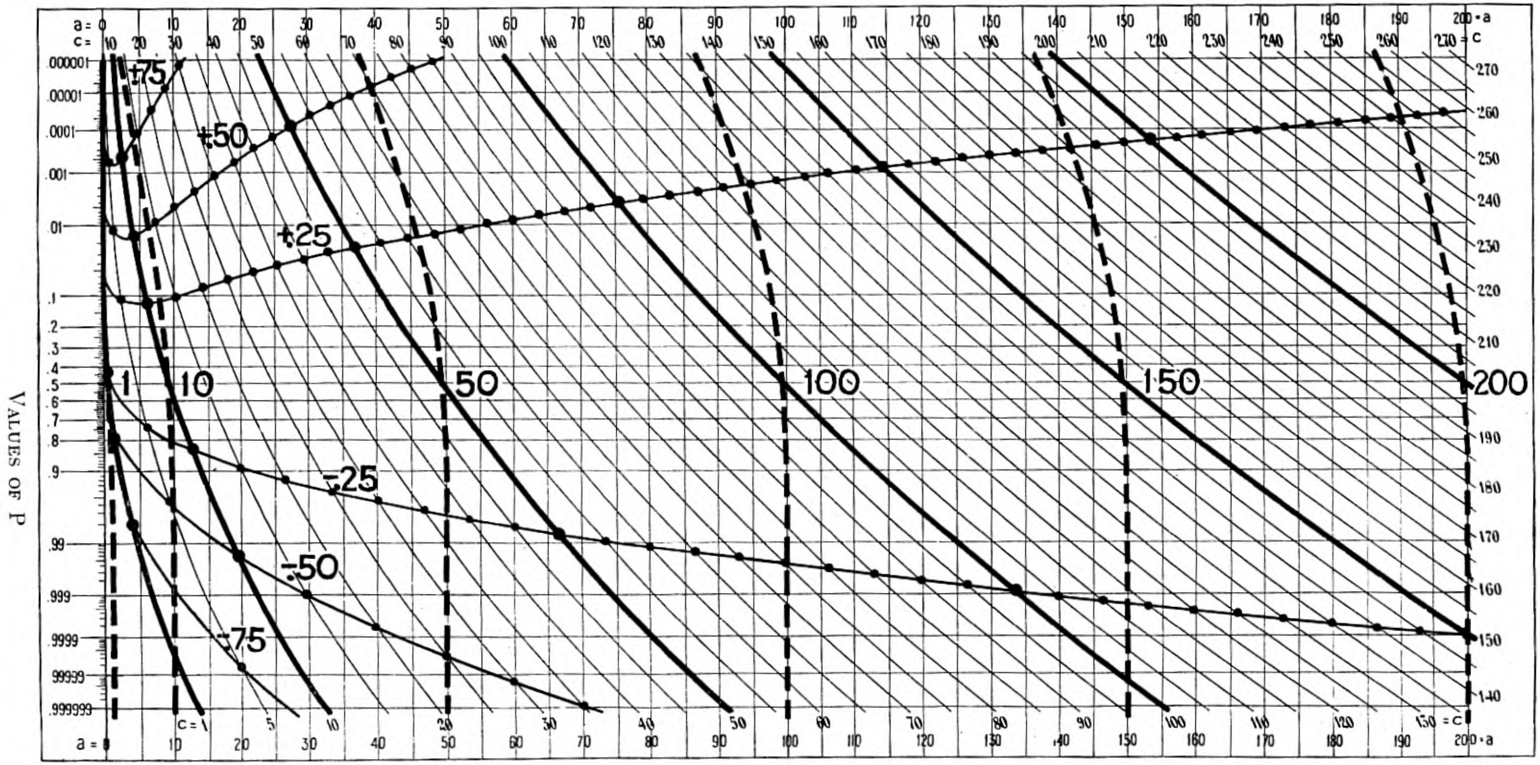


Fig. 3—Extreme curves of the point binomial; the heavy line curves are for the Poisson exponential $P(c,\infty,a)$ and the dashed curves are for $P(c,c,a)=(a/c)^c$ which are the special cases $n=\infty$ and $n=c$ of the general point binomial,

$$P(c,n,a)=\frac{\Gamma(n+1)}{\Gamma(c)\Gamma(n-c+1)}\int_0^{a/n} x^{c-1}(1-x)^{n-c}dx.$$

The beaded curves show the corresponding relative increment in the average, $\Delta a/a=[a(c,\,c,\,P)-a(c,\infty,P)]/a(c,\infty,P)$.

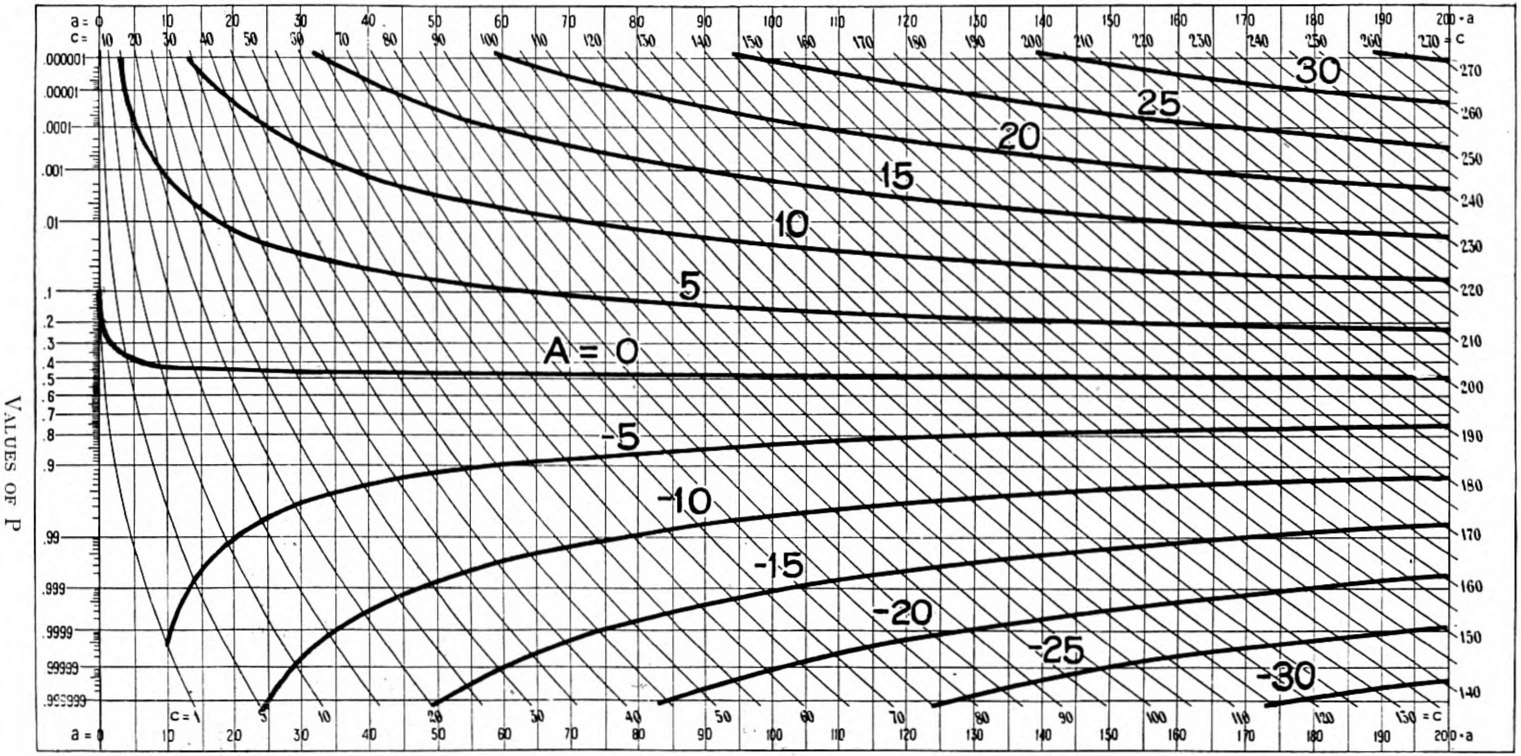


Fig. 4—Curves showing the initial rate of change, $[da/d(1/n)]/a$, as n decreases from infinity, in the point binomial probability curve. The values A attached to the curves are the value of the first coefficient in the following expansion for the relative change in the average,

$$\frac{a(c,n,P)-a(c,\infty,P)}{a(c,\infty,P)} = An^{-1} + \frac{1}{12}[14A^2+(3a+2)A+a]n^{-2}+\ldots, \text{ where } A=\tfrac{1}{2}(c-a-1) \text{ and } a=a(c,\infty,P).$$

n is an integer which cannot be less than the average a or the number of occurrences c.

Fig. 3, making use of Fig. 2 as a background, shows for $c=1$, 10, 50, 100, 150 and 200, the curves of the point binomial with $n=\infty$ and $n=c$, as heavy full and dashed lines, respectively. Each pair of curves, with the exception of the first, crosses in the neighborhood of $P=\frac{1}{2}$, and, except near this crossing, all of the intermediate curves of each family of c curves lie between these extreme curves. The relative change in the probability P or $1-P$, when these probabilities are small, due to reducing n to this lower limit c, for the c curve, is great, but the relative increase in the average a is only moderate over the greater part of the range covered by Fig. 3. The extreme relative change in the average a is shown by dots placed on each of the Poisson exponential curves, each dot being located at the point where the extreme relative increase in the average is $\pm.25$, $\pm.50$, or $\pm.75$. The relative increment in the average ranges, for Fig. 3, from a decrease of 93 per cent at $P=.999999$ on $c=1$ to an increase of 97 per cent at $P=.000001$ on $c=9$ and 10, but the greater part of the field is included between the beaded curves for ± 50 per cent. Having thus obtained, by examining Fig. 3, a general idea of the relative and absolute numerical magnitudes of the extreme changes to which the probability curves are subject, we are in a better position to make practical use of the curves of Figs. 4 and 5 for the small initial shift in the curves occurring when the number of trials is finite but still large compared with c.

The rate at which the probability curves start to shift, when the number of trials is decreased from infinity, is shown by Fig. 4, which gives the value of the first coefficient A in the expansion, in descending powers of n, for the relative increment in the average. In the upper part of the curves the shift is to the right and in the lower part of the curves it is to the left. The point at which the curve remains initially at rest is shown by the intersection of the c curve with the curve for $A=0$. Since $A=35$ is the largest arithmetical value occurring on Fig. 4 and $n=700$ will make the first term of the series equal 1/20, and the next term is then still smaller, it follows that Fig. 2 redrawn for 700 trials would not show a difference of more than about 5 per cent in any value of the average. For Fig. 1 the corresponding number of trials is 220; it may be shown by direct computation that n may even be reduced to the lower limit 1 with only a small percentage change in the abscissas of the upper portion of the curve $c=1$.

Curves similar to Fig. 4 showing the exact number of trials producing a given relative or absolute shift in the average would be useful. Still another variation is shown by Fig. 5 where the curves give the first coefficient in the expansion, in descending powers of n, of the ratio of the increments in probability, due to a decrease in n and to unit increase in c. These curves therefore show the initial rate at which any c curve approaches the $c+1$ curve above it, if the scale of ordinates were made linear; below the curve $A=0$ the initial shift is downward as indicated by the negative sign for the A's. If sets of curves corresponding to Figs. 1 and 2 were drawn for the number of trials $n=400$ and 2000, respectively, no curve would be shifted by as much as the original distance between the curves shown, since the maximum values on Fig. 5 up to $a=15$ and 200 are 400 and 10,000, respectively; Fig. 2 shows only every fifth curve; the second term of the series indicates that the initial maximum rate of shift is not maintained as n decreases at these points.

The second question arising in connection with the use of the curves is their accuracy. Fig. 1 was drawn with the greatest care on a scale somewhat larger than that of the reproduction, and errors are believed to be only of the order of uncertainty of reading such curves with the unaided eye. Fig. 2 was drawn with less skill and shows larger deviations but it has proved accurate enough for ordinary applications.[2]

The third question which may arise is that of going beyond the curves either in range or in accuracy.[3] The exact calculated values employed in plotting the curves up to $c=101$ are contained in Table II, every entry having been independently checked by two persons. The greater part of the table was calculated by means of a new formula which so expresses the average in terms of P and c as to readily give accurate results for the central range of P with large values of c, which

[1] Cf. Soper, H.E., The Numerical Evaluation of the Incomplete B-Function, 1921, p. 41, and Fisher, A., Mathematical Theory of Probabilities, 2nd Edition, 1922, p. 276.

[2] These claims for the accuracy of the curves of Figs. 1 and 2 have been confirmed by comparison with Pearson's Tables of the Incomplete Γ-Function, 1922, which has been received during the proof-reading of this paper. His tabulated function $I(u, p)$ is, in the notation of the present paper, the probability P corresponding to the average $a=u\sqrt{p+1}$ and the number of occurrences $c=p+1$.

[3] When c is not greater than 51, Pearson's tables may be employed. If the probability is assigned, as in many practical engineering problems, finding the corresponding average from the tables requires interpolation. Formula (1) of the present paper gives the average directly, that is, it gives the inverse incomplete gamma function. The following formula gives c in terms of a:

$$c=a\left[1-ta^{-\frac{1}{2}}+\tfrac{1}{6}(t^2+2)a^{-1}+\frac{1}{72}(t^3+2t)a^{-\frac{3}{2}}+\ldots\right]$$

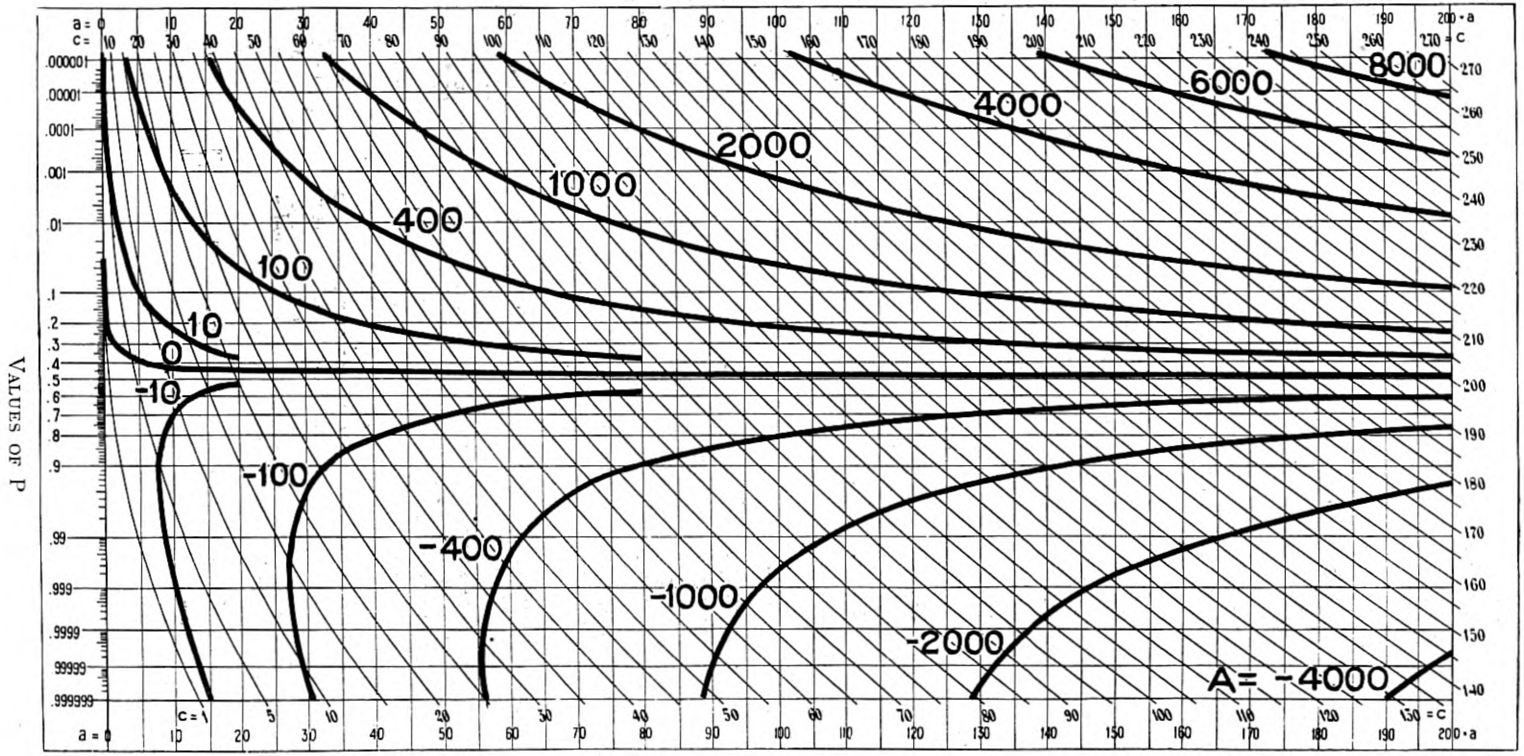


Fig. 5—Curves showing the initial rate of change in the probability, $dP/d(1/n)$, as n decreases from infinity, divided by the increment in P due to unit increase in c. The values of A attached to the curves are the value of the first coefficient in the expansion,

$$\frac{P(c,n,a)-P(c,\infty,a)}{P(c+1,\infty,a)-P(c,\infty,a)}=An^{-1}-\frac{c}{24}(3D^3-7aD+a^2-c^2+1)n^{-2}+\ldots,$$ where $A=\frac{1}{2}c(c-a-1)$, and $D=(c-a-1)$.

is the domain in which the ordinary formulas are not convenient for calculation. This is formula (1) below which involved transforming the normal probability integral to fit the skew probability summation of Poisson's exponential binomial limit. The reason for thinking that this transformation would prove useful is made clear by noting that in Figs. 1 and 2 the curves become more and more uniformly spaced with increasing values of the average a and thus the probability approximates more and more closely to the normal probability integral, since this is the scale employed for the ordinates. The results of the mathematical work are summed up in the following formula:

For Poisson's exponential binomial limit the average a is expressed as a function of the probability P of at least c occurrences by the infinite series

$$a = c\sum_{n=0}^{\infty} Q_n c^{-\frac{1}{2}n}, \qquad (1)$$

where the coefficients Q_n are functions of the argument t corresponding to the probability P expressed in the form of the normal probability integral,

$$P = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{1}{2}t^2}dt; \qquad (2)$$

twelve of these coefficients are given in the following table:

TABLE I COEFFICIENTS IN FORMULA (1) FOR THE AVERAGE

n	Q_n
0	1
1	t
2	$(t^2-1)/3$
3	$(t^3-7t)/2^2 3^2$
4	$(-3t^4-7t^2+16)/2^1 3^4 5$
5	$(9t^5+256t^3-433t)/2^5 3^5 5$
6	$(12t^6-243t^4-923t^2+1{,}472)/2^3 3^6 5^1 7$
7	$(-3{,}753t^7-4{,}353t^5+289{,}517t^3+289{,}717t)/2^7 3^8 5^2 7$
8	$(270t^8+4{,}614t^6-9{,}513t^4-104{,}989t^2+35{,}968)/2^4 3^9 5^2 7$
9	$(-5{,}139t^9-547{,}848t^7-2{,}742{,}210t^5+7{,}016{,}224t^3+37{,}501{,}325t)/2^{11}3^{10}5^2 7$

10 $(-364{,}176t^{10}+6{,}208{,}146t^{8}+125{,}735{,}778t^{6}+303{,}753{,}831t^{4}$
$-672{,}186{,}949t^{2}-2{,}432{,}820{,}224)/2^{7}3^{13}5^{3}7^{1}11$

11 $(199{,}112{,}985t^{11}+1{,}885{,}396{,}761t^{9}-31{,}857{,}434{,}154t^{7}$
$-287{,}542{,}736{,}226t^{5}-556{,}030{,}221{,}167t^{3}+487{,}855{,}454{,}729t)/$
$2^{13}3^{14}5^{3}7^{2}11$

For any given value of P the corresponding value of t in (2) can be found from tables of the probability integral. The value of a for this value of P and for any value of c can then be determined by (1). In this way values of a were calculated for every integral value of c from 1 to 101 and for eleven particular values of P: 0.000001, 0.0001, 0.01, 0.1, 0.25, 0.5, 0.75, 0.9, 0.99, 0.9999, 0.999999. These results are presented in Table II. The numerical values of the coefficients Q_1 to Q_7, corresponding to the particular values of P used in Table II, are given in Table VII.

From the information given in Table II, two sets of curves were drawn, Figs. 1 and 2, the first for each integral value of c in the range $a=0$ to $a=15$ and $P=0.000001$ to $P=0.999999$, and the second for every fifth integral value of c in the range $a=0$ to $a=200$ and the same range of P. From these curves any one of the variables (P, c, a) may be found corresponding to assigned values of the other two, subject to the practical condition that c is to be an integer.

Proof

The well-known expressions for the summation of Poisson's exponential binomial limit are:

$$P=\frac{a^ce^{-a}}{c!}+\frac{a^{c+1}e^{-a}}{(c+1)!}+\frac{a^{c+2}e^{-a}}{(c+2)!}+\ldots$$

$$=\sum_{s=c}^{\infty}\frac{a^se^{-a}}{s!}$$

$$=1-\left[1+\frac{a}{1!}+\frac{a^2}{2!}+\ldots+\frac{a^{c-1}}{(c-1)!}\right]e^{-a}$$

$$=1-\sum_{s=0}^{c-1}\frac{a^se^{-a}}{s!}$$

$$=\frac{1}{\Gamma(c)}\int_0^a a^{c-1}e^{-a}da. \tag{3}$$

The series expansion (1) is determined by equating the integrands of (2) and (3),

$$\frac{1}{\Gamma(c)} a^{c-1} e^{-a} da = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt, \tag{4}$$

and solving for positive values of a with the condition that $t = -\infty$ when $a = 0$.

$$\left.\begin{aligned} &\text{Let} \quad c = \frac{1}{b^2}, \qquad a = \frac{1}{b^2} Q, \qquad Q = e^L, \\ &\frac{\Gamma(c)}{\sqrt{2\pi}} = b(b^2 e)^{-1/b^2} e^M, \\ &R = \frac{Q - L - 1}{b^2} - \tfrac{1}{2}t^2 + M. \end{aligned}\right\} \tag{5}$$

Substituting these values (5) in equation (4),

$$L' = be^R, \tag{6}$$

where L' is written for dL/dt.

$$\text{Let } L = \sum_{s=0}^{\infty} L_s b^s, \quad M = \sum_{s=0}^{\infty} M_s b^s, \quad R = \sum_{s=0}^{\infty} R_s b, \quad Q = \sum_{s=0}^{\infty} Q_s b^s, \tag{7}$$

where the coefficients are polynomials in t (constants in the case of the series for M). Upon substituting these series expansions for the functions in the last equality of (5) and equating coefficients of like powers of b, we obtain

$$\begin{aligned} 0 &= Q_0 - L_0 - 1, \\ 0 &= Q_1 - L_1, \\ R_0 &= Q_2 - L_2 - \tfrac{1}{2}t^2 + M_0, \\ R_1 &= Q_3 - L_3 + M_1, \\ R_2 &= Q_4 - L_4 + M_2, \\ &\ldots\ldots\ldots\ldots \\ R_n &= Q_{n+2} - L_{n+2} + M_n, \ (n = 1, 2, 3 \ldots). \end{aligned} \tag{8}$$

From (5) we obtain $Q_o = e^{L_o}$, and then $L_o + 1 = e^{L_o}$, the only real solution of which is $L_o = 0$, and therefore, $Q_o = 1$.

Utilizing these initial values we obtain

$$Q_1 = L_1,$$
$$Q_2 = L_2 + \tfrac{1}{2} L_1 Q_1,$$
$$Q_3 = L_3 + \tfrac{2}{3} L_2 Q_1 + \tfrac{1}{3} L_1 Q_2,$$
$$\ldots\ldots\ldots\ldots$$
$$Q_n = \sum_{s=o}^{n-1} \frac{n-s}{n} L_{n-s} Q_s, \quad (n=1,\ 2,\ 3\ \ldots). \tag{9}$$

From (6) we obtain $L_1' = e^{R_o}$, and from that $L_1' = e^{\frac{1}{2}L_1^2 - \frac{1}{2}t^2 + M_0}$.

Since L_1 is a polynomial in t, and M_0 a constant, we must have $L_1' = 1$, $L_1 = \pm t$, $M_0 = 0$, that is, $L_1 = t$, and hence $Q_1 = t$.

Then $L_2' = R_1$,

$$L_3' = R_2 + \tfrac{1}{2} R_1 L_2',$$
$$L_4' = R_3 + \tfrac{2}{3} R_2 L_2' + \tfrac{1}{3} R_1 L_3',$$
$$\ldots\ldots\ldots\ldots$$
$$L_{n+1}' = \sum_{s=o}^{n-1} \frac{n-s}{n} R_{n-s} L_{s+1}', \quad (n=1,\ 2,\ 3\ \ldots). \tag{10}$$

The next set of coefficients can now be deduced, as follows:

$$Q_3 = L_3 + \tfrac{2}{3} L_2 Q_1 + \tfrac{1}{3} L_1 Q_2,$$
$$R_1 = Q_3 - L_3 + M_1,$$
$$R_1 = \tfrac{2}{3} L_2 Q_1 + \tfrac{1}{3} L_1 Q_2 + M_1,$$
$$L_2' = R_1,$$
$$Q_2 = L_2 + \tfrac{1}{2} L_1 Q_1,$$
$$L_2' = \tfrac{2}{3} L_2 Q_1 + \tfrac{1}{3} L_1 L_2 + \tfrac{1}{6} L_1^2 Q_1 + M_1,$$
$$L_1 = t,$$
$$Q_1 = t,$$
$$L_2' = L_2 t + \tfrac{1}{6} t^3 + M_1.$$

But M_1 is a constant, and L_2 is a polynomial in t. Let

$$L_2 = c_2 t^2 + c_1 t + c_o,$$

it being evident that L_2 is of the second degree. Then

$$L_2' = 2c_2 t + c_1.$$

Substituting and equating coefficients of like powers of t

$$c_2+\tfrac{1}{6}=0,\ c_1=0,\ c_o=2c_2,\ M_1=c_1.$$

Hence
$$\left.\begin{aligned} &L_2=(-t^2-2)/6, && R_1=-t/3,\\ &M_1=0, && Q_2=(t^2-1)/3. \end{aligned}\right\} \quad (11)$$

Starting with these initial values equations (8)–(10) are sufficient to determine as many coefficients in the expansions (7) as are required. In order to demonstrate this, assume that all the coefficients up to and including L_k, M_{k-1}, R_{k-1}, Q_k have been determined. It can then be shown that the next coefficient in each expansion can be obtained from these data, as follows:

For $n=k+2$, equation (9) can be written

$$Q_{k+2}=L_{k+2}+\frac{k+1}{k+2}L_{k+1}t+\sum_{s=2}^{k}\frac{k+2-s}{k+2}L_{k+2-s}Q_s+\frac{1}{k+2}Q_{k+1}t, \quad (12)$$

where Q_{k+1}, Q_{k+2}, L_{k+1}, L_{k+2} are the unknown quantities. For $n=k$ and $n=k-1$, equation (8) assumes the forms

$$R_k=Q_{k+2}-L_{k+2}+M_k, \quad (13)$$

and

$$R_{k-1}=Q_{k+1}-L_{k+1}+M_{k-1}, \quad (14)$$

respectively, where all the quantities are unknown except R_{k-1} and M_{k-1}. For $n=k$, equation (10) can be written in the form

$$L_{k+1}=R_k+\sum_{s=1}^{k-1}\frac{k-s}{k}R_{k-s}L'_{s+1}, \quad (15)$$

where L_{k+1} and R_k are the unknown quantities. Substituting in (12) the value of $(Q_{k+2}-L_{k+2})$ found from (13), and then substituting the value of R_k found from (15) and the value of Q_{k+1} from (14),

$$L'_{k+1}=M_k+\frac{k+1}{k+2}L_{k+1}t+\frac{1}{k+2}(R_{k-1}+L_{k+1}-M_{k-1})t$$
$$+\sum_{s=2}^{k}\frac{k+2-s}{k+2}L_{k+2-s}Q_s+\sum_{s=1}^{k-1}\frac{k-s}{k}R_{k-s}L'_{s+1}. \quad (16)$$

This is a linear differential equation in L_{k+1} as a function of t, all the coefficients being known functions of t with the exception of M_k which is an undetermined numerical constant. By a suitable choice of the constant M_k, (16) may be solved for L_{k+1} as a polynomial in

t of the $(k+1)$st degree. R_k may then be determined by (15) and Q_{k+1} by (14). From these results, the next set of coefficients may be found, and so on. The values of the coefficients for $k=2$ (L_2, M_1, R_1, Q_2) have been found, and equations (8)–(10) are valid for the particular values of n utilized in the above method. Hence the next set of coefficients (L_3, M_2, R_2, Q_3) may be found, and in the same way, as many more as are desired. The detailed work of the first step is indicated below:

Substituting $k=2$ in (16),

$$L_3' = M_2 + \tfrac{3}{4}L_3 t + \tfrac{1}{4}(R_1 + L_3 - M_1)t + \tfrac{1}{2}L_2 Q_2 + \tfrac{1}{2}R_1 L_2'. \tag{17}$$

Substituting in (17) the values known from (11),

$$L_3' = L_3 t + (-t^4 - 2t^2 + 2)/36 + M_2. \tag{18}$$

Let L_3 be a polynomial of the form $(A_3t^3 + A_2t^2 + A_1t + A_0)$ and substitute in (18). Upon equating coefficients of like powers of t, we find that $A_3 = 1/36$, $A_2 = 0$, $A_1 = 5/36$, $A_0 = 0$, and $M_2 = 1/12$. R_2 is then obtained by substituting these values in (15) and Q_3 from (14). The results are as follows:

$$\left.\begin{aligned} L_3 &= (t^3 - 5t)/36, & R_2 &= (t^2 - 5)/36, \\ M_2 &= 1/12, & Q_3 &= (t^3 - 7t)/36. \end{aligned}\right\} \tag{19}$$

The actual work of computing these coefficients has been performed up to and including $k=11$ (L_{11}, M_{10}, R_{10}, Q_{11}). These results are presented in the attached tables: Q_n in I, L_n in III, L_n' in IV, R_n in V, and M_n in VI. From this information the next coefficient in the series (1), Q_{12}, can be computed by the method outlined above.

It may be pointed out in conclusion that the expansion of M presented in Table VI is the asymptotic series obtained in Stirling's expansion of $\Gamma(c)$, as is to be expected from equations (5). This in itself constitutes a partial check upon the determination of the coefficients.

Additional Properties of the Curves

At the probability $P=0.5$, the difference $(c-a)=1/3$, approximately.[4] Discrepancies are so small as not to be positively dis-

[4] This recalls the approximate rule that the median lines one-third of the distance from the mean towards the mode. (Yule, Theory of Statistics, 1911, p. 121.) But in the Poisson exponential the median never lies between the mean and the mode; the median occurs at the first integer above or below the mean, whichever integer corresponds to the c curve cutting $P=0.5$ next below the mean, while the mode is always at the first integer less than the mean. For the range of cases having a given mode, however, the mean and the median are, on the average, greater than the mode by $\frac{1}{2}$ and approximately $\frac{1}{3}$, respectively; thus the median must line one-third of the distance from the mean towards the mode in the case of the corresponding heterogeneous samplings.

cernible on Fig. 1, but Table II gives for $c=1, 2, 3 \ldots 100$, the differences $(c-a)=0.3069, 0.3217, 0.3259, \ldots 0.3331$, which differ but little from $0.3333 \ldots$, which is approached more and more closely for large values of c.

At $P=0.5$ and large values of c the derivative along the c curve is $dP/da=1/\sqrt{2\pi c}$, as found by differentiating (3), substituting $a=c-1/3$ and Stirling's expression for the gamma function. Thus, for large values of c the slope of the curve at $P=0.5$ decreases numerically as the square root of c increases. For large values of c the curves are approximately straight over the wide range of probability shown in the figures. This, in connection with the additional fact that the standard deviation $\sqrt{npq}$ is always equal to $\sqrt{a}$ for the Poisson exponential, is an alternative way of arriving at the expression for the derivative given above.

I am indebted to Miss Edith Clarke for extending the series of formula (1) to seven terms, for making all of the original computations and for drawing Fig. 1, and to Miss Sallie E. Pero for extending the formula to its present eleven terms, and for checking all of the preceding work; the single error which she found occurred in the seventh term of the expansion where it was without effect on the final numerical results. Finally, the work was entirely rechecked, without discovering additional errors, by Mr. Ronald M. Foster, who also put the mathematical work into its present form, pointed out the asymptotic nature of the expansion and compared the overlapping numerical results with those obtained by direct summation by Miss Lucy Whitaker[5] and more recently by Mr. E. C. Molina, as well as with his earlier table.[6]

[5] Tables for Statisticians and Biometricians, 1914, Table LII.

[6] Computation Formula for the Probability of an Event Happening at Least C Times in N Trials, *American Mathematical Monthly*, XX, June, 1913, p. 193.

TABLE II. VALUES OF THE AVERAGE a CORRESPONDING TO GIVEN P AND c IN POISSON'S EXPONENTIAL SUMMATION

$P =$	0.000001	0.0001	0.01	0.1	0.25	0.5	0.75	0.9	0.99	0.9999	0.999999
$c =$ 1	0.00*	0.000*	0.010	0.1054	0.2877	0.6931	1.3863	2.3026	4.605	9.210	13.82
2	0.00*	0.014	0.149	0.5318	0.9613	1.6783	2.6926	3.8897	6.638	11.756	16.69
3	0.02	0.086	0.436	1.1021	1.7273	2.6741	3.9204	5.3223	8.406	13.928	19.13
4	0.07	0.232	0.823	1.7448	2.5353	3.6721	5.1094	6.6808	10.045	15.914	21.35
5	0.17	0.444	1.279	2.4326	3.3686	4.6709	6.2744	7.9936	11.605	17.782	23.43
6	0.31	0.714	1.785	3.1519	4.2192	5.6702	7.4227	9.2747	13.108	19.567	25.41
7	0.50	1.030	2.330	3.8948	5.0827	6.6696	8.5585	10.5321	14.571	21.290	27.32
8	0.73	1.387	2.906	4.6561	5.9561	7.6692	9.6844	11.7709	16.000	22.962	29.16
9	0.99	1.778	3.507	5.4325	6.8376	8.6690	10.8024	12.9947	17.403	24.595	30.96
10	1.28	2.198	4.130	6.2213	7.7259	9.6687	11.9138	14.2060	18.783	26.193	32.71
11	1.60	2.643	4.771	7.0208	8.6198	10.6685	13.0196	15.4066	20.145	27.762	34.43
12	1.94	3.111	5.428	7.8293	9.5186	11.6684	14.1206	16.5981	21.490	29.306	36.11
13	2.31	3.600	6.099	8.6459	10.4217	12.6682	15.2173	17.7816	22.821	30.829	37.77
14	2.69	4.106	6.782	9.4696	11.3286	13.6681	16.3102	18.9580	24.139	32.331	39.41
15	3.10	4.629	7.477	10.2996	12.2388	14.6680	17.3999	20.1280	25.446	33.816	41.02
16	3.52	5.167	8.181	11.1353	13.1521	15.6679	18.4865	21.2924	26.743	35.286	42.62
17	3.96	5.718	8.895	11.9761	14.0680	16.6679	19.5704	22.4516	28.031	36.741	44.19
18	4.42	6.281	9.616	12.8217	14.9865	17.6678	20.6518	23.6061	29.310	38.182	45.75
19	4.88	6.856	10.346	13.6715	15.9073	18.6677	21.7310	24.7563	30.581	39.612	47.30
20	5.36	7.442	11.082	14.5253	16.8301	19.6677	22.8080	25.9025	31.845	41.031	48.83
21	5.86	8.037	11.825	15.3827	17.7550	20.6676	23.8831	27.0451	33.103	42.440	50.34
22	6.36	8.641	12.574	16.2436	18.6816	21.6676	24.9564	28.1843	34.355	43.839	51.85
23	6.87	9.255	13.329	17.1076	19.6099	22.6675	26.0281	29.3203	35.601	45.229	53.35
24	7.40	9.876	14.088	17.9746	20.5397	23.6675	27.0982	30.4533	36.841	46.610	54.83
25	7.93	10.505	14.853	18.8443	21.4710	24.6675	28.1668	31.5836	38.077	47.984	56.30
26	8.47	11.141	15.623	19.7167	22.4038	25.6674	29.2340	32.7112	39.308	49.351	57.77
27	9.02	11.783	16.397	20.5915	23.3378	26.6674	30.3000	33.8364	40.535	50.711	59.23
28	9.57	12.432	17.175	21.4687	24.2730	27.6674	31.3647	34.9593	41.757	52.064	60.67
29	10.14	13.088	17.957	22.3480	25.2094	28.6674	32.4283	36.0799	42.975	53.411	62.11
30	10.71	13.748	18.742	23.2294	26.1469	29.6673	33.4907	37.1985	44.190	54.752	63.55
31	11.29	14.415	19.532	24.1128	27.0855	30.6673	34.5521	38.3151	45.401	56.087	64.97
32	11.87	15.086	20.324	24.9981	28.0250	31.6673	35.6126	39.4298	46.609	57.417	66.39
33	12.46	15.763	21.120	25.8852	28.9655	32.6673	36.6720	40.5427	47.813	58.742	67.81
34	13.06	16.444	21.919	26.7740	29.9069	33.6673	37.7306	41.6540	49.015	60.062	69.21

*These values which require more decimals are $a = 0.0000010$ and 0.0001000 for $c = 1$ and 0.0014149 for $c = 2$.

TABLE II. (*Continued*)

$P =$	0.000001	0.0001	0.01	0.1	0.25	0.5	0.75	0.9	0.99	0.9999	0.999999
$c = 35$	13.66	17.130	22.721	27.6645	30.8492	34.6672	38.7883	42.7635	50.213	61.377	70.61
36	14.27	17.821	23.525	28.5565	31.7923	35.6672	39.8452	43.8715	51.409	62.688	72.01
37	14.88	18.515	24.333	29.4500	32.7361	36.6672	40.9013	44.9780	52.601	63.995	73.40
38	15.50	19.214	25.143	30.3449	33.6808	37.6672	41.9566	46.0831	53.791	65.298	74.79
39	16.13	19.916	25.955	31.2413	34.6262	38.6672	43.0112	47.1868	54.979	66.596	76.16
40	16.75	20.622	26.770	32.1389	35.5723	39.6672	44.0651	48.2891	56.164	67.891	77.54
41	17.39	21.332	27.587	33.0379	36.5190	40.6672	45.1184	49.3902	57.347	69.183	78.91
42	18.02	22.045	28.406	33.9380	37.4664	41.6671	46.1709	50.4900	58.528	70.470	80.28
43	18.66	22.762	29.228	34.8394	38.4145	42.6671	47.2229	51.5886	59.707	71.755	81.64
44	19.31	23.481	30.051	35.7419	39.3631	43.6671	48.2742	52.6861	60.884	73.036	83.00
45	19.95	24.204	30.877	36.6455	40.3123	44.6671	49.3250	53.7825	62.059	74.314	84.35
46	20.61	24.930	31.704	37.5502	41.2621	45.6671	50.3752	54.8778	63.231	75.589	85.70
47	21.26	25.659	32.534	38.4560	42.2125	46.6671	51.4248	55.9721	64.402	76.860	87.05
48	21.92	26.391	33.365	39.3627	43.1633	47.6671	52.4739	57.0654	65.571	78.129	88.39
49	22.58	27.125	34.198	40.2704	44.1147	48.6671	53.5225	58.1576	66.738	79.396	89.73
50	23.25	27.862	35.032	41.1791	45.0666	49.6671	54.5706	59.2490	67.903	80.659	91.06
51	23.92	28.602	35.869	42.0886	46.0190	50.6671	55.6182	60.3394	69.067	81.920	92.40
52	24.59	29.344	36.707	42.9991	46.9718	51.6670	56.6654	61.4290	70.230	83.179	93.72
53	25.27	30.089	37.546	43.9104	47.9251	52.6670	57.7121	62.5177	71.390	84.435	95.05
54	25.95	30.836	38.387	44.8226	48.8788	53.6670	58.7584	63.6055	72.549	85.688	96.37
55	26.63	31.585	39.229	45.7355	49.8330	54.6670	59.8042	64.6926	73.707	86.940	97.69
56	27.31	32.337	40.073	46.6493	50.7876	55.6670	60.8496	65.7788	74.863	88.189	99.01
57	28.00	33.090	40.918	47.5638	51.7425	56.6670	61.8946	66.8643	76.018	89.435	100.33
58	28.69	33.846	41.765	48.4791	52.6979	57.6670	62.9392	67.9490	77.172	90.680	101.64
59	29.38	34.604	42.612	49.3951	53.6537	58.6670	63.9835	69.0330	78.324	91.922	102.95
60	30.08	35.364	43.462	50.3118	54.6098	59.6670	65.0273	70.1163	79.475	93.163	104.25
61	30.77	36.126	44.312	51.2292	55.5663	60.6670	66.0708	71.1989	80.625	94.401	105.56
62	31.47	36.890	45.164	52.1473	56.5232	61.6670	67.1139	72.2808	81.773	95.638	106.86
63	32.17	37.656	46.016	53.0661	57.4804	62.6670	68.1567	73.3620	82.921	96.873	108.16
64	32.88	38.423	46.870	53.9855	58.4380	63.6670	69.1991	74.4426	84.067	98.106	109.45
65	33.59	39.193	47.726	54.9055	59.3959	64.6670	70.2412	75.5226	85.212	99.337	110.75
66	34.29	39.964	48.582	55.8262	60.3541	65.6670	71.2830	76.6020	86.355	100.566	112.04
67	35.01	40.736	49.439	56.7474	61.3126	66.6670	72.3245	77.6807	87.498	101.793	113.33
68	35.72	41.511	50.298	57.6692	62.2714	67.6670	73.3656	78.7589	88.640	103.019	114.62

TABLE II. (*Continued*)

P =	0.000001	0.0001	0.01	0.1	0.25	0.5	0.75	0.9	0.99	0.9999	0.999999
c=69	36.43	42.287	51.157	58.5917	63.2306	68.6670	74.4065	79.8365	89.781	104.244	115.90
70	37.15	43.065	52.017	59.5146	64.1900	69.6670	75.4470	80.9135	90.920	105.466	117.19
71	37.87	43.844	52.879	60.4382	65.1497	70.6669	76.4873	81.9900	92.059	106.687	118.47
72	38.59	44.625	53.741	61.3622	66.1097	71.6669	77.5273	83.0659	93.197	107.907	119.75
73	39.31	45.407	54.604	62.2868	67.0700	72.6669	78.5670	84.1413	94.333	109.125	121.03
74	40.04	46.191	55.469	63.2119	68.0306	73.6669	79.6064	85.2162	95.469	110.341	122.30
75	40.77	46.976	56.334	64.1375	68.9914	74.6669	80.6456	86.2906	96.604	111.556	123.58
76	41.49	47.763	57.200	65.0636	69.9525	75.6669	81.6845	87.3645	97.738	112.770	124.85
77	42.22	48.551	58.067	65.9902	70.9139	76.6669	82.7231	88.4379	98.871	113.982	126.12
78	42.96	49.341	58.935	66.9173	71.8755	77.6669	83.7615	89.5108	100.003	115.193	127.39
79	43.69	50.131	59.803	67.8448	72.8373	78.6669	84.7997	90.5833	101.135	116.402	128.66
80	44.42	50.923	60.673	68.7728	73.7994	79.6669	85.8376	91.6553	102.265	117.610	129.92
81	45.16	51.717	61.543	69.7013	74.7617	80.6669	86.8753	92.7268	103.395	118.817	131.18
82	45.90	52.511	62.414	70.6302	75.7243	81.6669	87.9127	93.7980	104.524	120.023	132.45
83	46.64	53.307	63.286	71.5595	76.6871	82.6669	88.9499	94.8686	105.652	121.227	133.71
84	47.38	54.104	64.159	72.4893	77.6501	83.6669	89.9869	95.9389	106.779	122.430	134.97
85	48.12	54.903	65.032	73.4194	78.6133	84.6669	91.0237	97.0087	107.906	123.632	136.22
86	48.87	55.702	65.906	74.3500	79.5768	85.6669	92.0602	98.0781	109.032	124.833	137.48
87	49.62	56.503	66.781	75.2810	80.5404	86.6669	93.0966	99.1471	110.157	126.033	138.73
88	50.36	57.305	67.657	76.2124	81.5043	87.6669	94.1327	100.2158	111.281	127.231	139.99
89	51.11	58.107	68.533	77.1442	82.4684	88.6669	95.1686	101.2840	112.405	128.428	141.24
90	51.86	58.911	69.410	78.0763	83.4326	89.6669	96.2043	102.3518	113.528	129.625	142.49
91	52.61	59.716	70.288	79.0088	84.3971	90.6669	97.2398	103.4193	114.650	130.820	143.74
92	53.37	60.523	71.166	79.9418	85.3618	91.6669	98.2752	104.4864	115.772	132.014	144.99
93	54.12	61.330	72.045	80.8750	86.3266	92.6669	99.3103	105.5531	116.893	133.207	146.23
94	54.88	62.138	72.925	81.8086	87.2917	93.6669	100.3452	106.6195	118.014	134.399	147.48
95	55.63	62.947	73.805	82.7426	88.2569	94.6669	101.3800	107.6855	119.133	135.590	148.72
96	56.39	63.757	74.686	83.6770	89.2224	95.6669	102.4146	108.7512	120.252	136.780	149.96
97	57.15	64.568	75.568	84.6116	90.1880	96.6669	103.4490	109.8165	121.371	137.969	151.20
98	57.91	65.381	76.450	85.5466	91.1537	97.6669	104.4832	110.8815	122.489	139.157	152.44
99	58.67	66.194	77.333	86.4820	92.1197	98.6669	105.5172	111.9462	123.606	140.344	153.68
100	59.44	67.008	78.216	87.4176	93.0858	99.6669	106.5511	113.0105	124.723	141.530	154.92
101	60.20	67.823	79.100	88.3536	94.0521	100.6669	107.5848	114.0745	125.839	142.715	156.16

TABLE III

n	L_n
0	0
1	t
2	$(-t^2-2)/2^1 3$
3	$(t^3+5t)/2^2 3^2$
4	$(-6t^4-59t^2-58)/2^2 3^4 5$
5	$(9t^5+232t^3+599t)/2^5 3^5 5$
6	$(24t^6-45t^4-817t^2+592)/2^4 3^6 5^1 7$
7	$(-3{,}753t^7-44{,}853t^5-149{,}683t^3-418{,}583t)/2^7 3^8 5^2 7$
8	$(540t^8+12{,}396t^6+77{,}283t^4+226{,}939t^2+217{,}112)/2^5 3^9 5^2 7$
9	$(-5{,}139t^9-416{,}952t^7-4{,}411{,}314t^5-17{,}022{,}320t^3-24{,}039{,}619t)/2^{11} 3^{10} 5^2 7$
10	$(-728{,}352t^{10}-2{,}418{,}858t^8+84{,}239{,}766t^6+514{,}580{,}817t^4+428{,}031{,}517t^2-2{,}293{,}097{,}728)/2^8 3^{13} 5^3 7^1 11$
11	$(199{,}112{,}985t^{11}+4{,}293{,}113{,}877t^9+28{,}888{,}236{,}342t^7+124{,}692{,}719{,}238t^5+654{,}335{,}303{,}761t^3+2{,}373{,}932{,}511{,}173t)/2^{13} 3^{14} 5^3 7^2 11$

TABLE IV

n	L'_n
0	0
1	1
2	$-t/3$
3	$(3t^2+5)/2^2 3^2$
4	$(-12t^3-59t)/2^1 3^4 5$
5	$(45t^4+696t^2+599)/2^5 3^5 5$
6	$(72t^5-90t^3-817t)/2^3 3^6 5^1 7$
7	$(-26{,}271t^6-224{,}265t^4-449{,}049t^2-418{,}583)/2^7 3^8 5^2 7$
8	$(2{,}160t^7+37{,}188t^5+154{,}566t^3+226{,}939t)/2^4 3^9 5^2 7$
9	$(-46{,}251t^8-2{,}918{,}664t^6-22{,}056{,}570t^4-51{,}066{,}960t^2-24{,}039{,}619)/2^{11} 3^{10} 5^2 7$
10	$(-3{,}641{,}760t^9-9{,}675{,}432t^7+252{,}719{,}298t^5+1{,}029{,}161{,}634t^3+428{,}031{,}517t)/2^7 3^{13} 5^3 7^1 11$
11	$(2{,}190{,}242{,}835t^{10}+38{,}638{,}024{,}893t^8+202{,}217{,}654{,}394t^6+623{,}463{,}596{,}190t^4+1{,}963{,}005{,}911{,}283t^2+2{,}373{,}932{,}511{,}173)/2^{13} 3^{14} 5^3 7^2 11$

TABLE V

n	R_n
0	0
1	$-t/3$
2	$(t^2+5)/2^2 3^2$
3	$(t^3-43t)/2^2 3^4 5$
4	$(-21t^4-49t^2+112)/2^4 3^5 5$
5	$(45t^5+488t^3+787t)/2^5 3^6 7$
6	$(-1{,}056t^6-32{,}103t^4-145{,}639t^2-150{,}452)/2^5 3^8 5^2 7$
7	$(-2{,}727t^7+34{,}773t^5+500{,}803t^3+1{,}282{,}103t)/2^7 3^9 5^2 7$
8	$(9{,}990t^8+112{,}614t^6+62{,}577t^4-1{,}193{,}539t^2-1{,}732{,}352)/2^8 3^{10} 5^2 7$
9	$(-28{,}663{,}299t^9-723{,}162{,}744t^7-4{,}907{,}564{,}946t^5$ $-14{,}409{,}113{,}392t^3-22{,}453{,}298{,}291t)/2^{11} 3^{13} 5^3 7^1 11$
10	$(12{,}763{,}008t^{10}+897{,}127{,}182t^8+11{,}273{,}606{,}766t^6$ $+58{,}618{,}777{,}197t^4+161{,}552{,}157{,}577t^2+172{,}910{,}387{,}072)/$ $2^9 3^{14} 5^3 7^2 11$

TABLE VI

n	M_n
0	0
1	0
2	1/12
3	0
4	0
5	0
6	−1/360
7	0
8	0
9	0
10	1/1260

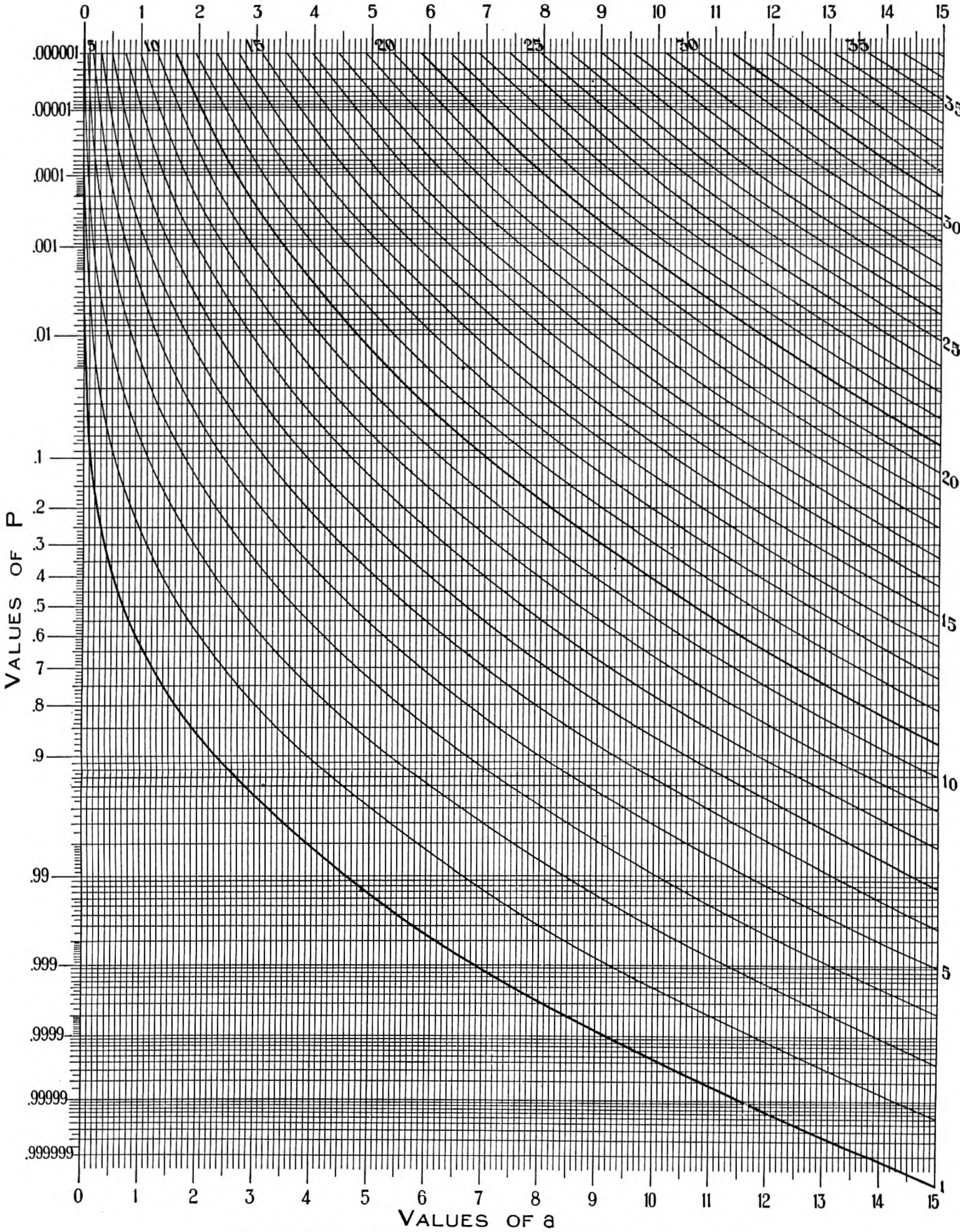


Fig. 1.—Probability Curves Showing Poisson's Exponential Summation $P=1-\left(1+\frac{a}{1!}+\frac{a^2}{2!}+\ldots\ldots+\frac{a^{c-1}}{(c-1)!}\right)e^{-a}$

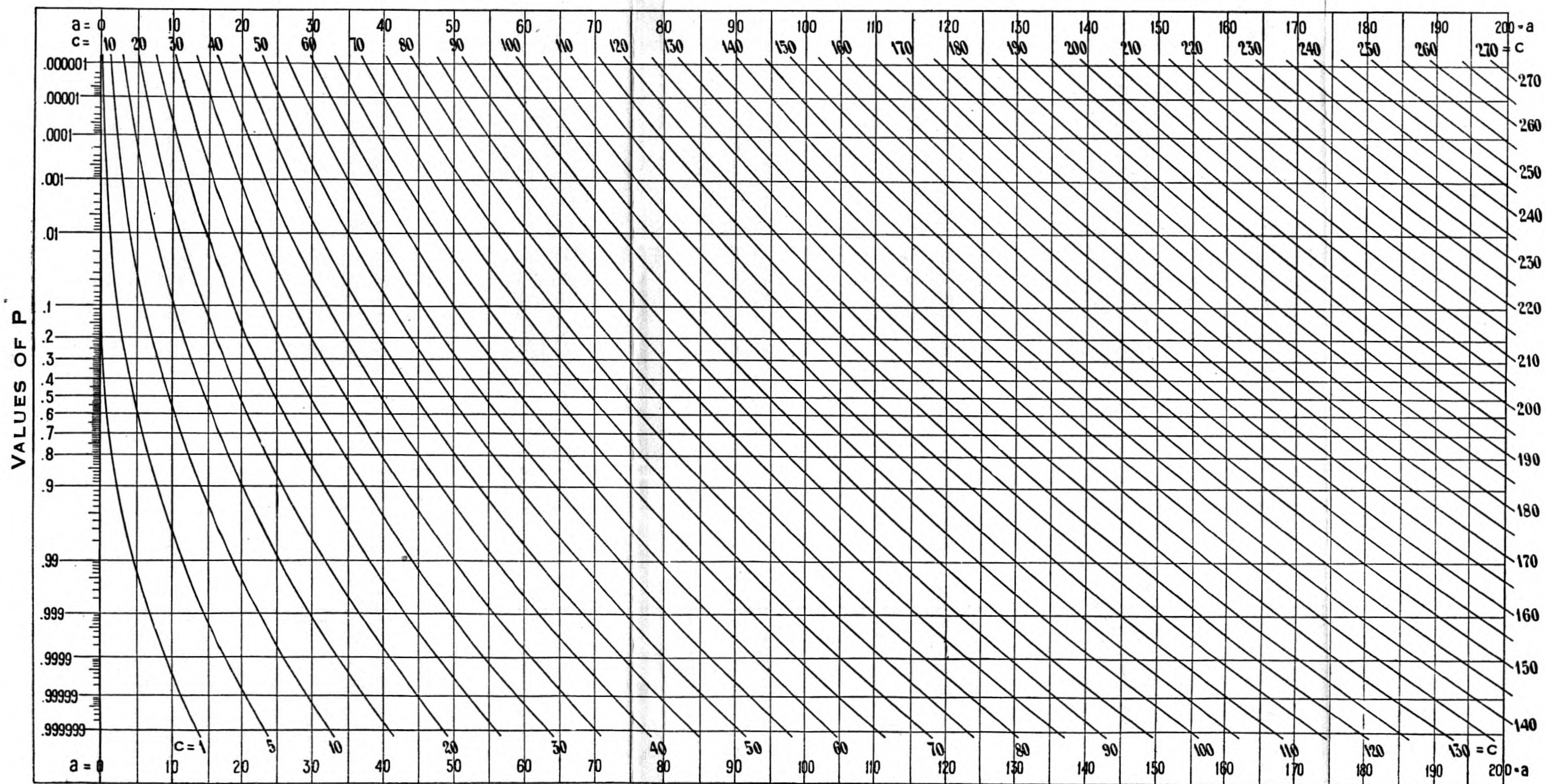


Fig. 2.—Probability Curves Showing Poisson's Exponential Summation $P=1-\left(1+\frac{a}{1!}+\frac{a^2}{2!}+\ldots+\frac{a^{c-1}}{(c-1)!}\right)e^{-a}$

TABLE VII.

P	θ	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
0.5	0	0	−0.333333	0	0.019753	0	0.007211	0
0.75	0.47693628	0.67448975	−0.181688	−0.122627	0.015055	−0.005459	0.004913	0.001928
0.9	0.90619380	1.2815516	0.214125	−0.190724	−0.004431	0.000386	−0.003166	0.006425
0.99	1.6449764	2.3263479	1.470632	−0.102625	−0.135493	0.072761	−0.042809	0.017953
0.999	2.1851242	3.0902323	2.849845	0.218852	−0.400528	0.225125	−0.093337	−0.012844
0.9999	2.6297418	3.7190165	4.277028	0.705692	−0.808289	0.461955	−0.127519	−0.163690
0.99999	3.0157332	4.2648908	5.729764	1.325587	−1.362810	0.789917	−0.115119	−0.535981
0.999999	3.3611785	4.7534242	7.198347	2.059162	−2.066386	1.216005	−0.024576	−1.251181

NOTE—By substituting $t=\theta\sqrt{2}$, equation (2) may be written $2P-1=\frac{2}{\sqrt{\pi}}\int_0^{\theta} e^{-\theta^2}d\theta$. Values of θ were found directly from tables of the probability integral. Upon changing P to $1-P$, θ, Q_1, Q_3, Q_5, and Q_7 change sign, while Q_2, Q_4, and Q_6 remain unchanged.

Bell System Sleet Storm Map

By J. N. KIRK

MANY, no doubt, have seen copies of the Bell System Sleet Storm Map which has been prepared to show the relative intensities and frequencies of sleet storms throughout the United States. In the midst of the current sleet storm season it may be of interest to discuss some of the factors which have led to the preparation of this map, and to outline in a general way, the means by which its indications are utilized in the design, construction and upkeep of the pole and aerial wire plant.

A sleet storm to be destructive to telephone plant must be accompanied by such atmospheric conditions as will cause either a relatively heavy deposit of ice with no wind or a deposit of ice with a considerable amount of wind. It is neither difficult nor expensive to construct the aerial plant so that it will withstand winds of relatively high velocities provided the wires are free of sleet. A slight deposit of sleet, however, rapidly increases the "sail area" against which the force of the wind is directed and the resulting load constitutes one of the most formidable and most difficult to anticipate of any of the destructive agencies with which wire using companies have to contend. Maintenance difficulties unfortunately, are not necessarily at an end with a lessening or cessation of the wind, for this change may often increase the precipitation of sleet, which undisturbed by the wind builds up around the wires, frequently stressing them beyond the breaking point. It is not uncommon for a wire of approximately 1/10 of an inch in diameter to accumulate sleet or ice under favorable atmospheric conditions to the extent of a cylindrical coating from one to two inches or more in diameter. Some idea of the destructive effects of such ice loads upon both poles and wire may be gathered from Fig. 1 in which ice coatings as much as 2½ inches thick without wind were responsible for practically a complete collapse of the pole and open wire plant.

Any medium through which information on the past performance of pole and wire plant when subjected to storm conditions, may be collected, analyzed and arranged in convenient form for reference is a very valuable aid in the design and construction of new plant which may be subjected to similar conditions. It would be both impracticable and uneconomical to attempt to design open wire pole lines to withstand such loads as are imposed by the occasional and unusual storm, such for example, as was responsible for the damage indicated in Fig. 1. It is, however, possible with proper data available to economically design the telephone lines so that they will withstand the more frequent storms of an average severity pertaining to the areas in which the lines are to be located. In predicting such average storms in different localities, a sleet storm map reflecting past performance as to storms and resulting damage is of material assistance.

In 1911 arrangements were made to collect data relative to sleet storms occurring in the territories of the various Associated Companies and to graphically indicate the cumulative results of such data

Fig. 1

with particular reference to the intensity and frequency of the storms recorded. In so far as possible the collection of these data was made retroactive so that, at least for those sections of the country where storms of a destructive nature had occurred, it was possible to start with an accumulation of reasonably accurate past performance data to which has annually been added very carefully collected information with regard to all subsequent storms of consequence. These data have made possible the preparation of a map, Fig. 2, in which the cumulative effect of storms extending over a considerable period of time is shown by various colors, markings and groupings of pins in such manner that for relatively large areas the future storm exposure to which the telephone plant is likely to be subjected, is indicated.

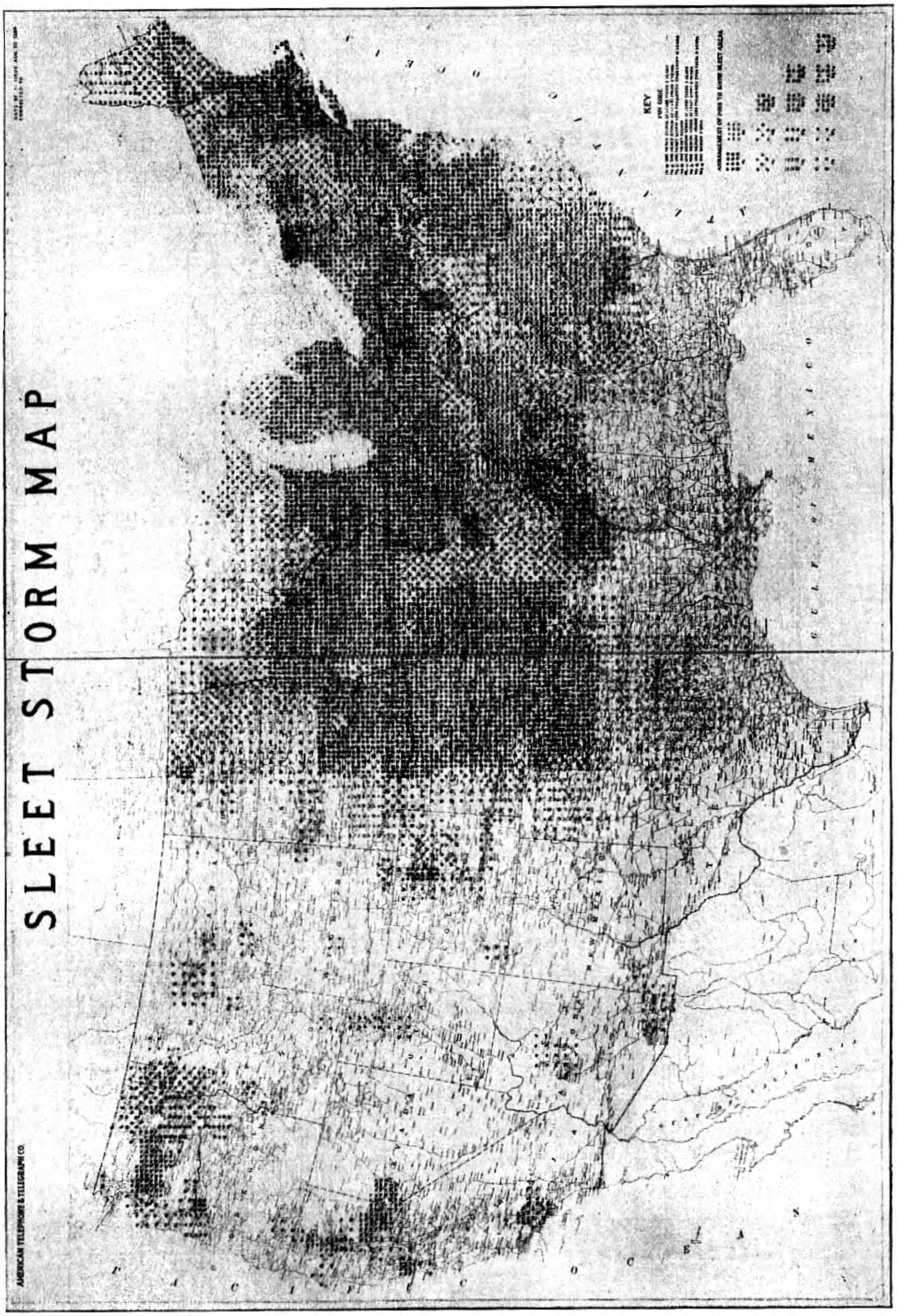


Fig. 2

A particularly important feature of this map in so far as its application to the design of telephone plant is concerned, is that it represents actual destructive effects upon existing telephone plant and is not directly governed by the extent of precipitation measured by the Weather Bureau or otherwise, which in certain areas may have been very heavy but from which no destructive effects were experienced.

For convenience in preparing and interpreting the sleet storm map, all storms have been classified as either heavy or medium and the frequency of their occurrence has been considered in four sub-divisions. The heavy storm was defined as one in which the diameter of the ice covering the wires was 3/4″ or greater and is represented on the map by the dark pins. The medium storm was defined as one in which the diameter of the ice covering the wires was less than 3/4″ yet sufficient to cause appreciable damage to the aerial plant, and is represented on the map by the light pins. In certain cases where the thickness of ice was not recorded, sleet storms have been classified by the amount of damage caused, commensurate with storms in which the ice deposit was known.

The four sub-divisions of storm frequency considered are represented by different markings and groupings of the pins corresponding to the class of storm experienced, light marking being used on the dark pins and dark marking on the light pins. At least one storm every three years is represented by unmarked, closely spaced pins; at least one storm every six years is represented by a single mark across the face of the pins the latter being more widely spaced and staggered; and storms occurring less frequently than every six years are represented by a cross on the face of the pins which are more closely spaced vertically than horizontally on the map. The fourth subdivision is in the nature of an exception to the third in that it covers those cases in which only one storm has been recorded. These single storms are represented by a single dot on the face of the pins, the latter being widely but evenly spaced. These various arrangements may occur singly or in combination resulting in the differently shaded areas presented by the map shown in Fig. 2. No shading or markings appear in those areas in which no sleet storms have been reported either because no aerial plant is maintained or because that area does not experience sleet.

The effect of the wind has been taken into consideration in the preparation of the sleet storm map only when such consideration would change the classification of the area involved. In the design of the aerial plant, however, the horizontal force exerted by the wind on the wires and therefore on the poles, is an important factor. For convenience in engineering studies and in the design of the pole and wire plant the country has been divided into areas according to three intensities of storm loading, designated as heavy, medium and light. Although not defined by the same limits, these classifications for all practical purposes agree with the relative severities indicated by the sleet storm map.

The heavy storm loading as used in engineering studies is defined as the load caused by a 1/2″ radial deposit of sleet on the wires, strand, etc., combined with a horizontal wind exerting a pressure of 8 pounds per square foot upon the projected areas of cylindrical surfaces. The medium storm loading is defined as 2/3 of the heavy loading. The light storm loading is defined as 2/3 of the medium or 4/9 of the heavy and is in general considered as applying to those areas in which no appreciable sleet storm damage has been recorded. This so-called light loading is, however, in the case of small wires such as those used in telephone plants, considerably in excess of the load created by high wind velocities with no sleet deposit.

The effective wind pressures used in defining these storm loadings are considered as being produced by steady winds of uniform velocity. The dynamic forces and cumulative loads which might be developed by sudden gusts of wind and vibration of the line are not considered because experience has indicated that aerial plant designed to withstand the more readily determined static forces is satisfactory.

A brief discussion of the effect of various sleet and wind loads upon the tension in telephone wires will emphasize the value of information as to probable storm loads, in the design of the aerial wire and pole plant.

Design of the Wire Plant

In the design of the wire plant the horizontal component and the vertical component of the storm load must be considered. The weight of the ice covered wire represents the vertical component. The wind pressure upon the projected area of the ice covered wire represents the horizontal component.

The curves in Fig. 3, show the relative magnitude of the loads upon the wires caused by (a) winds of various velocities with no sleet on the wires, (b) various ice coatings with no wind, and (c) the combination of an 8-pound (73.6 miles per hour indicated velocity) wind with the same ice coatings as in (b). These curves indicate that the wind exerts a relatively small load upon the wire plant even at high velocities and that the load caused by ice accumulation increases very rapidly as the radial coating of the ice increases.

Design of the Pole Plant

A pole is subjected to vertical and transverse loads. The vertical load comprises the combined weights of the pole, crossarms, wires and any snow and sleet that may adhere to them. The transverse load is considered to be caused by a horizontal wind pressure upon the projected area of pole, crossarms and ice covered wires.

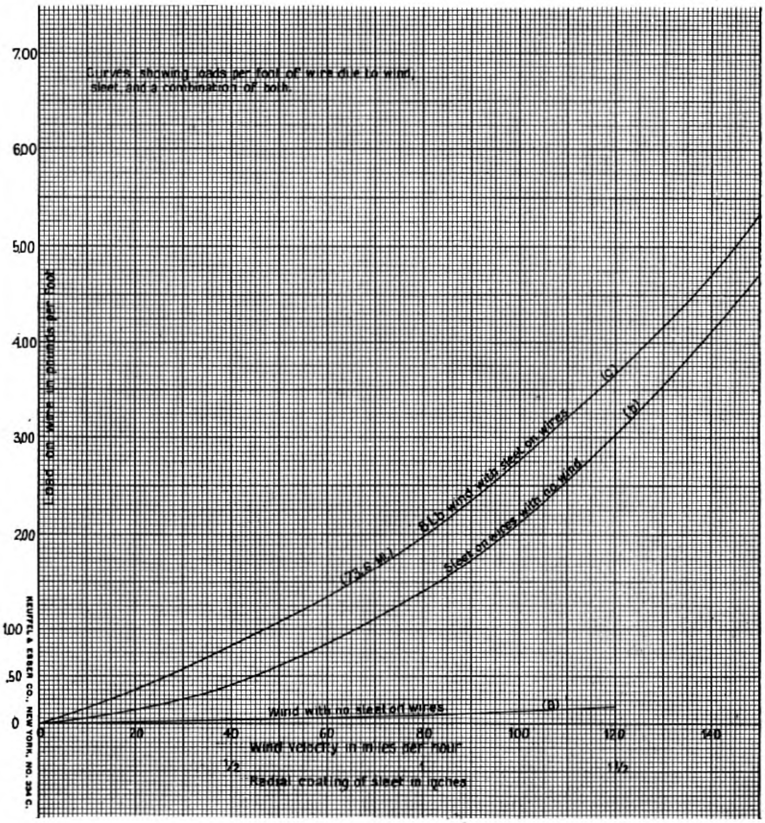


Fig. 3

The moment about the ground line section, due to the vertical loads is practically negligible due to the fact that the greater part of the weight of the pole, fixtures and attachments is balanced and therefore produces no moment upon the pole. Any small unbalanced weight acts through such a short lever arm that it produces a very small moment upon the pole. In any particular storm loading area, therefore, the pole should be designed to resist only the transverse load corresponding to that area. In calculating this load, the wind pressure is considered as acting upon the projected areas of the pole and of the ice-coated wires, strand, etc., for 1/2 of each adjacent span.

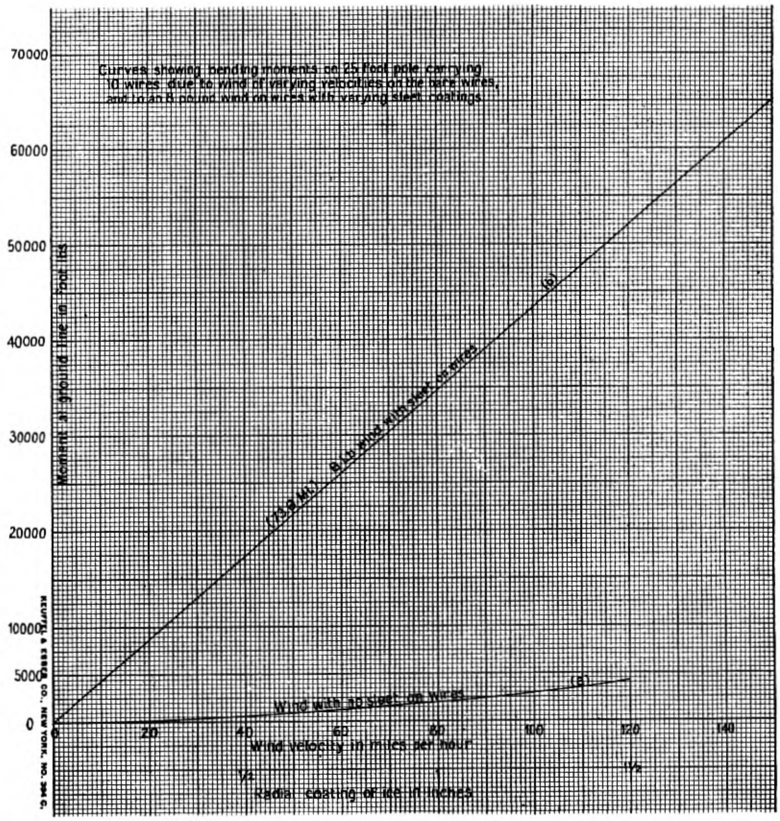


Fig. 4

Fig. 4 shows the relative magnitude of the bending moments at the ground line of a 25-foot pole carrying 10 wires, caused by (a) winds of various velocities with no ice on the wires, and (b) an 8-pound wind with various ice coatings on the wires. In the case of the pole design, it is evident that the wind pressure on the increased "sail area" created by the ice on the wires represents the predominant portion of the storm load.

There are in the Bell System approximately 14 million poles, carrying nearly 4 million miles of open wire of which it is estimated that about 9½ million poles and 2½ million miles of wire are located in what is known as the heavy loading area, some 2½ million poles and over 700,000 miles of wire in the medium area and about 2 million poles and nearly 600,000 miles of wire in the light or no sleet area. Nearly 70 per cent of the poles and aerial wire in the Bell System are, therefore, periodically exposed to storms of heavy loading intensity against which they must be designed, constructed and maintained in order that there may be minimum interruption of service consistent with the practicable and economical design of the plant as a whole.

Measurements on the Gases Evolved from Glasses of Known Chemical Composition

By J. E. HARRIS and E. E. SCHUMACHER

SYNOPSIS: This paper has a very direct bearing upon the pumping or exhausting of the vacuum tubes used in telephone repeaters and similar thermionic tubes. The telephone repeater bulb, as is well known, holds a vacuum of the order of 10^{-6} mm. of mercury. In order to produce this vacuum it is necessary during the manufacture of the tube to not only remove the air from the space within the bulb but also to allow very considerable amounts of various gases to diffuse out from within the glass of the bulb and the metal parts of the tube structure. The volume of gas which is frequently removed from the metal plates, for instance, may be roughly estimated as 100 times the volume of the plates themselves, the volume of the gas being measured at atmospheric pressure. To remove these gases from the bulb and metal parts, it is necessary to maintain during the pumping process a temperature which is far above the normal temperature and a fair degree of vacuum within the bulb for a period of time which varies from a few minutes to an hour or more depending upon the type of tube.

With a view to simplifying the pumping process, the authors have found that a glass relatively free from absorbed gases can be produced by using special precautions in manufacture. The authors have also measured and analyzed the gases evolved from glasses of various composition. Seven different samples representing four distinct types of glass have been experimented with. Six of these samples of glass have been carefully analyzed and definite relations found between the amounts and kinds of gases evolved and the chemical composition of the glasses.—*Editor.*

THIS investigation of the gases evolved by certain glasses when heated was undertaken with a view to securing a glass which, after an initial period of heating, should cease to give off appreciable quantities of gas. The use of such a gas-free glass would obviously be desirable in the experimental investigation of vacuum tubes, in which the filament and other parts within the tube may be affected adversely by gases evolved from the heated glass during manufacture of the tube.

The work of Guichard (11), Langmuir (19), Sherwood (30), Washburn (37), and others has established the following points with respect to the gases evolved by the glass when heated. The gases may be held as an adsorbed film or dissolved throughout the glass; the adsorbed gases are evolved readily at temperatures less than 300°C., whereas the dissolved gas, although it begins to come out of the surface layers at 200°C., comes out slowly, by reasons of the slowness of diffusion through the glass, even at much higher temperatures. Consequently the total gas evolved at each of a series of temperatures is a maximum somewhere between 200°C. and 400°C.; this decreases to a minimum, and rises again at temperatures approaching the softening range of the glass.

The main gases held by the glass are carbon dioxide (from the carbonates and possibly from the furnace gases), water (from the materials and the furnace gases), with smaller quantities of sulphur dioxide, oxygen and nitrogen. The amounts of the gases may correspond to a real equilibrium under the condition of melting; but more usually the viscosity of the melt has been such as to prevent the gases present in solution from escaping completely during the melting period. Commerical glasses consequently retain some gas which escapes slowly when the glass is reheated.

In most of the investigations published in the past the gases have been divided into three fractions: (a) condensable above –78°C. (b) condensable between –78°C. and –190°C., (c) not condensable at –190°C. These fractions represent with fair accuracy (a) water vapor, (b) carbon dioxide, (c) the permanent gases; oxygen, nitrogen, hydrogen. In general water vapor is the most abundant, followed by carbon dioxide; but, owing to lack of the necessary data, it is not possible to correlate these findings with the composition of the glass, still less with the mode of its melting.

Apparatus and Method

The apparatus used for determining the gases evolved from the glass on being heated is shown in Fig. 1.

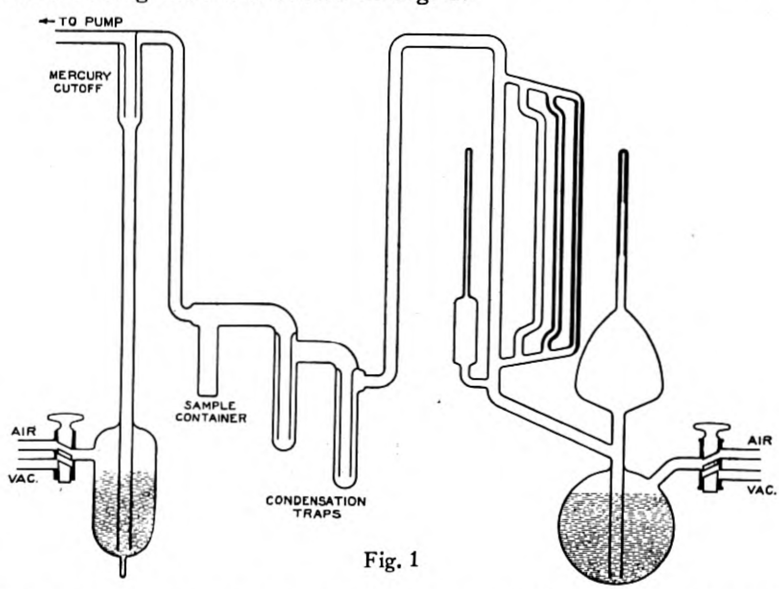


Fig. 1

The water vapor was removed by surrounding the condensation trap by a mixture of frozen and liquid acetone contained in a Dewar vessel; this mixture being obtained by adding liquid air to acetone until its freezing point –95°C. was reached. The carbon dioxide was removed similarly by means of liquid air. Readings were taken on the McLeod gauge (1) with no condensing agent, this giving the pressure of the total gas evolved; (2) with frozen acetone as condensing agent, this giving the pressure of gases less the water vapor, and (3) with liquid air as condensing material, corresponding to the pressure of the permanent gases. From the several pressures thus obtained the respective volumes could readily be calculated.

For the purpose of determining the amount of gases given up by the various types of glass, the glass was first cleaned with chromic acid, and washed thoroughly with water. It was then placed in the sample container (made of the same glass as that being tested) and the pump operated for several hours in order to dry the sample thoroughly. The pump used (a mercury vapor pump in conjunction with two oil pumps) was operated until the pressure within the apparatus after being trapped off from the pump remained constant at about 1 x 10^{-6} mm. for a period of at least two hours. The glass was then heated to a temperature as close to the softening range as it was possible to go without causing the container to collapse; it was kept at this temperature until the gas pressure became constant, when the volumes of the gases were determined in the manner outlined above. The period required for completely driving off the gases ranged from sixty-five to eighty hours.

Types of Glass Used

The types of glass worked with, together with their chemical compositions are given in Table I.

TABLE I.

Chemical Analysis of the Glasses Used

	1	2	3	4	5	6
SiO_2	69.93	69.40	64.64	61.50	72.05	65.47
Al_2O_3	1.54	.78	.20	.57	2.21	2.99
Fe_2O_3	.19	.14	.04	.11	.05	.51
PbO	1.44	Trace	21.66	22.55	6.11	20.20
CaO	3.17	5.15	.02	.21	.06	.22
MgO	.03	4.09	.02	.36	.09	.13
Na_2O	21.02	16.67	9.10	8.14	4.23	6.40
K_2O	.10	.20	3.20	3.76	1.12	3.59
P_2O_5	.08	.16	.75	.34	Trace	Trace
Sb_2O_3	.05	.10	....	....	Trace	....
MnO_2	.09	.19	....	.19	.01	.073
F_2	....	....	Trace	....	Trace	....
B_2O_3	2.36*	3.12*	.37*	2.27*	14.07*	....
SO_3	....	....	....	....	....	.013
BaO	....	....	....	....	....	Trace

* By difference.

It will be observed that glasses No. 1 and No. 2 are soda-lime glasses; 3, 4 and 6 are soda-lead glasses; 5 is a boro-silicate of lead and soda.

Experimental Results

The results of the determinations are shown in Table II.

TABLE II.

Gases Evolved from Glass of Known Chemical Composition During Heat Treatment in Vacuo

I	II	III	IV	V	VI		VII	VIII	IX
No. of Sample	% Alkali in the Glass	Vol. of Sample cc	Surface Area Cm^2	Temp. to which Glass Was Heated °C.	Total Gas Vol. cc		Comp. of Gas %	Vol. in cc Per Cm^2 x 10^4	Vol. Per cc Glass x 10^4
1	21.12	37.2	580	400	H_2O	2.70	88.5	46.6	726
					CO_2	.32	10.5	5.5	86
					P.G.	.03	1.0	.5	8
						3.05		52.6	820
2	16.87	31.9	540	400	H_2O	1.62	92.6	30.0	508
					CO_2	.11	6.3	2.0	34
					P.G.	.02	1.1	.4	6
						1.75		32.4	548
3	12.30	26.5	565	400	H_2O	1.34	96.4	23.7	506
					CO_2	.03	2.2	.5	11
					P.G.	.02	1.4	.4	8
						1.39		24.6	525
4	11.90	24.1	540	400	H_2O	1.37	97.2	25.4	568
					CO_2	.02	1.4	.4	8
					P.G.	.02	1.4	.4	8
						1.41		26.2	584
5	5.35	25.2	469	500	H_2O	.03	33.3	.6	12
					CO_2	.04	44.5	.9	16
					P.G.	.02	22.2	.4	8
						.09		1.9	36
6	9.99	15.6	292	400	H_2O	.03	49.6	1.0	19.
					CO_2	.03	49.6	1.0	19.
					P.G.	.0005	.8	.02	.3
						.0605		2.02	38.3

* P.G. = Permanent Gases.

If we may assume that the low melting glasses (1-4) worked with in this investigation received approximately the same heat treatment during their manufacture, then the data presented in Table II would indicate that there is a definite relation between gas evolved and the chemical composition of the glass. The amount of gas given up per square centimeter of glass surface is found to be closely parallel to the alkali content except in the case of glass number 6. This parallelism holds more closely for the amount of water vapor than it does for the other gases, although the relation does hold to a less striking extent for carbon dioxide. In a paper by Niggli (24), that treats of the phenomena of equilibrium between R_2O, $Si\ O_2$ and CO_2 ($R_2O = Na_2O, K_2O, Li_2O$), in melts at temperatures of 900 to 1000°C. under a pressure of one atmosphere CO_2, it is interesting to note, that the results obtained show the amount of CO_2 in the melt, when equilibrium is reached, to decrease as the composition of the melt becomes less alkaline. The amount of permanent gases evolved is roughly the same for the various types of glass. The reason for glass number 6 falling out of line with the other glasses will be discussed later. The authors, after making inquiries at several glass manufacturing concerns, feel that the statement can be made with a fair degree of justice that ordinary commercial glasses of like composition usually receive approximately the same heat treatment in their manufacture. Preservation of melting pots, saving in fuel, etc., make it very essential for the glass manufacturer to know the lowest temperature that can be used with the assurance of producing good glass. In all the glass factories where we made inquiries we found that this minimum temperature was about constant in cases where glasses having about the same melting points were being made. In these factories we were also told that the furnaces were usually held as near this minimum temperature as possible throughout the entire melting process.

Glass number 5 undoubtedly received a higher heat treatment in the melting process, because of its higher melting point and greater viscosity, than did the other glasses that were tested. The data given in Table II shows that this glass gave off very little gas when subjected to heat treatment in vacuo. The high heat treatment that this glass received in its manufacture is probably responsible in part for this small evolution of gas, but its low alkali content is probably equally responsible. It will be noted when reference is made to Table II that this glass was heated to 500°C. whereas the other glasses were only subjected to a heat treatment of 400°C.

EVOLUTION OF ADSORBED AND ABSORBED GAS

To determine the relations of adsorbed gases to absorbed gases in the six samples of glass, the pressures of the gases evolved were determined at intervals of 100°C. from 100°C. to the softening point of the glass. In addition to the six samples of glass, a run was taken on

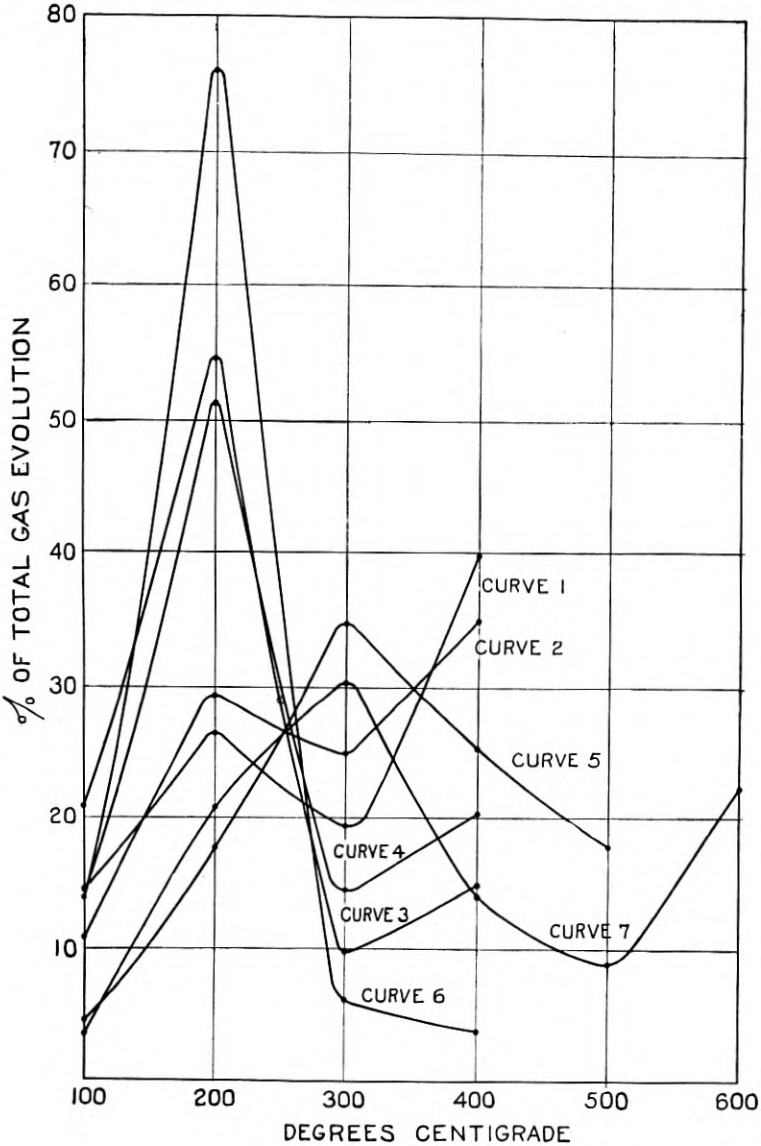


Fig. 2

a Pyrex Glass which will be indicated in the experiment as number 7. No analysis was made of this glass but it is known to be a boro-silicate glass practically free from alkali and heavy metals. From the pressures of the gases obtained, the distribution of gas evolution over the various intervals was determined. Expressed in terms of percentage of the total gas evolved over the entire range, these results are shown graphically in Fig. II.

It will be noted first of all that these curves with the exception of number 6, are in general similar to those obtained by Sherwood (loc. cit.) in that they show both maximum and minimum points. Glass No. 6 was found in this experiment as in the preceding one to behave quite differently from numbers 3 and 4, although chemically it is very similar. The explanation for this difference will appear later. The curves show very clearly the distinction between the adsorbed and absorbed gases. It is seen that the adsorbed gases for the lime and lead glasses are practically all given up at a temperature of 200°C. while 300°C. is required in the cases of the boro-silicate glasses. The absorbed gases begin to come off at the softening points of the various glasses; 400°C. for the lead and lime glasses and 600°C. for the boro-silicate glasses. The steeper slopes of the curves 1 and 2 above 300°, as compared to 3 and 4 are undoubtedly due to the fact that these glasses (lime glasses) become fluid more rapidly than do the lead glasses and therefore give up their dissolved gases at a more rapidly increasing rate than do the others. In this connection it should be stated that the amount of absorbed gases found in the above experiments represent only that portion of the dissolved gases which lie nearest the surface of the glass. Owing to the great viscosity of the glass at the temperatures used, the rate of diffusion of the gas would be altogether too slow to permit any considerable portion to reach the surface.

Some data that were taken in some of the experiments carried on in the investigation of the adsorbed gases are interesting in that they seem to throw considerable light on the question of the manner in which these gases are held to glass. Warburg and Ihmori (36) have maintained that the gases are held by chemical forces (primary valence forces) while other writers have maintained that the gases are held primarily by physical forces (secondary valence forces). The measurements of Warburg and Ihmori were made with water vapor and while no analyses of the glasses were made, it was found that only those glasses that contain alkali were capable of taking up water vapor.

Glasses Nos. 1, 2, 3, 4, 5 and 6 were experimented with in this connection. The glasses were heated to a temperature high enough to drive off all the adsorbed gases as indicated by the curves in Fig. 1 (200° for Nos. 1, 2, 3 and 4 and 300° for No. 5.) and the amount of carbon dioxide and permanent gases determined. The results are shown in Table III.

TABLE III.

Showing the Relation Existing Between Adsorbed Carbon Dioxide and Percentage Alkali in the Glass

Glass No.	Vol. of Adsorbed CO_2 in cm^3 x 10^5 per cm^2	Vol. of Adsorbed Permanent Gas in cm^3 x 10^5 per cm^2	% Na_2O+K_2O in Glass
1	19.3	1.30	21.12
2	10.4	1.80	16.87
3	7.5	2.65	12.30
4	7.5	2.35	11.90
5	4.5	.48	5.35
6	6.5	.009	9.99

These data show in a remarkable way the relation existing between the adsorbed carbon dioxide and the alkali content of the glass. This experiment would seem to indicate that adsorbed carbon dioxide is held by primary valence forces. Many other glasses must be tested, however, before such a statement can be accepted as a fact. It seems very plausible, however, to believe that carbon dioxide is taken up by a film of sodium hydroxide which has been formed by the slow hydrolysis of the glass.

This coincides with the view of Kraus (17) who found, in the case of a single glass which was high in soda, that no carbon dioxide was adsorbed when glass and gas were both carefully dried and also that on carbon dioxide is adsorbed by the glass if it has been washed with boiling water to remove the film of alkali on the surface. The above results do not necessarily indicate, however, that all gas held on the surface of the glass is held by primary valence forces. Indeed a determination that was made of the permanent gases given up by the glass up to 200°C. would indicate that these gases are held to the glass primarily by secondary valence forces. Table III shows that while appreciable quantities of these gases were evolved, no correlation existed between these quantities and the chemical composition of the glass. A similar determination was not made for water vapor because the capacity of the McLeod gauge used in the experiment was too small to take care of the large quantities of water vapor evolved; if a determination had been made in this connection, it seems highly probable that a definite relationship, existing between the adsorbed water vapor and the alkali content of the glass, would have been noted.

Manufacture of Relatively Gas Free Glass

In our first experiment we melted some ordinary commercial glass in vacuo. The temperature within the vacuum furnace was kept considerably above the melting point of the glass under examination and was held there for a period of about one hour. For a portion of this time huge quantities of gas were evolved. After this evolution subsided the furnace was cooled to room temperature and the melt was removed. Some of the glass obtained from this melt was later reheated in vacuo at various temperatures up to 500°C. At the lower temperatures from 20°C. to 200°C. a measurable amount of gas was evolved, but above 200°C. there was practically no gaseous evolotion. The results obtained from this and other similarly conducted experiments make it seem probable that a vacuum furnace process for the manufacture of certain kinds of glass is possible. The expense incurred, however, in the manufacture of gas free glass by this method would undoubtedly always prohibit its general use in the industries. After realizing that owing to the difficulties of making it, a gas free glass could be used only in a very limited field, we sought next to determine if it would not be possible to modify the ordinary standard procedure now pursued in glass making in such a way as to make a relatively gas free glass.

At our request, the Corning Glass Works agreed to undertake some experiments to produce glass that would be more nearly gas-free than that obtainable on the market.

In one of the most successful experiments and the only one that will be recorded here, materials that would produce a low melting point glass were subjected to a heat treatment of between 1500°C. and 1600°C. for a period of one hour. For a chemical analysis of the glass produced and for its behavior upon being heated in vacuo see Tables I and II—Glass 6.

Attention has been called to the fact that this glass behaved quite differently in the above experiments from the other glasses of the same type in two particulars (1) the total amount of gas given up by the glass is very much less than for the other glasses of like composition and (2) of the gas given up by this glass, 90% is given up at 200°C. indicating that this proportion of the gas is adsorbed gas. We may conclude then that the special treatment of this glass in its manufacture was very efficient in so far as the removal of the absorbed or dissolved gas was concerned. It could not, however, prevent the adsorption of gases by the glass on standing. The adsorbed gases, however, are rather unimportant from the standpoint of vacuum tube work since such gases can be fairly readily removed by a preliminary heating.

Summary of Results

The results of our investigation may be summarized as follows:

(1) Glasses whose compositions run high in alkali give off more gas during their heat treatment than do those of lower alkali content.

(2) A definite relation appears to exist between the amount of water vapor held by a glass and its alkali content.

(3) A relation, although not as pronounced as that mentioned above, appears to exist between the amount of carbon dioxide held by a glass and its alkali content.

(4) Adsorbed carbon dioxide seems to be held to glass primarily by primary valence forces.

(5) Adsorbed permanent gases seem to be held to glass primarily by secondary valence forces.

(6) Glass relatively free from absorbed gas can be produced by means of heating the glass during its melting process to a sufficiently high temperature.

In conclusion the authors wish to express their thanks to Dr. J. Johnston to whom they are especially indebted for assistance and advice.

BIBLIOGRAPHY

(1) Allen, E. T. & Zies, E. G. J. Am. Coram. Soc. *1* 739–790, 1918.
(2) Briggs, H. Nature, *107*, 285–286, 1921.
(3) Bunsen, R., Ueber die Verdichtung der CO_2 an blanken Glasflächen. Ann. d. Phys. (3) *20*², 545–560, 1883.
(4) ————Ueber die langsame Verdichtung der CO_2 an blanken Glasflächen und Kayser's Einwürfe dagegen (3), *22*, 145–152, 1884.
(5) ————Ueber capillare Gasabsorption. Ann. d. Phys. (3) *24*, 321–347, 1885.
Campbell, N. R. See Research Staff of G. E. Co.
(6) Campbell Swinton, A. A. The occlusion of the residual gas by the glass walls of vacuum tubes. Proc. Roy. Soc. A79, 134–137, 1907.
(7) ————The occlusion of the residual gas and the fluorescence of the glass walls of cathode tubes. Proc. Roy. Soc. *A81*, 453–59, 1908.
(8) Chappius, P., Die Verdichtung der Gase auf Glasoberflächen, Ann. d. Phys. (3) *8*, 1–29, 1879.
(9) Giesen, J. Einige Versuche mit der Salvionischen Mikrowage. Ann. d. Phys. (4) *10*, 830–844, 1903.
(10) Gouy, Sur la penetration des gaz dans les parois de verre des tubes de Crookes. C. R. 122, 775–76, 1896.
(11) Guichard, Sur les gaz degages des parois des tubes de verre. Bull. Soc. Chim. Paris, (4) *9*, 438–42, 1911.
(12) Hill, S. E. The absorption of gas in vacuum tubes. Proc. Phys. Soc. London, *25*, 35–43, 1912.

(13) Hughes, A. Ll. Dissociation of hydrogen and nitrogen by electron Impacts. Phil. Mag. (6) *41*, 778–798, 1921.
(14) Ihmori, T. Ann. Physik (3) *31*, 1006–1014, 1889.
(15) Jamin, L. and Bertrand, A. Note sur la condensation des gaz a la surface des corps solides. C. R. *36*, 994–98, 1853.
(16) Kayser, H. Ueber die langsame Verdichtung der CO_2 an blanken Glasflächen Ann. d. Phys. (3) 21, 495–498, 1884.
(17) Krause, H. Ueber Adsorption and Condensation. *V.* CO_2 an blanken Glasflächen. Ann. d. Phys. (3) 36, 923–36, 1889.
(18) Langmuir, I. Tungsten lamps of high efficiency. A. I. E. E. Trans. *32*, 1913–33, 1913.
(19) Langmuir, I. Adsorption of gases on plane surfaces of glass, mica, and platinum. J. Am. Chem. Sec. *40*, 1361–1403, 1918.
(20) Magnus, G. Ueber die Verdichtung der Gase an d. Oberfläche glatter Körper. Ann. d. Phys. (2) *89*, 604–10, 1853.
(21) Mehlhorn, F. Ueber die von feuchten Glasoberflächen fixirten permanenten Gase, Verh. d. deutsch. Phys. Ges. *17*, 123–28, 1898.
(22) Menzies, A. W. C. J. Am Chem. Soc. *42*, 978, 1920.
(23) Mulfärth, P. Adsorption of Gases by Glass Powder. Ann. d. Phys. (4), *3*, 328–52, 1900. Sci. Abs. 4, I 162, 1901.
(24) Niggii, P. The Phenomena of Equilibria Between Silica and the Alkali Carbonates, J. Am. Chem. Soc. *35*, 1693–1727, 1913.
(25) Parks, J. Thickness of the liquid film formed by condensation at the surface of a solid. Phil. Mag. (6) *5*, 517–523, 1903.
(26) Pettijohn, J. Am. Chem. Soc. *4*, 477–486, 1919.
(27) Pohl, R. Die Bildung von Gasblasen in den Wändenerhitzter Entladungsröhre. Verh. d. deutsch, Phys. Ges. *5*, 306–314, 1907.
(28) Research Staff of G. E. Co., London. The disappearance of gas in the electric discharge. Part I. Phil. Mag. (6) *40*, 585–611, 1920.
(29) ——————Part II. Phil. Mag. (6) *41*, 685–706, 1921.
(30) Sherwood, R. G. Effects of heat on chemical glassware. J. Am. Chem. Soc., *40*, 1645–53, 1918.
(31) Sherwood, R. G. Gases and vapors from glass. Phys. Rev. (2) *12*, 448–458, 1918.
(32) Shrader, J. E. Residual gases and vapors in highly exhausted glass bulbs. Phys. Rev. (2) *13*, 434–437, 1919.
(33) Soddy, F. and Mackenzie, T. D. The electric discharge in monatomic gases. Proc. Roy. Soc. *A80*, 92–109, 1907.
(34) Ulrey, D. Evolution and absorption of gases by glass. Phys. Rev. (2) *14*, 160–161, 1919.
(35) Vegard, L. On the electric discharge through *HC*1, *HBr*, *HI*, Phil. Mag. (6), *18*, 465–483, 1909.
(36) Warburg, E. and Ihmori, T. Ueber das Gewicht and die Ursache der Wasserhaut bei Glas and anderen Körpern. Ann. d. Phys. (3), *27*, 481–507, 1886.
(37) Washburn, E. W. Dissolved Gases in Glass. Univ. of Ill. Bulletin No. 118.
(38) Willows, R. S. On the absorption of gases in a Crookes tube. Phil. Mag. (6) *1*, 503–517, 1901.
(39) Willows, R. S. and George, H. T. The absorption of gas by quartz vacuum tubes. Proc. Phys. Soc. London, *28*, 124–131, 1916.

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